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ON WEAK IDEMPOTENT ELEMENTS AND WEAK IDEMPOTENT RINGS

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Abstract. Abstract. There are different approaches to defining and studying weak idempotent elements in ring theory. We provide a comparative study of three such approaches pertaining to weak idempotent elements and corresponding weak idempotent rings and obtain connections and differences among them. In addition, we study some aspects of weak idempotent rings and discuss their relations with tripotent and weakly tripotent rings.

Key-words: idempotent, weak idempotent element, weak idempotent ring, Boolean ring.
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1. Introduction

Let R be an associative ring. An element $a \in R$ is called idempotent if $a^2 = a$ [1-3]. A ring R is called an idempotent ring or a Boolean ring if every element of R is idempotent. Several authors have shown their interest in Boolean rings and their generalizations and studied rings like Boolean-like rings [4-5] and weakly idempotent rings [6-7].

In this paper, we consider the following three approaches which are used to define weak idempotent elements and undertake a systematic comparative study of these approaches.

- **(WAP 1)** An element a of a ring R is called a weak idempotent or weakly idempotent if $a^4 = a^2$. [6-7].
- **(WAP 2)** An element a of a ring R is called a weak idempotent or weakly idempotent if $a^n - a$ is idempotent [8] for some fixed positive integer n.
- **(WAP 3)** An element a of a ring R is called a weak idempotent or weakly idempotent if a is idempotent or $-a$ is idempotent [9].

It may be noted that $-a$ is idempotent means $(-a)^2 = -a$ which in turn implies $(a)^2 = -a$.

It should be emphasized that there is a research gap, as so far no one has studied whether these approaches to define and study weak idempotent elements are mathematically identical or different. What are the underlying connections among them? These questions motivate us to undertake a comparative study on different approaches which are used to define and study weak idempotent elements and weak idempotent rings.

As per (WAP1), a ring of characteristic two is called weak idempotent whose every element is a weak idempotent element.

We define that a ring R is called weak idempotent as per (WAP2) if every element of R is a weak idempotent as per (WAP2). Similarly, a ring R is a weak idempotent ring as per (WAP3) if each element of R is a weak idempotent ring as per (WAP3).

It may be noted that a ring R is called tripotent if for each $a \in R$ we have $a^3 = a$ and a ring R is called weakly tripotent as per (AP1) if for each $a \in R$ we have $a^3 = a$ or $(1 + a)^3 = 1 + a$ and a weakly tripotent ring as per (AP2) if $a^3 = a$ or $a^3 = -a$ for each $a \in R$ [10].

Among several other findings, we report that weak idempotent rings as per (WAP3) are not closed under the direct product of rings. There do not exist weak idempotent matrix rings of order $n \geq 2$ as per (WAP3) and (WAP2). In addition, we have established implication chains and non-implication results among three different approaches.

We provide our results in the next section.

2. Some Results

Proposition 2.1. *A weak idempotent element as per (WAP3) is a weak idempotent as per (WAP1) also, but the converse is not true.*

Proof. Let R be a ring and $a \in R$ is a weak idempotent element as per (WAP3). Now, as per [9] we have $a^2 = a$ or $(-a)^2 = -a$. If $a^2 = a$, then this implies that $a^4 = a^2$. Similarly if $(-a)^2 = -a$, then we obtain that $(-a)^4 = (-a)^2$. Therefore $a \in R$ is a weak idempotent as per (WAP1).

But the converse is not true. It can be seen as follows.

Let $R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \right\}$. One may easily verify that R is a ring under the addition and multiplication of matrices modulo four. Let $w = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \in R$.

Clearly $w^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $w^4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Thus $w^2 = w^4$. Therefore $w = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \in R$ is a weak idempotent element as per (WAP1). However, neither w nor $-w$ is idempotent. Hence the proof is complete. □

Proposition 2.2. *A weak idempotent element as per (WAP3) is a weak idempotent element as per (WAP2) also, but the converse is not true.*

Proof. Let R be a ring and a is a weak idempotent element as per (WAP3). If $a^2 = a$, then we have $a^3 = a$. Similarly if $(-a)^2 = -a$, then $(a)^3 = -a^2$. This implies $a^3 = a$. Thus for each $a \in R$ we have $a^3 = a$. Therefore $a^n - a$ is idempotent for each $a \in R$, here $n = 3$. Hence $a \in R$ is a weak idempotent element as per (WAP2) also.

Conversely, let us consider the ring $R = \frac{Z}{5Z}$. Here $\mathbb{Z}$ is the ring of integers. Then $2 \in R$ is a weak idempotent element as per (WAP2), but it is not a weak idempotent element as per (WAP3).

□

Proposition 2.3. *The concepts of weak idempotent element as per (WAP1) and as per (WAP2) are quite different.*

Proof. The proof follows directly from the result of Proposition 2.5 given below. □

Proposition 2.4. *Let R be a Boolean ring. Then R is a weak idempotent ring as per (WAP1) and as per (WAP2) also.*

Proof. Let R be a Boolean ring. Then R is a ring of characteristic two. Also for each $a \in R$, we have $a^2 = a$. This gives $a^4 = a^2$. Therefore, R is a weak idempotent ring as per (WAP1). Further $a^2 = a$ implies that $a^2 - a$ is idempotent. So R is a weak idempotent ring as per (WAP2). Hence, if R be a Boolean ring, then R is a weak idempotent ring as per (WAP1) and as per (WAP2) also. □

Note. *Let R be a weak idempotent ring of characteristic two as per (WAP3). Then R is a weak idempotent ring as per (WAP1). But the converse is not true.*

Proposition 2.5. *Let R is a weak idempotent ring as per (WAP2), then R is a reduced ring. However, if R is a weak idempotent ring as per (WAP1), then it is not necessarily a reduced ring.*

Proof. Let R is a weak idempotent ring as per (WAP2). Let R is a non-reduced ring. Let $a \in R$ is a non-zero element and there exists a least positive integer n such that $a^n = 0$. But in this case $a^n - a$ cannot be idempotent [8]. This is a contradiction, and so R is a reduced ring. It may be noted that for any integer $m < n$, $a^m - a$ cannot be idempotent if $a^n = 0$ for some positive integer n [8].

Let R is a weak idempotent ring as per (WAP1), then it is not necessarily a reduced ring. For such a ring, we consider the following ring.

Let $R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$. One may easily verify that R is a ring under the matrix addition and multiplication modulo two.

One may note that $a = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in R$ is a non-zero nilpotent element of R. Therefore, it is not a reduced ring. □

Proposition 2.6. *Let R be a weakly tripotent ring of characteristic two as per (AP1). Then R is not necessarily a weak idempotent ring as per (WAP2) but R is always a weak idempotent ring as per (WAP1).*

Proof. The proof of this Proposition directly follows from Proposition 2.4 and noting that each weakly tripotent ring of characteristic two as per (AP1) satisfies (WAP1). □

Proposition 2.7. *Every tripotent ring as well as a weakly tripotent ring as per (AP2) is a weak idempotent ring as per (WAP2).*

Proof. It has been noted in [10] that each weakly tripotent ring as per (AP2) and each tripotent ring satisfy $a^5 = a$ for $a \in R$. This leads to the conclusion that $a^5 - a$ is idempotent. Hence every tripotent ring as well as a weakly tripotent ring as per (AP2) is a weak idempotent ring as per (WAP2). □

Corollary 2.8. *Let R is a weakly tripotent ring of characteristic two as per (AP2). Then R is a weak idempotent ring as per (WAP2).*

Proposition 2.9. *The converse of Proposition 2.7 is not true.*

Proof. Let us consider a ring of characteristic two. Let $R = \{0, 1, a, b\}$. Here we have $a^2 = a$, $b^2 = a$, $a^3 = 1 = b^3$, $a^4 = a$, $b^4 = b$, $a + b = 1$, $1 + a = b$, $1 + b = a$. Now, after some computation, one may verify that R is not a weakly tripotent ring. Also, R is not a tripotent ring. However, it is a weak idempotent ring as per (WAP2). □

Proposition 2.10. *A weakly idempotent ring as per (WAP1) is not necessarily a weakly tripotent ring as per (AP1).*

Proof. Let $R = \{0, 1, a_2, a_3\}$. Then R is a commutative ring of characteristic two under addition and multiplication defined as follows.

$$a_2 + a_2 = a_3 + a_3 = 0, \quad 1 + a_2 = a_3, \quad 1 + a_3 = a_2, \quad a_2 + a_3 = 1,$$
$$a_3^2 = a_3, \quad a_2^2 = a_2, \quad a_2 a_3 = a_3 a_2 = 0.$$

Let us take $G = \{1, y, y^2 = 1\}$ which is an abelian group under multiplication. It is easy to note that the group ring RG consists of the elements. $0, 1, a_2$, a_3, ya_2y, a_3y, $1 + y, a_2 + y$, $a_3 + y$, $1 + a_2y$, $a_2 + a_2y$, $a_3 + a_2y$, $1 + a_3y$, $a_2 + a_3y$, $a_3 + a_3y$.

Now one may verify that RG is a weak idempotent ring as per (WAP1), but it is not a weakly tripotent ring as per (AP1). Hence the proof is complete. □

Note. *In order to know more about group rings, one may refer to [11].*

Proposition 2.11. *Let R be a field. Then R is a weak idempotent ring as per (WAP1), (WAP2) as well as (WAP3) also iff R is isomorphic to the field of order two.*

Proof. The proof follows from the definition of a weak idempotent ring as per (WAP1) and Proposition 2.4 and Note 2. □

Proposition 2.12. *Let R be a field. Then R is a weak idempotent ring as per (WAP2) as well as (WAP3) also iff R is isomorphic to the field of order two or the field of order three.*

Proof. Let R is a field. This implies that R is a commutative indecomposable ring. Let R is a weak idempotent ring as per (WAP3). This implies that for each $a \in R$ we have $a^2 = a$ or $a^2 = 2a - 1$. Rurther $a^2 = a$. Therefore $a(a-1) = 0$ or $(a-1)^2 = 0$. Similarly $a^2 = a$ implies that $a = 0, 1$. This gives $R \cong \mathbb{Z}_2$ or $R \cong \mathbb{Z}_3$. It can be easily seen that a field of order two and order three both are weak idempotent rings as per (WAP2). The converse easily follows. □

Proposition 2.13. *Let R be any ring. Then there does not exist any weak idempotent matrix ring $M_n(R)$, $n \geq 2$ as per (WAP3).*

Proof. Let R be any ring. Let $M_n(R)$, $n \geq 2$ be a matrix ring defined over R. Let $a \in R$ and $n = 2$. Then $A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \in M_2(R)$. Clearly A is neither idempotent nor weak idempotent. Therefore $M_2(R)$ is not a weak idempotent ring as per (WAP3). Similarly, we can prove for $n > 2$. Hence there does not exist any weak idempotent matrix ring $M_n(R)$, $n \geq 2$ as per (WAP3). □

Proposition 2.14. *Let R be any ring. Then there does not exist any weak idempotent matrix ring $M_n(R)$, $n \geq 2$ as per (WAP2).*

Proof. The proof is similar to the proof of Proposition 2.13. □

Proposition 2.15. *Let R_1 and R_2 are any two nonzero weak idempotent rings as per (WAP3). Then the direct product of R_1 and R_2 is not a weak idempotent ring as per (WAP3) provided $0 \neq a \in R_1$ and $0 \neq b \in R_2$ such that $a^2 = a$ and $b^2 = -b$.*

Proof. Let R_1 and R_2 are any two nonzero weak idempotent rings as per (WAP3). Let $0 \neq a \in R_1$ and $0 \neq b \in R_2$ such that $a^2 = a$ and $b^2 = -b$. Clearly $(a, b) \in R_1 \times R_2$. Now we have $(a\ b)^2 = (a - b)$. We note that neither $(ab)^2 = (a\ b)$ nor $(a\ b)^2 = -(a\ b)$. Therefore, $(a\ b)$ is not a weak idempotent element as per (WAP3). Thus $R_1 \times R_2$ is not a weak idempotent ring as per (WAP3). This completes the proof. □

Corollary 2.16. *Let R_1 and R_2 are any two nonzero weak idempotent rings as per (WAP3). Then the direct product of R_1 and R_2 is not a weak idempotent ring as per (WAP3) provided $0 \neq a \in R_1$ and $0 \neq b \in R_2$ such that $a^2 = -a$ and $b^2 = b$.*

Proof. The proof is similar to the proof of Proposition 2.15. □

Corollary 2.17. *Let R_1 and R_2 are any two nonzero weak idempotent rings as per (WAP3). Then the direct product of R_1 and R_2 need not a weak idempotent ring as per (WAP3).*

Proof. Let us consider the ring $R = \frac{Z}{6Z}$. Here Z is the ring of integers. It is easy to verify that R is a weak idempotent ring as per (WAP3). We note that $R_1 \times R_1$ is not a weak idempotent ring as per (WAP3) because $(3, 5) \in R_1 \times R_1$ is not a weak idempotent element as per (WAP3). Here $R = R_1 = R_2$. □

3. Conclusion

We have systematically compared three distinct definitions of weak idempotent elements. We have established implication chains and non-implication results among three different approaches. We have given structural properties of weak idempotent rings under (WAP1), (WAP2) and (WAP3). Weak idempotent rings as per (WAP3) is not closed under the direct product of rings. There do not exist weak idempotent matrix rings of order $n \geq 2$ as per (WAP3) and (WAP2).

This work may motivate future research on diverse approaches used to study different aspects of rings with certain conditions on ring elements and to establish underlying connections among them and related structures.

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