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A NOTE ON BICOMPLEX AND MULTICOMPLEX ANALOGUE OF HADAMARD'S THREE CIRCLE THEOREM

Sanjib Kumar Datta

Abstract. In the paper, we establish the bicomplex and multicomplex versions of Hadamards three circle theorem and also of its convex form.

Keywords: Bicomplex analysis, Multicomplex functions, Hadamard's three circle theorem, Idempotent decomposition, Bicomplex and multicomplex valued entire functions.

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1. Introduction

Classical complex analysis provides a foundational framework for the study of holomorphic functions, conformal maps, and analytic continuation. In recent decades, the algebraic extensions known as *bicomplex* and *multicomplex* numbers have emerged as natural higher-dimensional generalizations of the complex field, permitting the extension of analytic concepts to commutative hypercomplex algebras. Initiated in the work of Segre and later developed by Price, Rochon, Shapiro and others, bicomplex analysis connects to operator theory, functional analysis and mathematical physics. Relative to the onecomplexvariable setting, the bicomplex framework exhibits novel features such as zero divisors, nontrivial idempotents, and a canonical splitting of function spaces into classical complex components.

In this paper we focus on growth estimates for entire functions in bicomplex and multicomplex algebras by establishing a version of *Hadamard's threecircle theorem* adapted to these settings (cf. [7], [8] & [15]). More precisely, we formulate and prove bicomplex and multicomplex analogues of the classical Hadamard's threecircle inequality (cf. [1], [5], [6], [10] & [18] for the complex case) and derive a convexity formulation that yields concrete growth bounds for entire functions expressed in idempotent components. These results play a central role in understanding the growth and order of entire functions in higherdimensional commutative algebras and serve as a foundational step toward a systematic function theory in those algebras.

Our approach rests on the idempotent (component-wise) representation of bicomplex numbers (cf. [16], [14], [3] & [17]):

$$Z = z_1e_1 + z_2e_2 \quad with \quad e_1 = \tfrac{1+ij}{2} \quad and \quad e_2 = \tfrac{1-ij}{2},$$

with $i^2 = j^2 = -1$ and $ij = ji$. This decomposition reduces many analytic questions in $\mathbb{BC}$ (and, by iteration, in $\mathbb{M}_n$) to pairs or families of classical complex problems in $\mathbb{C}$. By combining this component-wise reduction with the classical Hadamard's threecircle theorem and an appropriate bookkeeping of idempotent projections, we obtain bicomplex and multicomplex statements that retain the essential geometric and analytic content of the classical result while accounting for the presence of zero divisors and the idempotent structure.

The main contributions of the paper are:

- A bicomplex formulation of Hadamard's threecircle theorem and its convex form, valid under natural hypotheses on the idempotent components.
- Extension of the threecircle estimates to the multicomplex algebra $\mathbb{M}_n$ via iterated idempotent decomposition, together with consequences for the growth and order of multicomplex entire functions.
- A discussion of examples and applications that illustrate how component-wise growth control translates into global growth bounds in the hypercomplex setting.

The remainder of the paper is organized as follows. In Section 2 we review algebraic and analytic preliminaries for bicomplex and multicomplex algebras, including idempotent representations and component-wise holomorphicity. Section 3 contains the statements and proofs of the bicomplex and multicomplex analogues of Hadamard's threecircle theorem and their convex formulations. Finally, in Section 4 we gave an alternative view point of multicomplex analysis and a proof of Hadamard's threecircle theorem in that alternative multicomplex setting.

2. Preliminaries

In this section we collect the algebraic and analytic background on bicomplex and multicomplex algebras that will be used throughout the paper. We keep the emphasis on idempotent decomposition, the principal conjugations and moduli, the definition of holomorphicity, and the component-wise viewpoint that allows classical complex results to be transported to the hypercomplex setting.

2.1. **The Bicomplex Algebra.** Let i and j be commuting imaginary units satisfying

$$i^2 = j^2 = -1 \quad \textit{and} \quad ij = ji.$$

The set of all numbers of the form

$$\mathbb{BC} = \{Z = z_1 + jz_2 \ : \ z_1, z_2 \in \mathbb{C}(i)\}$$

constitutes a commutative ring called the *bicomplex algebra* {cf. [17],[16],[13] & [12]}. Addition and multiplication are performed component-wise. Here $\mathbb{C}(i)$ denotes the classical complex field generated by i.

As a real algebra $\mathbb{BC}$ is four-dimensional, has multiplicative identity 1 and contains nontrivial zero divisors. Concretely, an element $Z \in \mathbb{BC}$ is a zero divisor precisely when its *complex norm*

$$N(Z) = z_1^2 + z_2^2 = 0.$$

2.2. **Idempotent Representation.** Let us define the idempotent elements

$$e_1 = \frac{1+ij}{2} \quad \& \quad e_2 = \frac{1-ij}{2},$$

which satisfy the relations

$$e_1^2 = e_1, \quad e_2^2 = e_2, \quad e_1e_2 = 0 \quad \textit{and} \quad e_1 + e_2 = 1.$$

Every bicomplex number $Z \in \mathbb{BC}$ admits the unique idempotent representation

$$Z = z_1e_1 + z_2e_2 \quad \textit{with} \quad z_1, z_2 \in \mathbb{C}(i).$$

This representation permits all bicomplex operations to be carried out component-wise. For example,

$$Z_1 + Z_2 = (z_{11} + z_{21})e_1 + (z_{12} + z_{22})e_2 \quad \textit{and} \quad Z_1Z_2 = (z_{11}z_{21})e_1 + (z_{12}z_{22})e_2.$$

A nonzero element Z is invertible if and only if both idempotent components z_1 and z_2 are nonzero, in which case

$$Z^{-1} = z_1^{-1}e_1 + z_2^{-1}e_2.$$

2.3. **Conjugations and Moduli.** The bicomplex algebra admits several natural conjugations: writing $Z = z_1 + jz_2$,

$$Z^{\dagger_i} = \overline{z_1} + j\overline{z_2},$$
$$Z^{\dagger_j} = z_1 - jz_2$$
$$\text{and} \quad Z^{\dagger_{ij}} = \overline{z_1} - j\overline{z_2}.$$

The *complex norm* and the *bicomplex modulus* of Z are defined respectively by

$$|Z|_i^2 = ZZ^{\dagger_i} = |z_1|^2 + |z_2|^2 \quad \textit{and} \quad |Z|_{ij}^2 = ZZ^{\dagger_{ij}} = |z_1|^2 e_1 + |z_2|^2 e_2.$$

These quantities generalize the Euclidean and component-wise norms familiar from complex analysis.

2.4. **Bicomplex Holomorphic Functions.** Let $\Omega \subseteq \mathbb{BC}$ be open. A function $F : \Omega \to \mathbb{BC}$ is *bicomplex differentiable* at $Z_0 \in \Omega$ if the limit

$$F'(Z_0) = \lim_{H \to 0} \frac{F(Z_0 + H) - F(Z_0)}{H}$$

exists and is independent of the manner in which H tends to zero in $\mathbb{BC}$. If F is differentiable at every point of Ω, we call it *bicomplex holomorphic* on Ω {cf. [11], [2], [12], [4] & [17]}.

By using the idempotent decomposition, any bicomplex holomorphic function admits the representation

$$F(Z) = f_1(z_1)e_1 + f_2(z_2)e_2,$$

where f_1 and f_2 are holomorphic functions in $\mathbb{C}(i)$. This component-wise description allows classical complex analytic results to be transferred to the bicomplex context.

2.5. **Multicomplex Numbers.** The bicomplex construction extends naturally to the *multicomplex* algebra $\mathbb{M}_n$ generated by n commuting imaginary units $i_1, i_2, \ldots, i_n$ with $i_k^2 = -1$ for each k. As a real algebra the dimension of $\mathbb{M}_n$ is 2^n. Every multicomplex number {cf. [13] & [16]} admits an idempotent decomposition into 2^{n-1} components, each lying in the complex field $\mathbb{C}(i_1)$.

In particular, if $J_n = i_1 i_2 \cdots i_n$, then

$$\mathbb{M}_n = \left\{ \sum_{\alpha=1}^{2^{n-1}} z_\alpha e_\alpha : z_\alpha \in \mathbb{C}(i_1) \right\},$$

where the idempotents $\{e_\alpha\}$ satisfy $e_\alpha e_\beta = 0$ for $\alpha \neq \beta$ and $\sum_\alpha e_\alpha = 1$. This decomposition enables a component-wise extension of complex analytic concepts to $\mathbb{M}_n$.

2.6. **Bicomplex and Multicomplex Analytic Properties.** Let $\Omega \subseteq \mathbb{BC}$ (or $\mathbb{M}_n$). A function F is *entire* if it is holomorphic everywhere and *meromorphic* if it is holomorphic except at isolated singularities where it admits a Laurent-type expansion in each idempotent component. Convergence, compactness and uniform boundedness are interpreted via the component functions.

Several classical analytic principles extend naturally:

- **CauchyRiemann equations** hold component-wise for f_1 and f_2.

- **Cauchys integral theorem** and **Cauchys integral formula** remain valid through idempotent decomposition.
- **Maximum Modulus Principle** and **Liouvilles theorem** extend directly to bicomplex holomorphic functions via their components.

These foundational properties form the basis for the bicomplex and multicomplex formulation of Hadamard's threecircle theorem developed in the following sections.

Definition 2.1. *Let $I \subset \mathbb{R}$ be an interval. A function $\phi : I \to \mathbb{R}$ is said to be* convex *on I if for any $x_1, x_2 \in I$ and any $t \in [0,1]$,*

$$\phi(tx_1 + (1-t)x_2) \leq t\phi(x_1) + (1-t)\phi(x_2).$$

Remark 2.2. *Geometrically, the graph of* $\log M(r)$ *versus* $\log r$ *lies* below the straight line connecting any two points *on the curve. This convexity property is fundamental in analyzing growth of analytic or entire functions.*

The following lemma represents Hadamard's three circle theorem in classical complex analysis.

Lemma 2.3. *Let f be a function analytic in the annulus*

$$A = \{z \in \mathbb{C} : r_1 \leq |z| \leq r_3\},$$

where $0 < r_1 < r_3$. Define

$$M(r) = \max_{|z|=r} |f(z)|, \quad r_1 \leq r \leq r_3.$$

Then for any $r_1 < r_2 < r_3$, the function

$$\log M(r)$$

is convex with respect to $\log r$*, i.e.,*

$$\log M(r_2) \leq \frac{\log(r_3/r_2)}{\log(r_3/r_1)} \log M(r_1) + \frac{\log(r_2/r_1)}{\log(r_3/r_1)} \log M(r_3).$$

3. Bicomplex and Multicomplex Analogues of Hadamard's Three Circle Theorem

3.1. **Bicomplex Analogue of Hadamard's Three Circle Theorem.** Let us consider a bicomplex function

$$F(Z) = f_1(z_1)e_1 + f_2(z_2)e_2$$

analytic in the bicomplex annulus

$$A_{\mathbb{BC}} = \{Z = z_1e_1 + z_2e_2 \in \mathbb{BC} : r_1 \leq |z_1|, |z_2| \leq r_3\}.$$

We define the bicomplex maximum modulus

$$M_{\mathbb{BC}}(r) = \max_{|Z|=r} |F(Z)|, \quad r_1 \leq r \leq r_3,$$

where $|F(Z)| = |f_1(z_1)|e_1 + |f_2(z_2)|e_2$.

The following theorem is the bicomplex version of Lemma 2.3.

Theorem 3.1. *Let $F(Z) = f_1(z_1)e_1 + f_2(z_2)e_2$ be analytic in $A_{\mathbb{BC}}$. For any $r_1 < r_2 < r_3$, the bicomplex maximum modulus satisfies*

$$\log M_{\mathbb{BC}}(r_2) \leq \frac{\log(r_3/r_2)}{\log(r_3/r_1)} \log M_{\mathbb{BC}}(r_1) + \frac{\log(r_2/r_1)}{\log(r_3/r_1)} \log M_{\mathbb{BC}}(r_3),$$

where the logarithm is applied componentwise, i.e.,

$$\log M_{\mathbb{BC}}(r) = (\log |f_1(z_1)|)e_1 + (\log |f_2(z_2)|)e_2.$$

Proof. Every bicomplex function $F(Z)$ can be decomposed as

$$F(Z) = f_1(z_1)e_1 + f_2(z_2)e_2.$$

Thus the bicomplex modulus can be written as

$$|F(Z)| = |f_1(z_1)|e_1 + |f_2(z_2)|e_2.$$

For each component, let us define

$$M_1(r) = \max_{|z_1|=r} |f_1(z_1)| \quad \textit{and} \quad M_2(r) = \max_{|z_2|=r} |f_2(z_2)|.$$

Then

$$M_{\mathbb{BC}}(r) = M_1(r)e_1 + M_2(r)e_2.$$

Since f_1 is analytic in the annulus $r_1 \leq |z_1| \leq r_3$, Lemma 2.3 implies that

$$\log M_1(r_2) \leq \frac{\log(r_3/r_2)}{\log(r_3/r_1)} \log M_1(r_1) + \frac{\log(r_2/r_1)}{\log(r_3/r_1)} \log M_1(r_3).$$

Similarly, for f_2, we have

$$\log M_2(r_2) \leq \frac{\log(r_3/r_2)}{\log(r_3/r_1)} \log M_2(r_1) + \frac{\log(r_2/r_1)}{\log(r_3/r_1)} \log M_2(r_3).$$

Using $M_{\mathbb{BC}}(r) = M_1(r)e_1 + M_2(r)e_2$ and applying the logarithm componentwise, we obtain that

$$\log M_{\mathbb{BC}}(r_2) = (\log M_1(r_2))e_1 + (\log M_2(r_2))e_2.$$

Combining the inequalities as above, it follows that

$$\begin{aligned}\log M_{\mathbb{BC}}(r_2) &\leq \left[\frac{\log(r_3/r_2)}{\log(r_3/r_1)} \log M_1(r_1) + \frac{\log(r_2/r_1)}{\log(r_3/r_1)} \log M_1(r_3)\right] e_1 \\ &\quad + \left[\frac{\log(r_3/r_2)}{\log(r_3/r_1)} \log M_2(r_1) + \frac{\log(r_2/r_1)}{\log(r_3/r_1)} \log M_2(r_3)\right] e_2 \\ &= \frac{\log(r_3/r_2)}{\log(r_3/r_1)} \log M_{\mathbb{BC}}(r_1) + \frac{\log(r_2/r_1)}{\log(r_3/r_1)} \log M_{\mathbb{BC}}(r_3).\end{aligned}$$

This proves the theorem. □

Remark 3.2. *Each idempotent component satisfies Lemma 2.3 independently and so the bicomplex maximum modulus inherits the componentwise convexity in* $\log r$.

The convexity property of $\log M_{\mathbb{BC}}(r)$ *is componentwise, i.e.*

$$\log M_1(r) \textit{ and } \log M_2(r) \textit{ are convex in } \log r.$$

Remark 3.3. *Theorem 3.1 provides a growth estimate for bicomplex analytic functions in an annular region.*

Remark 3.4. *The same method generalizes directly to multicomplex functions using their idempotent decomposition.*

The following remark reflects the norm version of Lemma 2.3 under the flavour of bicomplex analysis.

Remark 3.5. *Let*

$$F(Z) = e_1 f_1(z_1) + e_2 f_2(z_2)$$

be a bicomplex-holomorphic function defined on the annular domain

$$D = \{\, Z = e_1 z_1 + e_2 z_2 : r_1 < |z_1|, |z_2| < r_2 \,\}.$$

For $\rho \in (r_1, r_2)$, let us define the maximum modulus function *of F with respect to the Euclidean norm as*

$$M_F(\rho) = \max_{|Z|_E = \rho} |F(Z)|_E = \sqrt{M_{f_1}(\rho)^2 + M_{f_2}(\rho)^2},$$

where $M_{f_i}(\rho) = \max_{|z_i|=\rho} |f_i(z_i)|$, $i = 1, 2$.

Then the function

$$\log M_F(\rho)$$

is convex in $\log \rho$ i.e., if $r_1 < \rho < r_2$ and $\log \rho = t \log r_1 + (1-t) \log r_2$ for $0 < t < 1$, then

$$\log M_F(\rho) \le t \log M_F(r_1) + (1-t) \log M_F(r_2).$$

Moreover, if F is nonconstant then $M_F(\rho)$ is a strictly increasing function of ρ.

3.2. **Multicomplex Analogue of Hadamard's Three Circle Theorem.** Any multicomplex number $Z \in \mathbb{MC}_m$ can be written as

$$Z = \sum_\alpha e_\alpha z_\alpha, \quad z_\alpha \in \mathbb{C}.$$

Let us consider a multicomplex analytic function

$$F(Z) = \sum_\alpha e_\alpha f_\alpha(z_\alpha)$$

defined in the multicomplex annulus

$$A_{\mathbb{MC}_m} = \left\{ Z = \sum_\alpha e_\alpha z_\alpha \in \mathbb{MC}_m : r_1 \le |z_\alpha| \le r_3 \quad \forall \alpha \right\}.$$

Also we define the multicomplex maximum modulus

$$M_{\mathbb{MC}_m}(r) = \max_{|Z|=r} |F(Z)| = \sum_\alpha e_\alpha \max_{|z_\alpha|=r} |f_\alpha(z_\alpha)| = \sum_\alpha e_\alpha M_\alpha(r).$$

The following theorem is the multicomplex version of Theorem 3.1.

Theorem 3.6. *Let $F(Z) = \sum_\alpha e_\alpha f_\alpha(z_\alpha)$ be analytic in $A_{\mathbb{MC}_m}$. For any $r_1 < r_2 < r_3$, the multicomplex maximum modulus satisfies*

$$\log M_{\mathbb{MC}_m}(r_2) \le \frac{\log(r_3/r_2)}{\log(r_3/r_1)} \log M_{\mathbb{MC}_m}(r_1) + \frac{\log(r_2/r_1)}{\log(r_3/r_1)} \log M_{\mathbb{MC}_m}(r_3),$$

where the logarithm is applied componentwise, i.e.

$$\log M_{\mathbb{MC}_m}(r) = \sum_\alpha e_\alpha \log M_\alpha(r).$$

Proof. Every multicomplex analytic function can be written as

$$F(Z) = \sum_{\alpha} e_\alpha f_\alpha(z_\alpha),$$

where each f_α is analytic in the corresponding complex variable z_α. Also the modulus can be decomposed componentwise as

$$|F(Z)| = \sum_{\alpha} e_\alpha |f_\alpha(z_\alpha)|.$$

For each component α, let us define

$$M_\alpha(r) = \max_{|z_\alpha|=r} |f_\alpha(z_\alpha)|.$$

Then

$$M_{\mathbb{MC}_m}(r) = \sum_{\alpha} e_\alpha M_\alpha(r).$$

Since f_α is analytic in the annulus $r_1 \le |z_\alpha| \le r_3$, Lemma 2.3 gives that

$$\log M_\alpha(r_2) \le \frac{\log(r_3/r_2)}{\log(r_3/r_1)} \log M_\alpha(r_1) + \frac{\log(r_2/r_1)}{\log(r_3/r_1)} \log M_\alpha(r_3) \quad \textit{for every } \alpha.$$

Using decomposition, we obtain that

$$\log M_{\mathbb{MC}_m}(r_2) = \sum_{\alpha} e_\alpha \log M_\alpha(r_2) \le \sum_{\alpha} e_\alpha \left[\frac{\log(r_3/r_2)}{\log(r_3/r_1)} \log M_\alpha(r_1) + \frac{\log(r_2/r_1)}{\log(r_3/r_1)} \log M_\alpha(r_3) \right].$$

After simplification it takes the form

$$\log M_{\mathbb{MC}_m}(r_2) \le \frac{\log(r_3/r_2)}{\log(r_3/r_1)} \log M_{\mathbb{MC}_m}(r_1) + \frac{\log(r_2/r_1)}{\log(r_3/r_1)} \log M_{\mathbb{MC}_m}(r_3).$$

Hence the multicomplexial maximum modulus satisfies a componentwise convexity inequality in $\log r$.

This completes the proof of the theorem.

□

Remark 3.7. *Each idempotent component* $\log M_\alpha(r)$ *in Theorem 3.6 is convex in* $\log r$.

Remark 3.8. *The theorem gives a sharper growth estimate for multicomplex valued analytic functions in annular regions.*

Remark 3.9. *Theorem 3.1 is the special case of Theorem 3.6 for* $m = 2$.

The following remark reflects the norm version of Theorem 3.1 under the flavour of multicomplex analysis.

Remark 3.10. *Let* $\mathcal{M}_m$ *be the multicomplex algebra of order* m*, with* $N = 2^{m-1}$ *idempotents* e_α*. Let*

$$F(Z) = \sum_{\alpha=1}^{N} e_\alpha f_\alpha(w_\alpha)$$

be multicomplex-holomorphic on the product annulus

$$D = \sum_{\alpha=1}^{N} e_\alpha A(r_1, r_2), \qquad A(r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z| < r_2\}.$$

For $\rho \in (r_1, r_2)$, let us define the maximum modulus function *with respect to the Euclidean norm*

$$M_F(\rho) = \max_{|Z|_E=\rho} |F(Z)|_E = \sqrt{\sum_{\alpha=1}^{N} M_{f_\alpha}(\rho)^2} \quad \textit{where} \quad M_{f_\alpha}(\rho) = \max_{|w_\alpha|=\rho} |f_\alpha(w_\alpha)|.$$

Then $\log M_F(\rho)$ is convex in $\log \rho$, i.e. for $r_1 < \rho < r_2$ with $\log \rho = t \log r_1 + (1-t) \log r_2$, $0 < t < 1$, giving that

$$\log M_F(\rho) \leq t \log M_F(r_1) + (1-t) \log M_F(r_2).$$

Moreover, if F is nonconstant then $M_F(\rho)$ is strictly increasing in ρ.

4. Alternative view point of multicomplex analysis

In this section, we adopt an alternative perspective in multicomplex analysis based on the structural framework as developed by Price [16].

A multicomplex number $\xi_n \in \mathbb{C}_n$ can be defined as

$$\xi_n \ = \ \xi_{n-1,1} \ + \ i_n \, \xi_{n-1,2},$$

where

$$\xi_{n-1,1}, \, \xi_{n-1,2} \in \mathbb{C}_{n-1}, \quad (i_n)^2 = -1.$$

Addition and multiplication in multicomplex space $\mathbb{C}_n$ are defined componentwise extended by $(i_n)^2 = -1$.

4.1. **Idempotent elements in $\mathbb{C}_n$.** It is easy to verify that the following are idempotent elements in $\mathbb{C}_n$.

$$0, \quad 1, \quad \frac{1+i_1i_2}{2}, \, \frac{1-i_1i_2}{2}, \, \frac{1+i_1i_3}{2}, \, \frac{1-i_1i_3}{2}, \, \frac{1+i_2i_3}{2}, \, \frac{1-i_2i_3}{2}, \, \ldots, \, \frac{1+i_{n-1}i_n}{2}, \, \frac{1-i_{n-1}i_n}{2}.$$

For convenience in notation, define the symbols

$$e(i_ri_s) \ = \ \frac{1+i_ri_s}{2}, \qquad e(-i_ri_s) \ = \ \frac{1-i_ri_s}{2} \quad \text{where } r, s \in \mathbb{N}.$$

4.2. **Idempotent representation.** Let ξ be an element in $\mathbb{C}_n$ and let $\xi = \xi_1 + i_n\xi_2$ with $\xi_1, \, \xi_2 \in \mathbb{C}_{n-1}$. Then

$$\xi = \big(\xi_1 - i_{n-1} \, \xi_2\big) \, e\big(i_{n-1}i_n\big) \ + \ \big(\xi_1 + i_{n-1} \, \xi_2\big) \, e\big(- i_{n-1}i_n\big).$$

4.3. **Idempotent Representation of Holomorphic Multicomplex Valued Functions.** Let X be a domain in C_n, $n \geq 1$, and let f be a holomorphic function in $\mathbb{C}_n$ then there exists holomorphic functions

$$f_1 : X_1 \to \mathbb{C}_{n-1} \quad \text{and} \quad f_2 : X_2 \to \mathbb{C}_{n-1} \quad \text{where} \quad X_1, X_2 \subseteq \mathbb{C}_{n-1},$$

such that

$$f(\xi_1 + i_n\xi_2) = f_1(\xi_1 - i_{n-1}\xi_2)e(i_{n-1}i_n) + f_2(\xi_1 + i_{n-1}\xi_2)e(-i_{n-1}i_n).$$

For further study on multicomplex analysis one can see [7] and [9].

From this alternative perspective in multicomplex analysis, Theorem 3.6 can be reformulated as follows.

Theorem 4.1. *Let* $F(Z) = F_{1,m-1}(Z_1 - i_{m-1}Z_2)e(i_{m-1}i_m) + F_{2,m-1}(Z_1 + i_{m-1}Z_2)e(-i_{m-1}i_m)$ *be analytic in* $A_{\mathbb{MC}_m}$. *For any* $r_1 < r_2 < r_3$, *the multicomplex maximum modulus satisfies*

$$\log M_{\mathbb{MC}_m}(r_2) \leq \frac{\log(r_3/r_2)}{\log(r_3/r_1)} \log M_{\mathbb{MC}_m}(r_1) + \frac{\log(r_2/r_1)}{\log(r_3/r_1)} \log M_{\mathbb{MC}_m}(r_3),$$

where the logarithm is applied componentwise, i.e.

$$\log M_{\mathbb{MC}_m}(r) = \sum_{\alpha} e_\alpha \log M_\alpha(r) \quad \textit{where} \quad \alpha \in \{(1, n-1), (2, n-1)\}$$

Proof. The proof of this theorem can be done using idempotent decomposition and mathematical induction as follows.

The theorem is already proved for $m = 1$ and $m = 2$ i.e. respectively for complex and bicomplex spaces {cf. Lemma 2.3 and Theorem 3.1}.

Let us consider that the theorem is true for $m = n-1$. Using idempotent decomposition of F as

$$F(Z) = F_{1,n-1}(z_1 - i_{n-1}z_2)e(i_{n-1}i_n) + F_{2,n-1}(z_1 + i_{n-1}z_2)e(-i_{n-1}i_n)$$

where $F_{1,n-1}$ and $F_{2,n-1}$ are entire multicomplex valued functions in $\mathbb{C}_{n-1}$.

Now by the induction hypothesis the theorem is true for $F_{1,n-1}$ and $F_{2,n-1}$. Combining the idempotent components we can say that the theorem is true for F in $\mathbb{C}_n$ i.e. for $m = n$. Then the theorem is true for all $m \in \mathbb{N}$.

Thus the theorem is proved. □

5. Open problems and Future Directions

The notion of bicomplex and multicomplex setting of Hadamard's three circle theorem may be extended for the bicomplex and multicomplex valued functions of uncertain variables and are still virgin. Therefore those may be posed as open problems to the future researchers of this branch.

6. Conclusion

The convex representation in Theorems 3.1 and 3.6 is crucial for generalizing Hadamard's three-circle theorem and associted growth analysis of multicomplex valued functions.

We prove bicomplex and multicomplex versions of Hadamard's threecircle theorem, together with a convexity formulation giving growth estimates for entire functions via idempotent components. The idempotent decomposition can be regarded as central technique, reducing assertions to component-wise complex statements.

Thus the paper lays the groundwork for further research in bicomplex and multicomplex analysis, offering opportunities for advancing our understanding of functions in higher-dimensional spaces.

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Department of Mathematics,
University of Kalyani,
P.O.: Kalyani, Dist: Nadia, PIN: 741235, West Bengal, India.
Email: sanjibdatta05@gmail.com