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FIXED POINTS OF BICOMPLEX AND MULTICOMPLEX VALUED ENTIRE FUNCTIONS

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Abstract. We develop a systematic fixed point theory for entire functions defined on bicomplex and, more generally, multicomplex algebras. Using the idempotent decomposition of these algebras, we establish precise correspondences between bicomplex (or multicomplex) functional equations and their componentwise complex counterparts. This reduction allows us to derive necessary and sufficient conditions for the existence of fixed points. We classify them according to the behaviour of their multipliers and describe their local dynamics. We further obtain a bicomplex analogue of Schröder's linearization theorem, which shows that local conjugacy to linear maps holds componentwise under the usual non-resonance conditions. All results are proved here directly from first principles with complete and self-contained arguments that extend naturally to higher multicomplex settings.

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1. Preliminaries: bicomplex and multicomplex algebras

1.1. **Bicomplex algebra.** The *bicomplex algebra* is

$$\mathbb{BC} = \{z_1 + jz_2 \ : \ z_1, z_2 \in \mathbb{C}\},$$

with $i^2 = j^2 = -1$, $ij = ji$ and $k = ij$ satisfying $k^2 = 1$. The idempotents

$$e_1 = \frac{1+k}{2} \quad \textit{and} \quad e_2 = \frac{1-k}{2}$$

satisfy $e_1^2 = e_1,\ e_2^2 = e_2,\ e_1e_2 = 0$ and $e_1 + e_2 = 1$. Every $w \in \mathbb{BC}$ has the *idempotent decomposition*

$$w = e_1w_1 + e_2w_2 \quad \textit{with} \quad w_1, w_2 \in \mathbb{C}.$$

The bicomplex-valued entire function now can be defined in the following way.

Definition 1.1. *$f : \mathbb{BC} \to \mathbb{BC}$ is called a* bicomplex-valued entire *function (holomorphic in the bicomplex sense) if, when written in idempotent form,*

$$f(w) = e_1f_1(w_1) + e_2f_2(w_2),$$

each $f_i : \mathbb{C} \to \mathbb{C}$ is an ordinary entire function (holomorphic entire).

Remark 1.2. *The following example clears the notion in Definition 1.1.*

Example 1.3. *Let $Z = z_1e_1 + z_2e_2$ and we consider the function*

$$f(Z) = \sin(\pi Z) = \sin(\pi z_1)e_1 + \sin(\pi z_2)e_2.$$

Then f is a bicomplex-valued entire.

1.2. **Multicomplex algebra.** For $n \geq 1$, the *multicomplex algebra* $\mathbb{M}_n$ is generated by commuting imaginary units $i_1, \ldots, i_n$ with $i_k^2 = -1$. $\mathbb{M}_n$ admits 2^{n-1} idempotent elements $e_1, \ldots, e_{2^{n-1}}$ and every element decomposes as

$$w = \sum_{r=1}^{2^{n-1}} e_r w_r \quad \textit{with} \quad w_r \in \mathbb{C}.$$

multicomplex-valued entire functions split componentwise as

$$f(w) = \sum_{r=1}^{2^{n-1}} e_r f_r(w_r),$$

with each $f_r : \mathbb{C} \to \mathbb{C}$ being entire.

We do not explain in detail the standard theories of bicomplex and multicomplex analysis as those are available in [15]. For further details, one can follow [10], [11], [17] and [18].

2. LEMMAS

The following lemma indicates the fixed point criterion and its classification for complex valued entire functions.

Lemma 2.1. *Let $f : D \to D$ be a holomorphic function on a domain $D \subset \mathbb{C}$. If there exists a point $z_0 \in D$ such that*

$$f(z_0) = z_0 \qquad \textit{and} \qquad |f'(z_0)| < 1,$$

then z_0 is an attracting *fixed point, i.e. for all z sufficiently close to z_0,*

$$\lim_{n\to\infty} f^n(z) = z_0,$$

where f^n denotes the n-fold iterate of f.

More generally,

$|f'(z_0)| > 1$ and $|f'(z_0)| = 1$ *respectively imply that z_0 is repelling and neutral (i.e. neutral).*

The following lemma reflects the flavour of idempotent reduction of bicomplex-valued functional equations.

Lemma 2.2. *Let $f : \mathbb{BC} \to \mathbb{BC}$ be bicomplex entire with component functions $f_1, f_2 : \mathbb{C} \to \mathbb{C}$. For $w = e_1 w_1 + e_2 w_2$ and any bicomplex equation $F(f(w), w) = 0$ where F is $\mathbb{BC}$-linear in each slot (in particular $f(w) - w = 0$), the equation holds if and only if it holds componentwise. In other words*

$$F(f(w), w) = 0 \quad \textit{if and only if} \quad F(e_1 f_1(w_1), e_1 w_1) = 0 \text{ and } F(e_2 f_2(w_2), e_2 w_2) = 0,$$

which is equivalent to two complex equations in w_1 and w_2.

Proof. Since e_1, e_2 are orthogonal idempotents with $e_1 + e_2 = 1$, any bicomplex equality splits uniquely along the components e_1 and e_2.

Since F is linear and each f_i depends only on w_i for $i = 1, 2$, we may write $F(f(w), w) = e_1A_1(w_1) + e_2A_2(w_2)$.

So $F(f(w), w) = 0$ if and only if $A_1(w_1) = 0$ and $A_2(w_2) = 0$.

This completes the proof of the lemma. □

The following corollary is immediate from Lemma 2.2 and so its proof is omitted.

Corollary 2.3. *Any identity, root, fixed point or derivative condition for a bicomplex-valued entire function is equivalent to the corresponding pair of conditions on the complex components* f_1 and f_2.

3. Existence of fixed point criterion for bicomplex-valued entire functions

We start with the definition of fixed point of a bicomplexial entire function.

Definition 3.1 (cf. [2], [12])**.** *A point* $w_0 \in \mathbb{BC}$ *is a fixed point of* $f : \mathbb{BC} \to \mathbb{BC}$ *if* $f(w_0) = w_0$.

The following theorem provides a criterion for bicomplex-valued entire functions to have fixed point(s).

Theorem 3.2. *Let* $f : \mathbb{BC} \to \mathbb{BC}$ *be bicomplex-valued entire function with* $f(w) = e_1f_1(w_1) + e_2f_2(w_2)$. *A point* $w_0 = e_1\xi_1 + e_2\xi_2 \in \mathbb{BC}$ *is a fixed point of* f *if and only if*

$$f_1(\xi_1) = \xi_1 \quad \textit{and} \quad f_2(\xi_2) = \xi_2 \quad \textit{for} \quad \xi_1, \xi_2 \in \mathbb{C}.$$

Proof. Let us write $f(w_0) - w_0 = e_1\big(f_1(\xi_1) - \xi_1\big) + e_2\big(f_2(\xi_2) - \xi_2\big)$. By orthogonality of e_1 and e_2, the left hand side of the expression will be equal to zero if and only if each coefficient is zero. Thus $f(w_0) = w_0$ if and only if $f_i(\xi_i) = \xi_i$ for $i = 1, 2$. This proofs the theorem. □

Remark 3.3. *Theorem 3.2 can alternatively be stated as:*

There exists a bicomplexial fixed point of $f : \mathbb{BC} \to \mathbb{BC}$ *if and only if there exists* $\xi_i \in \mathbb{C}$ *with* $f_i(\xi_i) = \xi_i$ *for* $i = 1, 2$.

Remark 3.4. *Every entire function on* $\mathbb{C}$ *does not have a fixed point (e.g.* $g(z) = z + e^z$ *has no fixed point because* $g(z) - z = e^z \neq 0$ *for all* z*). Consequently, a bicomplexial entire function may have no fixed points. It has a fixed point if and only if* each *of its idempotent-component has a fixed point* {*cf.* [2], [12]}.

This can be seen more clearly from the following illustrative example.

Example 3.5. *Let* $f(w) = e_1(w_1 + e^{w_1}) + e_2(w_2 + e^{w_2})$. *Here* $f_1(z) = z + e^z$ *and* $f_2(z) = z + e^z$ *and neither* f_1 *nor* f_2 *has a complex fixed point. Therefore* f *has no fixed point in* $\mathbb{BC}$.

4. Classification of fixed points of bicomplex-valued functions via multipliers

The following definition reflects on bicomplex derivative via multiplier.

Definition 4.1. *Let $f(w) = e_1 f_1(w_1) + e_2 f_2(w_2)$ with $w_1, w_2 \in \mathbb{C}$ be a bicomplex-valued entire function and $w_0 = e_1\xi_1 + e_2\xi_2$ be a fixed point of it. We define the* multiplier *at w_0 by*

$$f'(w_0) = e_1 f_1'(\xi_1) + e_2 f_2'(\xi_2).$$

A precise classification of fixed points is given by the following definition.

Definition 4.2. *Let $\lambda_1 = f_1'(\xi_1)$ and $\lambda_2 = f_2'(\xi_2)$. The fixed point $w_0 = e_1\xi_1 + e_2\xi_2 \in \mathbb{BC}$ is*

- attracting *if $|\lambda_1| < 1$ and $|\lambda_2| < 1$;*
- repelling *if $|\lambda_1| > 1$ and $|\lambda_2| > 1$ and*
- neutral *i.e. indif and only iferent otherwise (i.e. it includes mixed cases where one component is attracting and the other repelling).*

The next theorem describes how the dynamical behaviour of bicomplexial fixed points is entirely governed by its complex components.

Theorem 4.3. *Let f and w_0 be as in Definition 4.1. Then the local dynamical behaviour of iterations $w_{n+1} = f(w_n)$ near w_0 is determined by the behaviours of f_1 near ξ_1 and f_2 near ξ_2. In particular,*

(i) If $|\lambda_i| < 1$ for $i = 1, 2$, then w_0 is attracting and any sufficiently close initial point w has $w_n \to w_0$ as $n \to \infty$.

(ii) If $|\lambda_i| > 1$ for $i = 1, 2$, then w_0 is repelling and nearby points move away under iteration.

(iii) If one $|\lambda_i| < 1$ and the other $|\lambda_j| > 1$, the fixed point is of mixed type *i.e. convergence in the e_i–component and divergence in the e_j–component for $i \neq j$.*

Proof. Let us write the points and iteration in idempotent form as

$$w_n = e_1 w_{1,n} + e_2 w_{2,n} \quad \text{and} \quad w_{i,n+1} = f_i(w_{i,n}).$$

Thus the bicomplex iteration decomposes into two independent complex iterations. Classical one-dimensional complex dynamics focusing local linearization, derivative criterion and stable manifold considerations give items (i)-(iii) componentwise. Recombining with idempotents we obtain the global bicomplex behaviour. Convergence of both the components yields convergence in $\mathbb{BC}$ while divergence of any component yields divergence in $\mathbb{BC}$ as because idempotents are orthogonal summands.

This proofs the theorem. □

5. Local linearization of bicomplex-valued entire functions

The following lemma represents Schrder linearization theorem in classical complex analysis.

Lemma 5.1. *If g is holomorphic near a fixed point ξ with multiplier $\lambda(\neq 0)$, not a root of unity, then there is a local holomorphic change of coordinate h with $h(\xi) = 0$, $h'(\xi) = 1$ that conjugates g to the linear map $z \mapsto \lambda z$.*

The following theorem establishes the bicomplex version of Schrder linearization theorem, obtained through idempotent decomposition.

Theorem 5.2. *Let $f : \mathbb{BC} \to \mathbb{BC}$ be bicomplex entire and let $w_0 = e_1\xi_1 + e_2\xi_2$ be a fixed point with multipliers $\lambda_i = f_i'(\xi_i)$. Also for $i = 1, 2$ let $\lambda_i \neq 0$ and λ_i is not a root of unity (i.e. $\lambda_i^m \neq 1$ for all positive integers m). Then there exist local biholomorphic maps (in respective idempotent components) h_i defined in neighborhoods of ξ_i with $h_i(\xi_i) = 0$, $h_i'(\xi_i) = 1$, such that*

$$h_i \circ f_i \circ h_i^{-1}(z) = \lambda_i z$$

for $i = 1, 2$. Further if $H = e_1h_1 + e_2h_2$, then

$$H \circ f \circ H^{-1}(w) = e_1\lambda_1 w_1 + e_2\lambda_2 w_2$$

holds locally, i.e. f is locally conjugate to the linear map $w \mapsto f'(w_0)\, w$ in the idempotent co-ordinates.

Proof. By applying Lemma 5.1 independently to each f_i at ξ_i we obtain h_i as stated for $i = 1$ and 2. Now let us define $H(w) = e_1h_1(w_1) + e_2h_2(w_2)$. Then the conjugacy follows by evaluating componentwise and using orthogonality of idempotents as

$$H\big(f(e_1z_1 + e_2z_2)\big) = e_1h_1(f_1(z_1)) + e_2h_2(f_2(z_2)) = e_1\lambda_1h_1(z_1) + e_2\lambda_2h_2(z_2).$$

Similar expression holds for H^{-1}. This proves the theorem. □

Remark 5.3. *Theorem 5.2 gives local linear models of bicomplex dynamics by combining the one-variable linearizations. If one component fails the non-resonance condition (i.e. λ is root of unity or $\lambda = 0$), Lemma 5.1 fails and one must use dif and only iferent normal forms, applied componentwise.*

6. Multicomplex generalizations

The statements and proofs in Sections 3, 4 and 5 can be generalized to $\mathbb{M}_n$ {cf. [9], [15] and [17]} by replacing two idempotents with 2^{n-1} idempotents of $\mathbb{M}_n$.

The following theorem is the multicomplex analogue of Theorems 3.2, 4.3 and 5.2 and in fact it is a fixed point criterion in $\mathbb{M}_n$.

Theorem 6.1. *Let $f : \mathbb{M}_n \to \mathbb{M}_n$ be multicomplex-valued entire with idempotent decomposition as*

$$f(w) = \sum_{r=1}^{2^{n-1}} e_r f_r(w_r) \quad \textit{with} \quad w = \sum_{r=1}^{2^{n-1}} e_r w_r.$$

Then $w_0 = \sum e_r\xi_r$ is a fixed point of f if and only if $f_r(\xi_r) = \xi_r$ for all $r = 1, \ldots, 2^{n-1}$. Lemma 2.1 holds componentwise in the same manner as respectively in Theorem 4.3 and Theorem 5.2.

Proof. The argument of the idempotent decomposition in Lemma 2.2 extends the fact that orthogonal idempotents split any multicomplex equality into 2^{n-1} complex equalities. Now applying Lemma 2.1 and Lemma 5.1 to each component f_r $(r = 1, 2, \cdots, 2^{n-1})$ independently and recombining the component results with idempotents we obtain the multicomplex conclusions. □

For further studies one can see [1], [3], [4], [5], [7], [13], [14], [16], [19] and [20].

7. Alternative view point of multicomplex analysis {cf. [15]}

A multicomplex number $\xi_n \in \mathbb{C}_n$ can be defined as

$$\xi_n \;=\; \xi_{n-1,1} \;+\; i_n\, \xi_{n-1,2},$$

where

$$\xi_{n-1,1},\, \xi_{n-1,2} \in \mathbb{C}_{n-1}, \quad (i_n)^2 = -1.$$

Addition and multiplication in multicomplex space $\mathbb{C}_n$ are defined componentwise extended by $(i_n)^2 = -1$.

7.1. **Idempotent elements in $\mathbb{C}_n$.** It is easy to verify that the following are idempotent elements in $\mathbb{C}_n$.

$$0, \quad 1, \quad \frac{1+i_1i_2}{2},\, \frac{1-i_1i_2}{2},\, \frac{1+i_1i_3}{2},\, \frac{1-i_1i_3}{2},\, \frac{1+i_2i_3}{2},\, \frac{1-i_2i_3}{2},\, \ldots,\, \frac{1+i_{n-1}i_n}{2},\, \frac{1-i_{n-1}i_n}{2}.$$

For convenience in notation, define the symbols

$$e(i_ri_s) \;=\; \frac{1+i_ri_s}{2}, \qquad e(-i_ri_s) \;=\; \frac{1-i_ri_s}{2} \quad \text{where } r, s \in \mathbb{N}.$$

7.2. **Idempotent representation.** Let ξ be an element in $\mathbb{C}_n$ and let $\xi = \xi_1 + i_n\xi_2$ with $\xi_1,\, \xi_2 \in \mathbb{C}_{n-1}$. Then

$$\xi = \big(\xi_1 - i_{n-1}\,\xi_2\big)\, e\big(i_{n-1}i_n\big) \;+\; \big(\xi_1 + i_{n-1}\,\xi_2\big)\, e\big(-\,i_{n-1}i_n\big).$$

7.3. **Idempotent Representation of Holomorphic multicomplex-valued Functions.** Let X be a domain in $\mathbb{C}_n$, $n \geq 1$, and let f be a holomorphic function in $\mathbb{C}_n$ then there exists holomorphic functions

$$f_1 : X_1 \to \mathbb{C}_{n-1} \quad \text{and} \quad f_2 : X_2 \to \mathbb{C}_{n-1} \quad \text{where} \quad X_1, X_2 \subseteq \mathbb{C}_{n-1},$$

such that

$$f(\xi_1 + i_n\xi_2) = f_1(\xi_1 - i_{n-1}\xi_2)e(i_{n-1}i_n) + f_2(\xi_1 + i_{n-1}\xi_2)e(-i_{n-1}i_n).$$

For further study on multicomplex analysis one can see [6] and [8].

In view of this alternative perspective in multicomplex analysis, Theorem 6.1 can be reformulated as follows.

Theorem 7.1. *Let $f : \mathbb{C}_n \to \mathbb{C}_n$ be multicomplex-valued entire with idempotent decomposition as*

$$f(\xi_1 + i_n\xi_2) = f_1(\xi_1 - i_{n-1}\xi_2)e(i_{n-1}i_n) + f_2(\xi_1 + i_{n-1}\xi_2)e(-i_{n-1}i_n).$$

Then $w_0 = w_{0,1}e(i_{n-1}i_n) + w_{0,2}e(-i_{n-1}i_n)$ is a fixed point of f if and only if $f_i(w_{0,i}) = w_{0,i}$ for $i = 1$ and 2. Lemma 2.1 holds componentwise in the same manner as respectively in Theorem 4.3 and Theorem 5.2.

Proof. The proof of Theorem 7.1 can be carried out using the idempotent decomposition and mathematical induction and therefore the same is omitted. □

8. Examples

In this section we present some examples of bicomplex-valued entire functions respectively having no fixed point and unique fixed point. Also a few examples on bicomplexial quadratic map and transcendental map having fixed points are provided.

Remark 8.1. *The following example demonstrates that a bicomplex holomorphic map may not have a fixed point.*

Example 8.2. *Let $f : \mathbb{BC} \to \mathbb{BC}$ be defined by $f(Z) = Z + e^Z$. By idempotent decomposition $f(Z)$ can be expressed as $f(Z) = f_1(z_1)e_1 + f_2(z_2)e_2$ where $f_1(z_1) = (z_1 + e^{z_1})$ and $f_2(z_2) = (z_2 + e^{z_2})$. Clearly the idempotent components f_1 and f_2 do not have any fixed point in $\mathbb{C}$. Therefore f has no fixed point in $\mathbb{BC}$.*

Remark 8.3. *Fixed point can be uniquely determined for certain linear map as we see in the following example.*

Example 8.4. *Let us consider the map $f : \mathbb{BC} \to \mathbb{BC}$ defined by $f(Z) = aZ + b$ where a and b are bicomplex numbers. Then $f(Z)$ can be decomposed as $f(Z) = f_1(z_1)e_1 + f_2(z_2)e_2$ where $f_1(z_1) = a_1z_1 + b_1$ and $f_2(z_2) = a_2z_2 + b_2$ with $a_1 = \frac{1}{2}$, $a_2 = 2$, $b_1 = 1$ and $b_2 = 3$.*

The only fixed point of f is clearly

$$Z = 2e_1 - 3e_2.$$

Remark 8.5. *Using the standard basis $\{1, i, j, ij\}$ with*

$$e_1 = \frac{1 + ij}{2} \quad \text{and} \quad e_2 = \frac{1 - ij}{2},$$

the fixed point $f(Z)$ in Example 8.4 can be expressed as

$$Z = 2\frac{1 + ij}{2} - 3\frac{1 - ij}{2} = \left(2 \cdot \frac{1}{2} - 3 \cdot \frac{1}{2}\right) \cdot 1 + \left(2 \cdot \frac{1}{2} - (-3) \cdot \frac{1}{2}\right) \cdot ij.$$

i.e.

$$Z = -\frac{1}{2} + \frac{5}{2}\, ij.$$

Remark 8.6. *The following example investigates the existence of fixed point for a bicomplex-valued quadratic map.*

Example 8.7. *Let us take* $f : \mathbb{BC} \to \mathbb{BC}$ *where* $f(Z) = Z^2 + c$ *and* c *is a bicomplex number. Then the bicomplex decomposition of* $f(Z)$ *is* $f(Z) = f_1(z_1)e_1 + f_2(z_2)e_2$

where $f_1(z_1) = z_1^2 + c_1$ *and* $f_2(z_2) = z_2^2 + c_2$

with $c = c_1e_1 + c_2e_2 = 0 \cdot e_1 + 1 \cdot e_2$.

To find out the fixed point(s) of $f(Z)$ *we should use the componentwise equations* $f_1(z_1) = z_1$ *and* $f_2(z_2) = z_2$, *i.e.*

$$z_1^2 - z_1 = 0 \quad \textit{and} \quad z_2^2 - z_2 + 1 = 0.$$

On solving both the equations, we obtain that

$$z_1 = 0, 1 \quad \textit{and} \quad z_2 = \frac{1 \pm i\sqrt{3}}{2}.$$

Now we take all the combinations of z_1 *and* z_2 *as above. So, there are* $2 \times 2 = 4$ *fixed points, which are given by*

$$Z_{1,\pm} = 0 \cdot e_1 + \frac{1 \pm i\sqrt{3}}{2}e_2 \quad \textit{and} \quad Z_{2,\pm} = 1 \cdot e_1 + \frac{1 \pm i\sqrt{3}}{2}e_2.$$

Remark 8.8. *Let us expand the fixed point* $Z = 1 \cdot e_1 + \dfrac{1 + i\sqrt{3}}{2}e_2$ *in Example 8.7 using the standard basis* $\{1, i, j, ij\}$.

Let $1 \cdot e_1 + \dfrac{1 + i\sqrt{3}}{2}e_2 = z_1e_1 + z_2e_2$. *Then*

$$z_1 = 1 \quad \textit{and} \quad z_2 = \frac{1 + i\sqrt{3}}{2}.$$

We use the relation connecting idempotent representation to the $1, ij$ *decomposition as*

$$Z = \frac{z_1 + z_2}{2} + \frac{z_1 - z_2}{2}\, ij.$$

Then

$$z_1 + z_2 = 1 + \frac{1 + i\sqrt{3}}{2} = \frac{3 + i\sqrt{3}}{2}$$

and

$$z_1 - z_2 = 1 - \frac{1 + i\sqrt{3}}{2} = \frac{1 - i\sqrt{3}}{2}.$$

Therefore

$$Z = \frac{3 + i\sqrt{3}}{4} + \frac{1 - i\sqrt{3}}{4}\, ij.$$

Now expand ij *into the* $\{1, i, j, ij\}$ *basis is already* ij, *so the full coordinate vector relative to* $\{1, i, j, ij\}$ *is obtained by writing the complex coefficients in terms of real and imaginary parts if desired. We write the real and* i *parts as*

$$\frac{3 + i\sqrt{3}}{4} = \frac{3}{4} + i\frac{\sqrt{3}}{4} \quad \textit{and} \quad \frac{1 - i\sqrt{3}}{4} = \frac{1}{4} - i\frac{\sqrt{3}}{4}.$$

Hence

$$Z = \underbrace{\frac{3}{4}}_{\textit{real part}} + \underbrace{\frac{\sqrt{3}}{4}}_{i\textit{-part}}\, i \;+\; 0 \cdot j \;+\; \left(\frac{1}{4} - i\frac{\sqrt{3}}{4}\right) ij.$$

If you wish to separate into real coefficients of $1, i, j, ij$*, note* ij *is independent and carries a complex coefficient, which itself splits into real and* i *components corresponding to* ij *and* $(i)(ij) = i^2 j = -j$*.*

Remark 8.9. *The existence of transcendental fixed point in case of a bicomplex-valued entire function is ensured by the following example.*

Example 8.10. *We take*

$$f(Z) = e^Z - 1.$$

Then

$$f(Z) = (e^{z_1} - 1)e_1 + (e^{z_2} - 1)e_2.$$

The fixed point(s) of $f(Z)$ *are obtained by solving the associated componentwise equations*

$$e^{z_k} - 1 = z_k \quad with \quad k = 1, 2.$$

By rearranging, we get

$$e^{z_k} = z_k + 1.$$

Let $u_k = z_k + 1$*. Then* $e^{u_k - 1} = u_k$*, so*

$$u_k e^{-u_k} = e^{-1} \quad i.e. \quad -u_k e^{-u_k} = -e^{-1}.$$

Thus

$$-u_k = W_m(-e^{-1}) \quad i.e. \quad u_k = -W_m(-e^{-1}),$$

where W_m *denotes the* m*th branch of the Lambert* W*-function* ($m \in \mathbb{Z}$) *which is a multi-valued function satisfying*

$$W(z)\, e^{W(z)} = z,$$

for any complex number z*. Therefore*

$$z_k = u_k - 1 = -1 - W_m(-e^{-1}).$$

Remark 8.11. *For each branch* m *of* W *in Example 8.10, there is a complex solution* z_k*. Taking arbitrary choices of branches for* $k = 1, 2$ *gives a countable family of bicomplex fixed points*

$$Z = \big(-1 - W_{m_1}(-e^{-1})\big)e_1 + \big(-1 - W_{m_2}(-e^{-1})\big)e_2,$$

indexed by $(m_1, m_2) \in \mathbb{Z}^2$*.*

We may select principal branches $m_1 = m_2 = 0$ *to obtain the principal bicomplex solution.*

Remark 8.12. *For certain multicomplex-valued linear maps, the fixed point is uniquely determined, as illustrated in the following example.*

Example 8.13. *Let* $F : \mathbb{M}_3 \to \mathbb{M}_3$ *is defined by* $F(W) = aW + b$*, where* $a, b \in \mathbb{M}_3$*.*

Let us write F *in idempotent decomposition form as* $F(W) = \sum_{k=1}^{4} F_k(w_k)e_k$ *where* $F_k(w_k) = a_k w_k + b_k$ *with* $a_1 = \frac{1}{2}, a_2 = 2, a_3 = 3$ *and* $a_4 = 0$ *and* $b_1 = 1, b_2 = 2, b_3 = 3$ *and* $b_4 = 4$*.*

To compute the fixed point(s) of $F(W)$, it is necessary to analyze the componentwise system of equations,

$$a_k w_k + b_k = w_k \quad i.e. \quad (1-a_k)w_k = b_k \quad for \quad k = 1, 2, 3 \text{ and } 4.$$

By solving each equations, we will get that

$$w_1 = 2, \quad w_2 = -2, \quad w_3 = -\tfrac{3}{2} \quad and \quad w_4 = 4.$$

Thus the unique tricomplex fixed point of $F(W)$ is

$$W = 2e_1 - 2e_2 - \tfrac{3}{2}e_3 + 4e_4.$$

Remark 8.14. *The following example analyzes the fixed point behaviour of a multicomplexial quadratic map.*

Example 8.15. *Let us consider the tricomplex valued map $F(W) = W^2 + c$ where c is a tricomplex point. By decomposing $F(W)$ in idempotent components we have*

$$F(W) = \sum_{k=1}^{4} F_k(w_k)e_k \quad where \quad F_k(w_k) = w_k^2 + c_k \quad with \quad k = 1, 2, 3 \text{ and } 4.$$

Now we take

$$c_1 = 0, \quad c_2 = 1, \quad c_3 = -1 \quad and \quad c_4 = i_1.$$

In order to determine the fixed point(s) of $F(W)$, we have to solve the equations $F_k(w_k) = w_k$ for $k = 1, 2, 3$ and 4,

i.e.

$$w_k^2 - w_k + c_k = 0.$$

After solving each equations, we get that

$$w_1 = 0, 1, \quad w_2 = \frac{1 \pm i\sqrt{3}}{2}, \quad w_3 = \frac{1 \pm \sqrt{5}}{2} \quad and \quad w_4 = \frac{1 \pm \sqrt{1 - 4i_1}}{2}.$$

Taking all possible combinations of w_k for $k = 1, 2, 3$ and 4 as above yields $2\times2\times2\times2 = 16$ fixed points.

Explicitly, all the fixed points are

$$W = w_1e_1 + w_2e_2 + w_3e_3 + w_4e_4,$$

with $w_1 \in \{0, 1\}$, $w_2 \in \{\frac{1\pm i\sqrt{3}}{2}\}$, $w_3 \in \{\frac{1\pm\sqrt{5}}{2}\}$ and $w_4 \in \{\frac{1\pm\sqrt{1-4i_1}}{2}\}$.

Remark 8.16. *All the above examples follow the same computational procedures, which are (1) to write the hypercomplex number in idempotent decomposition, (2) to solve the ordinary complex fixed-point equations componentwise, (3) to form all combinations of component solutions and if needed, to convert back to a chosen standard basis. These examples contain the detailed arithmetic steps required for concrete calculations.*

9. Open problems and Future Directions

The notion of bicomplex and multicomplex setting of Schrder linearization theorem and theorems describing the dynamical behaviour of existence of fixed points may be extended for the bicomplex and multicomplex-valued functions of uncertain variables and are still virgin. In this setting, the occurrence of parabolic or resonant multipliers in any component calls for a componentwise application of the classical Leau Fatou theory, whose recombination may yield richer global bicomplex behaviour. Similarly, global dynamical questions such as Fatou and Julia structures or escaping behaviour naturally begin with separate component analyses before examining how algebraic coupling merges these into new phenomena. These directions remain largely unexplored and offer ample scope for future investigation. Therefore those may be posed as open problems to the future researchers of this branch.

10. Conclusion

The methodologies developed in this paper rely fundamentally on the principle that every bicomplex or multicomplex functional equation reduces, via idempotent decomposition, to a family of independent complex equations. This componentwise correspondence enables the direct use of classical complex analytic techniques to determine fixed point existence, classify their dynamical behaviour and establish the bicomplex analogue of Schrder linearization. Consequently, the entire fixed point structure in the bicomplex or multicomplex setting is completely governed by the behaviour of the underlying complex components. All proofs carried out here are *componentwise reductions* to classical single-variable complex results via idempotent decomposition. This is in fact the standard and robust method for many multicomplex statements.

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