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COMPARATIVE GROWTH ANALYSIS OF COMPOSITE BICOMPLEX AND MULTICOMPLEX VALUED ENTIRE FUNCTIONS

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Abstract. In this paper, we extend the central index based growth properties of composite entire functions to the bicomplex and multicomplex settings by introducing (α, β, γ)-order and (α, β, γ)-lower order through idempotent decomposition. We also establish the corresponding central index driven growth estimates for compositions of entire functions and illustrate the results by examples.

Keywords: Bicomplex and multicomplex valued entire functions, central index, (α, β, γ)-order, growth estimate.

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1. Introduction

Growth of entire and meromorphic functions is a central topic in classical complex analysis. Several refined growth indicators such as iterated orders, (p, q)-orders, φ-orders, and recently (α, β, γ)-order have been introduced to capture diversified growth behaviours. The central index of an entire power series is another instrument for growth analysis. It measures the coefficient that dominates the maximum term at radius r. The present paper generalizes central index oriented growth analysis, specifically the results of *central index based growths of composite entire functions* in terms of (α, β, γ)-order to the bicomplex and multicomplex frameworks.

Also, we give the preliminaries and relevant definitions used in bicomplex & multicomplex analysis and develop analogous results. Proofs are adapted to the multi-component structure of bicomplex and multicomplex function theory and use componentwise maxima and central indices. The classical exposition in {[1], [8] & [16]} offers further insight on this topic.

2. Preliminaries

We briefly recall the algebraic models and notation used in the sequel. The aim is to keep the exposition on the growth notions as we need.

2.1. **Bicomplex numbers and functions.** We start with the definition of a bicomplex number.

Definition 2.1. *The algebra of bicomplex numbers* $\mathbb{BC}$ *is defined by*

$$\mathbb{BC} = \{z_1 + jz_2 \ : \ z_1, z_2 \in \mathbb{C}, \ j^2 = -1, \ j \text{ commutes with } i\},$$

often realized with two commuting imaginary units i and j, where $i^2 = j^2 = -1$ and $ij = ji$. An important representation uses idempotents

$$e_1 = \frac{1+ij}{2} \quad \textit{and} \quad e_2 = \frac{1-ij}{2},$$

satisfying $e_1^2 = e_1,\ e_2^2 = e_2,\ e_1e_2 = 0$ & $e_1 + e_2 = 1$. Every $w \in \mathbb{BC}$ has the idempotent decomposition

$$w = w_1e_1 + w_2e_2 \quad \textit{where} \quad w_1, w_2 \in \mathbb{C}.$$

The following definition reflects on a bicomplexial entire function.

Definition 2.2. *A function $F : \mathbb{BC} \to \mathbb{BC}$ is called bicomplex entire if in the idempotent representation of F*

$$F(w) = F_1(w_1)e_1 + F_2(w_2)e_2,$$

each F_k is an entire function of one complex variable $w_k \in \mathbb{C}$ for $k = 1$ & 2.

Remark 2.3. *The following two examples ensure Definition 2.1 and Definition 2.2, respectively.*

Example 2.4. *Consider $1 + 3i - ij = 1 + 3i + j(-i)$ is a bicomplex number where $1 + 3i$ & $-i \in \mathbb{C}$.*

Example 2.5. *Let $Z = z_1e_1 + z_2e_2$ and we consider the function*

$$f(Z) = \cos(\pi Z) = \cos(\pi z_1)e_1 + \cos(\pi z_2)e_2.$$

Then f is bicomplex valued entire.

Maximum modulus and maximum term of bicomplex valued entire functions are defined in the following manner.

Definition 2.6. *Let F be a bicomplex valued entire function with the decomposition as in Definition 2.2. Now the maximum modulus of F on $|\zeta| = r$ denoted by $M_F(r)$ is defined as*

$$M_F(r) = \max\{|F_1(\zeta)|, |F_2(\zeta)| \ : \ |\zeta| = r\} \quad \textit{for} \quad r > 0,$$

i.e., the maximum of the modulus of the two idempotent components.

Similarly, if each F_k has power series coefficients $a_n^{(k)}$, we define the maximum term and central index of F_k respectively denoted by $\mu_{F_k}(r)$ & $\nu_{F_k}(r)$ as

$$\mu_{F_k}(r) = \max_{n\geq 0} |a_n^{(k)}|r^n \quad \textit{and} \quad \nu_{F_k}(r) = \max\{n : \ |a_n^{(k)}|r^n = \mu_{F_k}(r)\}.$$

Then the bicomplexial maximum term $\mu_F(r)$ and the bicomplexial central index $\nu_F(r)$ of F on $|\zeta| = r$ are defined by

$$\mu_F(r) = \max\{\mu_{F_1}(r), \mu_{F_2}(r)\} \quad \textit{and} \quad \nu_F(r) = \max\{\nu_{F_1}(r), \nu_{F_2}(r)\}.$$

2.2. **Multicomplex numbers and functions.** Extending the notion as carried out in 2.1 we present some definitions under the flavour of multicomplex algebra.

Definition 2.7 (cf. [11])**.** *The multicomplex algebras* $\mathbb{MC}_n$ *are defined recursively as* $\mathbb{MC}_1 = \mathbb{C}$ *and* $\mathbb{MC}_{n+1} = \{z_1 + Jz_2 : \ z_1, z_2 \in \mathbb{MC}_n, \ J^2 = -1, \ J$ *commutes with multicomplex units in* $\mathbb{MC}_n\}$*. The elements of* $\mathbb{MC}_n$ *may be presented as the direct sums of* 2^{n-1} *complex components via idempotents. Generally, we denote the multicomplex algebra by* $\mathbb{MC}$ *when the dimension is not explicitly stated.*

The following definition extends the concept of entire function to the multicomplex domain.

Definition 2.8. *A function* $G : \mathbb{MC}_n \to \mathbb{MC}_n$ *is entire if all its complex component functions as presented in its idempotent decomposition are entire functions of a complex variable. Concretely, with decomposition into* $N = 2^{n-1}$ *complex branches, the expression*

$$G(w) = \sum_{k=1}^{N} G_k(w_k) e_k^{(n)} \quad \textit{with} \quad G_k : \mathbb{C} \to \mathbb{C} \textit{ being entire and}$$

$e_k^{(n)}$ *being the canonical idempotents holds.*

For detailed study one can see [3], [4], [5], [6], [9] & [11].

We now generalize the standard analytic quantities to multicomplex valued entire functions by the following definition.

Definition 2.9 (cf. [5])**.** *Let* G *be a multicomplex valued entire function with the decomposition as in Definition 2.8. Now the maximum modulus of* G *on* $|\zeta| = r$ *termed by* $M_G(r)$ *is defined as*

$$M_G(r) = \max_{1 \le k \le N} \max_{|\zeta| = r} |G_k(\zeta)|.$$

If each $G_k(z) = \sum_{n \ge 0} b_n^{(k)} z^n$*, we define* $\mu_{G_k}(r)$ *&* $\nu_{G_k}(r)$ *as usual and write*

$$\mu_G(r) = \max_{1 \le k \le N} \mu_{G_k}(r) \quad \textit{and} \quad \nu_G(r) = \max_{1 \le k \le N} \nu_{G_k}(r).$$

Remark 2.10. *Bicomplexial and multicomplexial growth questions reduce to coordinate-wise classical complexial growth by using the idempotent decomposition and taking maxima across the components. In fact, this type of standard practice is adopted in the study of growth analysis in these algebras.*

3. (α, β, γ)-ORDER AND LOWER ORDER IN BICOMPLEX AND MULTICOMPLEX SETTINGS

We first recall the classes of comparison functions and then give generalized definitions in both the bicomplex and multicomplex settings {cf. [7, 10, 12, 15]}.

Definition 3.1. *Let* L *be the class of continuous non-negative functions* α *on* $(-\infty, \infty)$ *such that* $\alpha(x) = \alpha(x_0) \ge 0$ *for* $x \le x_0$ *and* $\alpha(x) \to +\infty$ *as* $x \to +\infty$*. We define*

- $\alpha \in L_1$ *if* $\alpha \in L$ *and* $\alpha(a+b) \le \alpha(a) + \alpha(b) + c$ *for all* $a, b \ge R_0$ *with some fixed* $c > 0$*,*

- $\beta \in L_2$ *if* $\beta \in L$ *and* $\beta(x + O(1)) = (1 + o(1))\beta(x)$ *as* $x \to +\infty$ *and*
- $\gamma \in L_3$ *if* $\gamma \in L$ *and* $\gamma(a + b) \leq \gamma(a) + \gamma(b)$ *for* $a, b \geq R_0$ *(i.e. subadditive).*

Throughout the paper we take $\alpha \in L_1,\ \beta \in L_2$ *and* $\gamma \in L_3$.

The following two definitions reflect on (α, β, γ)-order (lower order) of bicomplex and multicomplex valued entire functions respectively.

Definition 3.2. *Let F be bicomplex entire and let $M_F(r)$ be defined as in Definition 2.6. The bicomplexial (α, β, γ)-order and (α, β, γ)-lower order of F is denoted and defined respectively by*

$$\rho^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F] = \limsup_{r\to+\infty} \frac{\alpha(\log\log M_F(r))}{\beta(\log(\gamma(r)))}$$

and

$$\lambda^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F] = \liminf_{r\to+\infty} \frac{\alpha(\log\log M_F(r))}{\beta(\log(\gamma(r)))}.$$

Definition 3.3. *Let G be multicomplex entire with $M_G(r)$ as in Definition 2.9. We respectively denote and define the multicomplexial (α, β, γ)-order and (α, β, γ)-lower order as*

$$\rho^{\mathbb{MC}}_{(\alpha,\beta,\gamma)}[G] = \limsup_{r\to+\infty} \frac{\alpha(\log\log M_G(r))}{\beta(\log(\gamma(r)))} \quad \text{and} \quad \lambda^{\mathbb{MC}}_{(\alpha,\beta,\gamma)}[G] = \liminf_{r\to+\infty} \frac{\alpha(\log\log M_G(r))}{\beta(\log(\gamma(r)))}.$$

Remark 3.4. *When $\alpha(r) = \beta(r) = \gamma(r) = r$, Definition 3.2 and Definition 3.3 will respectively reduce to order and lower order in bicomplex and multicomplex settings.*

Example 3.5. *Taking G on $\mathbb{MC}_n$ with each complex component $G_k(z) = e^{z^m}$ for some integer $m \geq 1$, we find that each component has classical order m. Therefore the order of G is also m.*

3.1. **Lemmas on central index.** The following lemma expresses the asymptotic equivalence between the growth of maximum modulus and central index of a complex valued entire function f.

Lemma 3.6 (cf. [2, 14])**.** *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function of finite nonzero order. Then*

$$\log\log M_f(r) \sim \log \nu_f(r)$$

holds.

We now present two lemmas indicating the definitions of growth indicators in terms of central index in bicomplex and multicomplex settings.

Lemma 3.7. *Let $F : \mathbb{BC} \to \mathbb{BC}$ be bicomplex entire. Then*

$$\rho^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F] = \limsup_{r\to+\infty} \frac{\alpha(\log \nu_F(r))}{\beta(\log(\gamma(r)))},$$

and

$$\lambda^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F] = \liminf_{r\to+\infty} \frac{\alpha(\log \nu_F(r))}{\beta(\log(\gamma(r)))}.$$

Proof. Since F is bicomplex entire then $F = F_1e_1 + F_2e_2$ and

$$\log\log M_F(r) = \log\log\left(\max\{M_{F_1}(r), M_{F_2}(r)\}\right).$$

Now for each component F_k for $k = 1$ & 2 we have

$$\log\log M_{F_k}(r) \sim \log \nu_{F_k}(r),$$

up to $o(1)$ comparisons appropriate for α, β in their classes.

Since $\nu_F(r) = \max\{\nu_{F_1}(r), \nu_{F_2}(r)\}$ and α, β, γ satisfy subadditivity, the lemma follows in view of Lemma 3.6 applied to each component F_k for $k = 1$ & 2. □

Lemma 3.8. *Let $G : \mathbb{MC} \to \mathbb{MC}$ be a multicomplex valued entire function. Then*

$$\rho^{\mathbb{MC}}_{(\alpha,\beta,\gamma)}[G] = \limsup_{r\to+\infty} \frac{\alpha(\log \nu_G(r))}{\beta(\log(\gamma(r)))}.$$

and

$$\lambda^{\mathbb{MC}}_{(\alpha,\beta,\gamma)}[G] = \liminf_{r\to+\infty} \frac{\alpha(\log \nu_G(r))}{\beta(\log(\gamma(r)))}.$$

Proof. Since $\nu_G(r) = \max_{1\le k\le N} \nu_{G_k}(r)$ and α, β, γ satisfy subadditivity, using the idempotent decomposition of G, the lemma follows in view of Lemma 3.6 applied to each $N = 2^{n-1}$ complex component. □

4. Main results

In this section we present the main results of the paper. In fact, we establish the bicomplex and multicomplex versions of several theorems concerning the growth of composite entire functions in terms of (α, β, γ)-order and central index.

4.1. **Notation for composition.** If F and G are both bicomplex entire, then the composition $F \circ G$ is defined pointwise on $\mathbb{BC}$ with idempotent decomposition leading to componentwise compositions, i.e., $(F \circ G)_k = F_k \circ G_k$ on each complex branch. A similar notion is valid for multicomplex components.

4.2. **Central index based growth theorems.**

Theorem 4.1. *Let F and G be bicomplex entire such that*

$$0 < \lambda^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F] \le \rho^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F] < +\infty,$$

and also let

$$\lambda^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F \circ G] = +\infty.$$

Then

$$\lim_{r\to+\infty} \frac{\alpha(\log \nu_{F\circ G}(r))}{\alpha(\log \nu_F(r))} = +\infty.$$

Proof. On the contrary, let us assume that the limit is finite. Then there exists $\Delta > 0$ and a sequence $r_m \to +\infty$ such that for those r_m

$$\alpha(\log \nu_{F\circ G}(r_m)) \le \Delta\ \alpha(\log \nu_F(r_m)). \tag{1}$$

From the assumed finiteness of the bicomplex (α,β,γ)-order and positive lower order of F, we obtain for sufficiently large r

$$\alpha(\log \nu_F(r)) \leq (\rho^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F]+\varepsilon)\ \beta(\log \gamma(r)), \tag{2}$$

using the componentwise estimates aggregated via the max function.

Now combining (1) and (2) along the sequence r_m, we get that

$$\alpha(\log \nu_{F\circ G}(r_m)) \leq \Delta(\rho^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F]+\varepsilon)\ \beta(\log \gamma(r_m)).$$

Thus

$$\frac{\alpha(\log \nu_{F\circ G}(r_m))}{\beta(\log \gamma(r_m))} \leq \Delta(\rho^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F]+\varepsilon),$$

which contradicts the hypothesis $\lambda^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F\circ G]=+\infty$.

This completes the proof of the theorem. □

Remark 4.2. *The multicomplex version of Theorem 4.1 follows by replacing the maxima over two components by the maxima over N components and using identical inequalities componentwise, as we can see in the following theorem.*

Theorem 4.3. *Let F and G be multicomplex entire such that*

$$0<\lambda^{\mathbb{MC}}_{(\alpha,\beta,\gamma)}[F] \leq \rho^{\mathbb{MC}}_{(\alpha,\beta,\gamma)}[F]<+\infty,$$

and

$$\lambda^{\mathbb{MC}}_{(\alpha,\beta,\gamma)}[F\circ G]=+\infty.$$

Then

$$\lim_{r\to+\infty}\frac{\alpha(\log \nu_{F\circ G}(r))}{\alpha(\log \nu_F(r))}=+\infty.$$

Proof. The proof of the theorem is identical to that of Theorem 4.1 just after replacing the two components by N components and maxima over $1\leq k\leq N$. Therefore, the proof is omitted. □

Remark 4.4. *If $\alpha(t)=\beta(t)=t$ & $\gamma(r)=r$, then Theorem 4.1 and Theorem 4.3 reduce to the classical central index based order relations for composite entire functions which are obtained by replacing $\alpha(\log\nu)$ with $\log\nu$ and (α,β,γ)-orders with classical orders.*

4.3. **Comparison inequalities for ratios of $\alpha(\log\nu)$.** The following theorem describes bounds for lim inf and lim sup of the quotient $\dfrac{\alpha(\log \nu_{F\circ G}(r))}{\alpha(\log \nu_F(r))}$ in terms of (α,β,γ)-orders of F and $F\circ G$ where F & G are bicomplex valued entire functions.

Theorem 4.5. *Let F,G be bicomplex entire such that*

$$0<\lambda^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F\circ G] \leq \rho^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F\circ G]<+\infty$$

and

$$0<\lambda^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F] \leq \rho^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F]<+\infty.$$

Then

$$\begin{aligned}\frac{\lambda^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F\circ G]}{\rho^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F]} &\leq \liminf_{r\to+\infty}\frac{\alpha(\log\nu_{F\circ G}(r))}{\alpha(\log\nu_F(r))}\\ &\leq \min\left\{\frac{\lambda^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F\circ G]}{\lambda^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F]},\ \frac{\rho^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F\circ G]}{\rho^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F]}\right\}\\ &\leq \max\left\{\frac{\lambda^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F\circ G]}{\lambda^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F]},\ \frac{\rho^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F\circ G]}{\rho^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F]}\right\}\\ &\leq \limsup_{r\to+\infty}\frac{\alpha(\log\nu_{F\circ G}(r))}{\alpha(\log\nu_F(r))} \leq \frac{\rho^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F\circ G]}{\lambda^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F]}.\end{aligned}$$

Proof. From the definitions of lower and upper (α,β,γ)-orders in terms of central index we have for arbitrary $\varepsilon>0$ and for all sufficiently large r,

$$\alpha\big(\log(\nu_{F\circ G}(r))\big) \geq (\lambda^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F\circ G]-\varepsilon)\,\beta(\log(\gamma(r))), \tag{3}$$

$$\alpha\big(\log(\nu_{F\circ G}(r))\big) \leq (\rho^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F\circ G]+\varepsilon)\,\beta(\log(\gamma(r))), \tag{4}$$

$$\alpha\big(\log(\nu_F(r))\big) \geq (\lambda^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F]-\varepsilon)\,\beta(\log(\gamma(r))) \tag{5}$$

and

$$\alpha\big(\log(\nu_F(r))\big) \leq (\rho^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F]+\varepsilon)\,\beta(\log(\gamma(r))). \tag{6}$$

Again, for a sequence of values of r tending to infinity,

$$\alpha\big(\log(\nu_{F\circ G}(r))\big) \leq (\lambda^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F\circ G]+\varepsilon)\,\beta(\log(\gamma(r))), \tag{7}$$

$$\alpha\big(\log(\nu_{F\circ G}(r))\big) \geq (\rho^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F\circ G]-\varepsilon)\,\beta(\log(\gamma(r))), \tag{8}$$

$$\alpha\big(\log(\nu_F(r))\big) \leq (\lambda^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F]+\varepsilon)\,\beta(\log(\gamma(r))) \tag{9}$$

and

$$\alpha\big(\log(\nu_F(r))\big) \geq (\rho^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F]-\varepsilon)\,\beta(\log(\gamma(r))). \tag{10}$$

Now from (3) and (6) it follows for all sufficiently large values of r that

$$\frac{\alpha\big(\log(\nu_{F\circ G}(r))\big)}{\alpha\big(\log(\nu_F(r))\big)} \geq \frac{\lambda^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F\circ G]-\varepsilon}{\rho^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F]+\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to+\infty}\frac{\alpha\big(\log(\nu_{F\circ G}(r))\big)}{\alpha\big(\log(\nu_F(r))\big)} \geq \frac{\lambda^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F\circ G]}{\rho^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F]}. \tag{11}$$

Combining (5) and (7), we have for a sequence of values of r tending to infinity that

$$\frac{\alpha\big(\log(\nu_{F\circ G}(r))\big)}{\alpha\big(\log(\nu_F(r))\big)} \leq \frac{\lambda^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F\circ G]+\varepsilon}{\lambda^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F]-\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary it follows that

$$\liminf_{r\to+\infty}\frac{\alpha\big(\log(\nu_{F\circ G}(r))\big)}{\alpha\big(\log(\nu_F(r))\big)}\leq\frac{\lambda^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F\circ G]}{\lambda^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F]}. \tag{12}$$

Again from (3) and (9), for a sequence of values of r tending to infinity, we get

$$\frac{\alpha\big(\log(\nu_{F\circ G}(r))\big)}{\alpha\big(\log(\nu_F(r))\big)}\geq\frac{\lambda^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F\circ G]-\varepsilon}{\lambda^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F]+\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to+\infty}\frac{\alpha\big(\log(\nu_{F\circ G}(r))\big)}{\alpha\big(\log(\nu_F(r))\big)}\geq\frac{\lambda^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F\circ G]}{\lambda^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F]}. \tag{13}$$

Now, it follows from (4) and (5), for all sufficiently large values of r that

$$\frac{\alpha\big(\log(\nu_{F\circ G}(r))\big)}{\alpha\big(\log(\nu_F(r))\big)}\leq\frac{\rho^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F\circ G]+\varepsilon}{\lambda^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F]-\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to+\infty}\frac{\alpha\big(\log(\nu_{F\circ G}(r))\big)}{\alpha\big(\log(\nu_F(r))\big)}\leq\frac{\rho^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F\circ G]}{\lambda^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F]}. \tag{14}$$

Now from (4) and (10), it follows for a sequence of values of r tending to infinity that

$$\frac{\alpha\left(\log\left(\nu_{F\circ G}(r)\right)\right)}{\alpha\left(\log\left(\nu_F(r)\right)\right)}\leq\frac{\rho^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F\circ G]+\varepsilon}{\rho^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F]-\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to+\infty}\frac{\alpha\left(\log\left(\nu_{F\circ G}(r)\right)\right)}{\alpha\left(\log\left(\nu_F(r)\right)\right)}\leq\frac{\rho^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F\circ G]}{\rho^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F]}. \tag{15}$$

So combining (6) and (8), we get for a sequence of values of r tending to infinity that

$$\frac{\alpha\left(\log\left(\nu_{F\circ G}(r)\right)\right)}{\alpha\left(\log\left(\nu_F(r)\right)\right)}\geq\frac{\rho^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F\circ G]-\varepsilon}{\rho^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F]+\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows that

$$\limsup_{r\to+\infty}\frac{\alpha\left(\log\left(\nu_{F\circ G}(r)\right)\right)}{\alpha\left(\log\left(\nu_F(r)\right)\right)}\geq\frac{\rho^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F\circ G]}{\rho^{\mathbb{BC}}_{(\alpha,\beta,\gamma)}[F]}. \tag{16}$$

Thus the theorem follows from (11), (12), (13), (14), (15) and (16). □

The following theorem is the multicomplex analogue of Theorem 4.5.

Theorem 4.6. *Let F, G be multicomplexial entire satisfying*

$$0<\lambda^{\mathbb{MC}}_{(\alpha,\beta,\gamma)}[F\circ G]\leq\rho^{\mathbb{MC}}_{(\alpha,\beta,\gamma)}[F\circ G]<+\infty$$

and

$$0<\lambda^{\mathbb{MC}}_{(\alpha,\beta,\gamma)}[F]\leq\rho^{\mathbb{MC}}_{(\alpha,\beta,\gamma)}[F]<+\infty.$$

Then

$$\begin{aligned}\frac{\lambda^{\mathbb{MC}}_{(\alpha,\beta,\gamma)}[F\circ G]}{\rho^{\mathbb{MC}}_{(\alpha,\beta,\gamma)}[F]} &\leq \liminf_{r\to+\infty}\frac{\alpha(\log\nu_{F\circ G}(r))}{\alpha(\log\nu_F(r))}\\ &\leq \min\left\{\frac{\lambda^{\mathbb{MC}}_{(\alpha,\beta,\gamma)}[F\circ G]}{\lambda^{\mathbb{MC}}_{(\alpha,\beta,\gamma)}[F]},\ \frac{\rho^{\mathbb{MC}}_{(\alpha,\beta,\gamma)}[F\circ G]}{\rho^{\mathbb{MC}}_{(\alpha,\beta,\gamma)}[F]}\right\}\\ &\leq \max\left\{\frac{\lambda^{\mathbb{MC}}_{(\alpha,\beta,\gamma)}[F\circ G]}{\lambda^{\mathbb{MC}}_{(\alpha,\beta,\gamma)}[F]},\ \frac{\rho^{\mathbb{MC}}_{(\alpha,\beta,\gamma)}[F\circ G]}{\rho^{\mathbb{MC}}_{(\alpha,\beta,\gamma)}[F]}\right\}\\ &\leq \limsup_{r\to+\infty}\frac{\alpha(\log\nu_{F\circ G}(r))}{\alpha(\log\nu_F(r))} \leq \frac{\rho^{\mathbb{MC}}_{(\alpha,\beta,\gamma)}[F\circ G]}{\lambda^{\mathbb{MC}}_{(\alpha,\beta,\gamma)}[F]}\end{aligned}$$

holds.

Proof. The proof of the theorem is omitted as it is identical to Theorem 4.5 just after replacing the two components by N components and maxima over $1 \leq k \leq N$. □

Discussions

- The results under the flavour of bicomplex analysis are immediate generalizations of the complex ones because bicomplex entire functions decompose into two complex entire functions under idempotent representation. The main technical point is to ensure that the (α, β, γ)-functions admit the same asymptotic manipulations under 'max' and $O(1)$ perturbations.
- For multicomplex algebras of higher rank, the same principle applies. There are $N = 2^{n-1}$ complex components and maxima across them which govern the global growth of composite entire functions. The inequalities are identical, replacing 2 by N.
- The statements of Theorem 4.1 and Theorem 4.3 are focused on central index because those provide a coefficient-level view of growth, often simplifying composite estimates.
- The results presented in the paper are stated with maxima across components. Alternatively, more refined results could attempt to track growth on specific idempotent branches, whereas for central index components of composite functions the maximal-behavior viewpoint is most natural and useful.

5. Conclusion

We extend central-index oriented growth analysis of composite entire functions to bicomplex and multicomplex settings in terms of (α, β, γ)-orders. The bicomplexial and multicomplexial central indices and maximum moduli reduce the problem to componentwise classical complex analysis using the notion of maxima. Several directions remain for further refinement, such as tracking branchwise growth, non-entire (meromorphic) bicomplex valued functions, and sharper estimations when components have different relative growth rates.

6. Open problems and future directions

The notion of bicomplex and multicomplex setting of central index based growth theorems and comparison inequalities for ratios of $\alpha(\log \nu)$ may be extended for the bicomplex & multicomplex valued meromorphic functions of uncertain variables and are still virgin. Therefore, those may be posed as open problems to the future researchers of this branch.

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