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TRAVELING WAVE SIMULATIONS OF FRACTIONAL ORDER SYSTEMS USING PERTURBATION TECHNIQUE

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Abstract. Abstract. This study evaluates and models fractional-order systems of partial differential equations commonly encountered in the study of chemistry and physics. We examine the fractional-order reaction-diffusion Brusselator system, which has extensive applications in modeling chemical reaction-diffusion phenomena. Also, we investigate the time-fractional generalized coupled Hirota Satsuma KdV system, which characterizes the interaction between two long waves possessing distinct dispersion relations. The fractional homotopy perturbation method is presented to estimate the solution of the proposed models. We estimate the solutions for various time-fractional orders and illustrate them in graphs and tables.

Keywords: Fractional reaction-diffusion Brusselator system, Fractional generalized coupled Hirota Satsuma KdV system, Homotopy Perturbation Method, Caputo derivative, Riemann-Liouville integral.

2010 AMS Subject Classification: 35K57, 35Q53, 26A33, 35C10.

1. Introduction

Fractional order partial differential equations play a crucial role in the study of all fields of physics, mathematics, engineering, economics, etc [3, 6, 16, 18, 22, 28, 33]. Traveling wave simulations in fractional-order systems represent a powerful approach to understanding dynamic processes where memory effects and hereditary properties dominate common in various fields, including viscoelasticity, anomalous diffusion, signal processing, plasma physics, nonlinear biological systems, acoustics, etc. [12, 20, 29]. The estimation of traveling wave-type solutions of fractional ordered differential equations has a vital role in the analysis of nonlinear behavior [10, 19, 31]. Exact solutions for many nonlinear fractional differential equations are rare. Therefore, researchers preferred tools like series solution methods.

The homotopy perturbation method is an appropriate technique for solving a wide class of fractional as well as classical differential systems [4, 11, 14, 15]. Several recent studies illustrate the role of perturbation methods and approximation schemes in traveling wave simulations of fractional-order systems. Deng and Zhang [7] provide a stability framework for fractional-order controllers, which has direct implications for analyzing traveling wave dynamics. Swarnakar [27] proposes finite-dimensional representations of fractional-order systems through bandlimited approximations, making the simulation of fractional traveling waves more practical. Abolvafaei et al. [1] develop a hybrid approach using the homotopy singular perturbation method to decompose nonlinear fractional systems into simpler subsystems while preserving their stability and wave-like properties. In applied contexts, Rico-Ramirez et al. [25] introduce optimization techniques for combined fractional-ordinary systems, which are particularly relevant for controlling traveling waves in engineering and biological models. Walia et al.[30] analyze approximation schemes in fractional-order circuits, demonstrating how wavelike behavior can be efficiently modeled and implemented in practical hardware.

In this paper, we extend this method to solve a system of fractional-order partial differential equations. Consider the time-fractional two-dimensional Brusselator system, which is presented

by a system of nonlinear partial differential equations as follows:

$$\begin{aligned} D_t^\alpha U &= B + U^2V - (A+1)U + \gamma\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) \\ D_t^\alpha V &= AU - U^2V + \gamma\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) \end{aligned} \tag{1}$$

where $0 < \alpha \leq 1$ and $U(x,y,t), V(x,y,t)$ are concentrations of chemical species in a twodimensional region Ω, subject to the initial boundary conditions $U(x,y,0) = U_0, V(x,y,0) = V_0$. The fractional-order introduces memory effects, making the model better suited for describing anomalous diffusion and nonlocal temporal behavior. This system can exhibit fractional oscillations, pattern formation, and wave-like behavior, which are not captured by the classical model. Qamar Din [8] demonstrated effective chaos control and bifurcation management in discrete fractional-order Brusselator systems, showing how fractional derivatives enrich dynamical analysis. Complementing this, Priyendhu et al. [23] provided exact solutions for coupled nonlinear time-fractional diffusion-reaction systems with time delays, which serve as analogues to fractional Brusselators. Extending to related models, Jagatheeshwari et al. [17] introduced a strong analytical technique for solving the multi-dimensional Schnakenberg system, an approach equally applicable to Brusselator-type systems. In the domain of oscillatory chemistry, Rehman et al. [24] conducted a comparative analysis of fractional Belousov-Zhabotinsky models, underscoring the role of fractional calculus in refining nonlinear oscillatory dynamics. Also, Alsallami et al. [2] explored time-fractional dynamics in the Belousov-Zhabotinsky system, showing how fractional derivatives improve the modeling of oscillations and chaotic patterns.

Furthermore, we study the time-fractional generalized coupled Hirota Satsuma KdV system as follows

$$\begin{aligned} D_t^\alpha U &= \frac{1}{2}\frac{\partial^3 U}{\partial x^3} - 3U\frac{\partial U}{\partial x} + 3W\frac{\partial V}{\partial x} + 3V\frac{\partial W}{\partial x} \\ D_t^\alpha V &= -\frac{\partial^3 V}{\partial x^3} + 3U\frac{\partial V}{\partial x} \\ D_t^\alpha W &= -\frac{\partial^3 W}{\partial x^3} + 3U\frac{\partial W}{\partial x} \end{aligned} \tag{2}$$

where $0 < \alpha \leq 1$ and $t > 0$. System (2) describes an interaction of two long waves with different dispersion relations, and they are well-suited for wave propagation phenomena. Shallow water equations provide an interesting and rich field of research. Recent works on the time-fractional generalized coupled Hirota-Satsuma KdV system have revealed significant advances in analytical solution techniques and soliton dynamics. Yin and Gao [32] introduced multiple analytical approaches, including the unified method, improved expansion method, and homogeneous balance method, for the time-fractional HS-KdV equation, yielding a rich family of solutions. Building on this, Mohanty [21] applied extended expansion techniques to derive abundant soliton structures, including solitary waves, kink-type waves, and multi-soliton solutions. Hassaballa et al. [13]further demonstrated the effectiveness of the Hirota bilinear method in analyzing fractional coupled nonlinear systems, providing both multiple and singular soliton solutions. Complementing these developments, Dutta et al. [9] analyzed solitons in the fractional Zakharov-Kuznetsov equation and showed how variations in fractional order significantly alter soliton structures-insights that are directly relevant to the generalizations of the HS-KdV system.

Further paper is organized as follows: Basic definitions and properties related to fractional derivatives and integrals are given in Section 2. In Section 3, we describe the homotopy perturbation method to solve a system of fractional differential equations. Section 4 contains the numerical experiments to show the efficiency and accuracy of the proposed method. The conclusion is given in the last section, followed by the references.

2. Preliminaries

In this section, we study some fundamental definitions and their properties of fractional calculus. There are a variety of definitional approaches given for fractional derivatives and integrals in the literature, notably including the Riemann-Liouville, Caputo, and Grünwald-Letnikov definitions, etc. In this paper, the Riemann-Liouville and Caputo definitions are considered for the fractional-order differential operators.

Definition 2.1. *The Riemann-Liouville fractional integral operator of a function f is defined as*

$$J^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{f(\zeta)}{(x-\zeta)^{1-\alpha}}d\zeta, \alpha > 0$$

Definition 2.2. *The Caputo fractional derivative of order β for the function $f(x)$ is defined as*

$$D^{\beta}f(x) = J^{m-\beta}f^{(m)}(x) = \frac{1}{\Gamma(m-\beta)}\int_0^x \frac{f^{(m)}(\zeta)}{(x-\zeta)^{1-m+\beta}}d\zeta$$

where $m = \lceil \beta \rceil$.

Lemma 2.3. *For a function f and $\alpha, \beta \geq 0$, we have*
(i) $J^{\alpha}J^{\beta}f(x) = J^{\alpha+\beta}f(x)$.
(ii) $J^{\alpha}D^{\alpha}f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{x^k}{k!}, x > 0$
(iii) $D^{\alpha}(C) = 0$, *where C is a constant.*
(iv) $D^{\alpha}x^{\gamma} = \begin{cases} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}x^{\gamma-\alpha}, & \gamma > \alpha - 1, \\ 0, & \gamma \leqslant \alpha - 1, \gamma \in \mathbb{N} \end{cases}$

In the next section, we discuss the fractional homotopy perturbation method for solving systems of fractional partial differential equations.

3. Homotopy Perturbation Method for Fractional Order System

In the literature, we can find that there are various methods are given to solve fractional order differential equations, but not many methods are available for solving systems of fractional differential equations. Homotopy Perturbation Method, due to its efficiency and effectiveness, will help us to solve the proposed problem. The general form of a system of fractional order partial differential equations can be considered as follows

$$D_t^{\alpha}U_i = N_i\left(U_1, U_2, \ldots U_n, x, t\right) + f_i(x,t), \quad i = 1, 2, \ldots n \tag{3}$$

with the initial conditions $U_i(x,0) = g_i(x), \quad i = 1, 2, \ldots n$. Here N_i are non-linear operators, which usually depend on the functions U_i and their derivatives. For solving system (3), we construct the following homotopies:

$$H_i\left(U_i, p\right) = D_t^{\alpha}U_i - D_t^{\alpha}U_i(x,0) + p\left[D_t^{\alpha}U_i(x,0) - N_i\left(U_1, U_2, \ldots U_n, x, t\right) - f_i(x,t)\right] = 0 \tag{4}$$

with initial approximation $U_i(x,0) = g_i(x), i = 1, 2, \ldots n$. We assume the solution of the equations (4) is expressed as a power series in p as follows

$$U_i = \sum_{n=0}^{\infty} p^n U_{in} = U_{i0} + pU_{i1} + p^2U_{i2} + p^3U_{i3} + \ldots \tag{5}$$

Now, we substitute equations (5) in equations (4). Equating like powers of p, and applying the integration operator, J_x^α, to both sides, we obtain

$$\begin{aligned}
p^0 &: U_{i0} = g_i(x) \\
p^1 &: U_{i1} = J_x^\alpha \left[D_t^\alpha U_{i0}(x,0) + N_i\left(U_1, U_2, \dots U_n, x, t\right) + f_i(x,t)\right] \\
p^2 &: U_{i2} = J_x^\alpha \left[D_t^\alpha N_i\left(U_1, U_2, \dots U_n, x, t\right) + f_i(x,t)\right] \\
p^2 &: U_{i3} = J_x^\alpha \left[D_t^\alpha N_i\left(U_1, U_2, \dots U_n, x, t\right) + f_i(x,t)\right] \\
&\vdots
\end{aligned}$$

Hence, setting $p = 1$ in equations (5), the solutions for equation (3) can be estimated as follows

$$U_i = \sum_{n=0}^{\infty} U_{in} = U_{i0} + U_{i1} + U_{i2} + + \dots$$

Convergence and uniqueness of the solution obtained by the homotopy perturbation technique have been proved and found in the literature [5, 26].

4. Numerical Results

Example 1. Consider the following time-fractional two-dimensional Brusselator system (1) with $A = 1, B = 0, \gamma = \frac{1}{4}$.

$$\begin{aligned}
D_t^\alpha U &= U^2V - 2U + \frac{1}{4}\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) \\
D_t^\alpha V &= U - U^2V + \frac{1}{4}\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right)
\end{aligned} \tag{6}$$

subject to the initial conditions $U(x,y,0) = e^{-x-y}, V(x,y,0) = e^{x+y}$.
In order to solve problem (6), we construct homotopies $H(u,p), H(v,p), : \mathbb{R} \times [0,1] \to \mathbb{R}$ as follows

$$H(u,p) = D_t^\alpha u - D_t^\alpha U(x,y,0) + p\left[D_t^\alpha U(x,y,0) - u^2v + 2u - \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right] = 0 \tag{7}$$

$$H(v,p) = D_t^\alpha v - D_t^\alpha V(x,y,0) + p\left[D_t^\alpha V(x,y,0) - u + u^2v + \frac{1}{4}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)\right] = 0 \tag{8}$$

subject to the initial conditions $U(x,y,0) = e^{-x-y}, V(x,y,0) = e^{x+y}$. Suppose the solution of the equation is expressed as a power series in p as follows

$$u = \sum_{n=0}^{\infty} p^n u_n, v = \sum_{n=0}^{\infty} p^n v_n \tag{9}$$

Now, we substitute (9) in equations (7) - (8). Equating like powers of p, we obtain

$$\begin{aligned}
p^0 &: u_0 = U(x,y,0) \\
&: v_0 = V(x,y,0) \\
p^1 &: u_1 = -J_x^\alpha\left(-\gamma\frac{\partial^2}{\partial x^2}u_0 - \gamma\frac{\partial^2}{\partial y^2}u_0 - u_0^2v_0 + Au_0 + D_t^\alpha u_0 - B + u_0\right) \\
&: v_1 = -J_x^\alpha\left(-\gamma\frac{\partial^2}{\partial x^2}v_0 - \gamma\frac{\partial^2}{\partial y^2}v_0 + u_0^2v_0 - Au_0 + D_t^\alpha v_0\right) \\
p^2 &: u_2 = -J_x^\alpha\left(-\gamma\frac{\partial^2}{\partial x^2}u_1 - \gamma\frac{\partial^2}{\partial y^2}u_1 - 2u_0u_1v_0 - u_0^2v_1 + Au_1 + u_1\right)
\end{aligned}$$

$$
\begin{aligned}
: v_2 =& - J_x^\alpha \left(2u_0u_1v_0 - \gamma\frac{\partial^2}{\partial x^2}v_1 - \gamma\frac{\partial^2}{\partial y^2}v_1 + u_0^2v_1 - Au_1 \right) \\
p^3 : u_3 =& - J_x^\alpha \left(-\gamma\frac{\partial^2}{\partial x^2}u_2 - \gamma\frac{\partial^2}{\partial y^2}u_2 - u_1^2v_0 - 2u_0u_2v_0 - 2u_0u_1v_1 - u_0^2v_2 + Au_2 + u_2 \right) \\
: v_3 =& - J_x^\alpha \left(u_1^2v_0 + 2u_0u_2v_0 + 2u_0u_1v_1 - \gamma\frac{\partial^2}{\partial x^2}v_2 - \gamma\frac{\partial^2}{\partial y^2}v_2 + u_0^2v_2 - Au_2 \right) \\
p^4 : u_4 =& - J_x^\alpha \left(-\gamma\frac{\partial^2}{\partial x^2}u_3 - \gamma\frac{\partial^2}{\partial y^2}u_3 - 2u_1u_2v_0 - 2u_0u_3v_0 - u_1^2v_1 - 2u_0u_2v_1 - 2u_0u_1v_2 \right. \\
& \left. -u_0^2v_3 + Au_3 + u_3 \right) \\
: v_4 =& -J_x^\alpha \left(2u_1u_2v_0 + 2u_0u_3v_0 + u_1^2v_1 + 2u_0u_2v_1 + 2u_0u_1v_2 - \gamma\frac{\partial^2}{\partial x^2}v_3 \right. \\
& \left. -\gamma\frac{\partial^2}{\partial y^2}v_3 + u_0^2v_3 - Au_3 \right) \\
p^5 : u_5 =& - J_x^\alpha \left(-\gamma\frac{\partial^2}{\partial x^2}u_4 - \gamma\frac{\partial^2}{\partial y^2}u_4 - u_2^2v_0 - 2u_1u_3v_0 - 2u_0u_4v_0 - 2u_1u_2v_1 - 2u_0u_3v_1 \right. \\
& \left. -u_1^2v_2 - 2u_0u_2v_2 - 2u_0u_1v_3 - u_0^2v_4 + Au_4 + u_4 \right)
\end{aligned}
$$

and so on. Now, we solve the above system of equations, starting with initial values $U(x,y,0) = e^{-x-y}, V(x,y,0) = e^{x+y}$. We get,

$$
\begin{aligned}
p^0 : u_0 &= e^{-x-y} \\
: v_0 &= e^{x+y} \\
p^1 : u_1 &= -\frac{t^\alpha e^{(-x-y)}}{2\Gamma(\alpha+1)} \\
: v_1 &= \frac{t^\alpha e^{(x+y)}}{2\Gamma(\alpha+1)} \\
p^2 : u_2 &= \frac{t^{2\alpha} e^{(-x-y)}}{4\Gamma(2\alpha+1)} \\
: v_2 &= \frac{t^{2\alpha} e^{(x+y)}}{4\Gamma(2\alpha+1)} \\
p^3 : u_3 &= \frac{\left(3\Gamma(\alpha+1)^2 - 2\Gamma(2\alpha+1)\right) t^{3\alpha} e^{(-x-y)}}{8\Gamma(3\alpha+1)\Gamma(\alpha+1)^2} \\
: v_3 &= \frac{\left(t^{3\alpha} e^{(2x+2y)}\Gamma(\alpha+1)^2 - 2\left(2\Gamma(\alpha+1)^2 - \Gamma(2\alpha+1)\right) t^{3\alpha}\right) e^{(-x-y)}}{8\Gamma(3\alpha+1)\Gamma(\alpha+1)^2} \\
&\vdots
\end{aligned}
$$

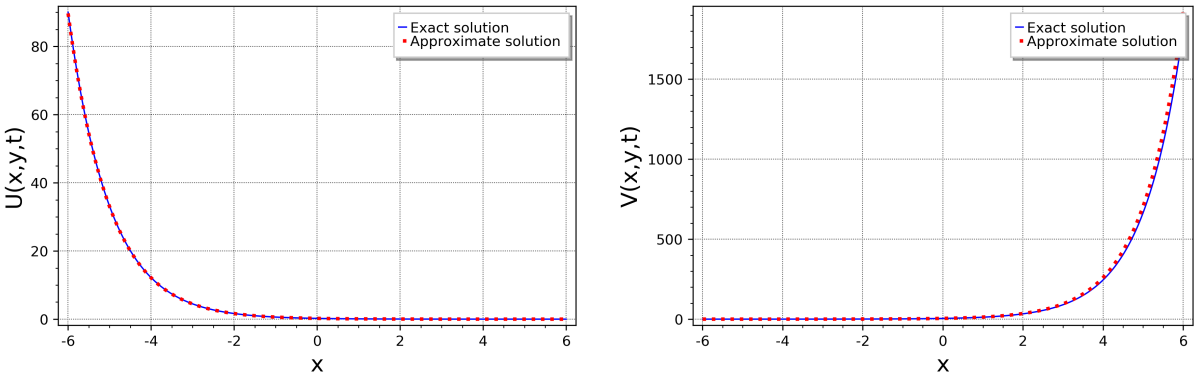
and so on. Hence, the approximate solution $U(x,y,t)$ and $V(x,y,t)$ can be express as follows

$$
\begin{aligned}
U(x,y,t) =& e^{-x-y} - \frac{t^\alpha e^{(-x-y)}}{2\Gamma(\alpha+1)} + \frac{t^{2\alpha} e^{(-x-y)}}{4\Gamma(2\alpha+1)} + \frac{\left(3(\Gamma(\alpha+1))^2 - 2\Gamma(2\alpha+1)\right) t^{3\alpha} e^{(-x-y)}}{8\Gamma(3\alpha+1)(\Gamma(\alpha+1))^2} + \dots \\
V(x,y,t) =& e^{x+y} + \frac{t^\alpha e^{(x+y)}}{2\Gamma(\alpha+1)} + \frac{t^{2\alpha} e^{(-x-y)}}{4\Gamma(2\alpha+1)} \\
& + \frac{\left(t^{3\alpha} e^{(2x+2y)}(\Gamma(\alpha+1))^2 - 2\left(2(\Gamma(\alpha+1))^2 - \Gamma(2\alpha+1)\right) t^{3\alpha}\right) e^{(-x-y)}}{8\Gamma(3\alpha+1)(\Gamma(\alpha+1))^2} + \dots
\end{aligned}
$$

The exact solution are $U(x,y,t) = e^{-x-y-\frac{t}{2}}$ and $V(x,y,t) = e^{x+y+\frac{t}{2}}$ respectively. The absolute error is estimated for different values of x, y, and t in Table 1, which shows that the obtained approximate solution is enormously agreed with the exact solution.

TABLE 1. Absolute error

	$(x,y) \to t \downarrow$	$(1,1)$	$(5,1)$	$(10,5)$	$(15,15)$
Error for $U(x,y,t)$	0.1	3.4894×10^{-8}	6.3910×10^{-10}	7.8872×10^{-14}	2.4127×10^{-20}
	0.2	5.5280×10^{-7}	1.0124×10^{-8}	1.2495×10^{-12}	3.8223×10^{-19}
	0.5	2.0970×10^{-5}	3.8408×10^{-7}	4.7399×10^{-11}	1.4499×10^{-17}
	1.0	3.1993×10^{-4}	5.8591×10^{-6}	7.2315×10^{-10}	2.2121×10^{-16}
	2.0	4.6753×10^{-3}	8.5631×10^{-5}	1.05677×10^{-8}	3.2326×10^{-15}
Error for $V(x,y,t)$	0.1	5.2921×10^{-11}	3.7231×10^{-7}	3.2792×10^{-7}	1.1529×10^{-7}
	0.2	2.1168×10^{-10}	7.4463×10^{-7}	6.5590×10^{-7}	2.3060×10^{-7}
	0.5	1.3230×10^{-9}	1.8615×10^{-6}	1.8615×10^{-6}	1.8615×10^{-6}
	1.0	5.2921×10^{-9}	3.7229×10^{-6}	3.2813×10^{-6}	1.1534×10^{-6}
	2.0	2.1168×10^{-8}	7.4453×10^{-6}	6.5670×10^{-6}	2.3080×10^{-6}



(A) Comparison of the exact solutions with the approximate solution at $t = 1,\ y = 1$.

(B) Comparison of the exact solutions with the approximate solution at $t = 1,\ y = 1$.

FIGURE 1.

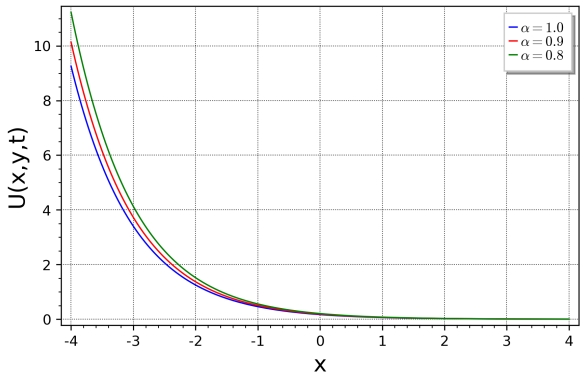
The comparison of the approximate solutions with the exact solutions are represented in Figure 1, which shows that the approximate solution is enormously close to the exact solutions.

In Figure 2, we examine the behavior of the solution for different values of fractional order α and observed that the amplitude of the wave decreases with time-fractional order α.

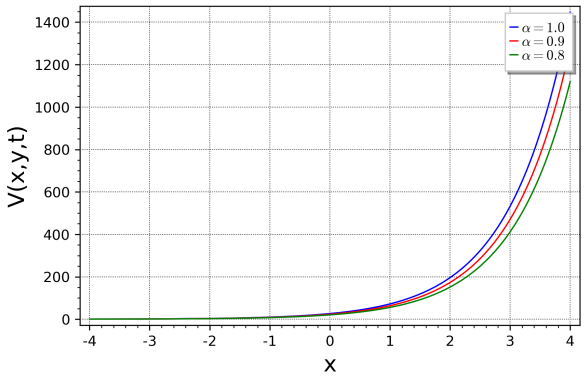
In Figure 3, we plot the profile of the approximate solution $U(x,y,t)$ and $V(x,y,t)$ given above for the parameters $\alpha = 0.95, -6 \le x \le 6, -6 \le t \le 6$, and observe that the obtained solution is a kink-type traveling wave solution.

Example 2. Consider the time-fractional generalized coupled Hirota Satsuma KdV system as follows

$$\begin{aligned} D_t^\alpha U &= \frac{1}{2}\frac{\partial^3 U}{\partial x^3} - 3U\frac{\partial U}{\partial x} + 3W\frac{\partial V}{\partial x} + 3V\frac{\partial W}{\partial x} \\ D_t^\alpha V &= -\frac{\partial^3 V}{\partial x^3} + 3U\frac{\partial V}{\partial x} \\ D_t^\alpha W &= -\frac{\partial^3 W}{\partial x^3} + 3U\frac{\partial W}{\partial x} \end{aligned} \tag{10}$$

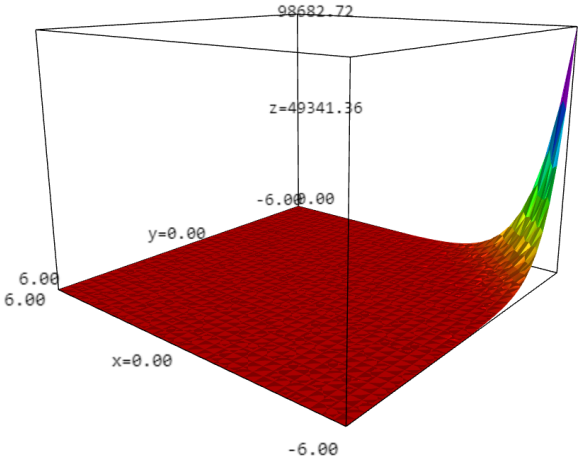


(A) Behavior of approximate solution for different values of α at $t = 1.5, y = 1$

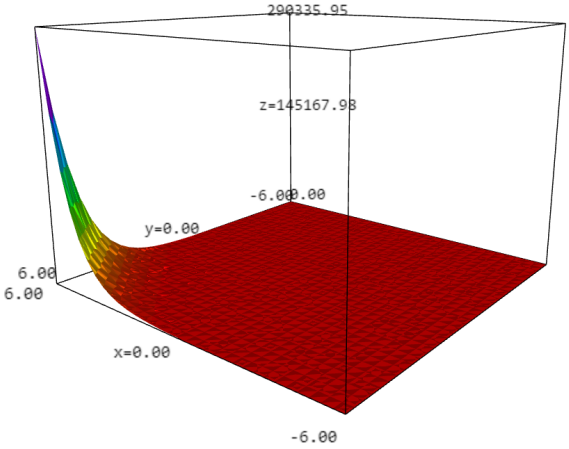


(B) Behavior of approximate solution for different values of α at $t = 1.5, y = 1$

FIGURE 2.



(A) Approximate solution at $t = 1, \alpha = 0.95$



(B) Approximate solution at $t = 5, \alpha = 0.95$

FIGURE 3. Kink-type traveling wave solution

with initial conditions $U(x,0) = \frac{\beta-2k^2}{3}2k^2\tanh^2(kx), V(x,0) = -\frac{4k^2c_0(\beta+k^2)}{3c_1^2} + \frac{4k^2(\beta+k^2)\tanh(kx)}{3c_1}$, $w(x,0) = c_0 + c_1\tanh(kx)$, where $k, c_0, \beta, c_1 \neq 0$ are constant.

Now, we construct homotopies $H(u,p), H(v,p), H(w,p) : \mathbb{R}\times[0,1] \to \mathbb{R}$ as follows

$$\begin{aligned} H(u,p) &= D_t^\alpha u - D_t^\alpha U_0 + p\left[D_t^\alpha U_0 - \frac{1}{2}\frac{\partial^3 u}{\partial x^3} + 3u\frac{\partial u}{\partial x} - 3w\frac{\partial v}{\partial x} - 3v\frac{\partial w}{\partial x}\right] = 0 \\ H(v,p) &= D_t^\alpha v - D_t^\alpha V_0 + p\left[D_t^\alpha V_0 + \frac{\partial^3 v}{\partial x^3} - 3u\frac{\partial v}{\partial x}\right] = 0 \\ H(w,p) &= D_t^\alpha w - D_t^\alpha W_0 + p\left[D_t^\alpha W_0 + \frac{\partial^3 w}{\partial x^3} - 3u\frac{\partial w}{\partial x}\right] = 0 \end{aligned} \tag{11}$$

with initial approximation $U_0 = U(x,0), V_0 = V(x,0)$ and $W_0 = W(x,0)$ respectively. Suppose the solution of the equation is expressed as a power series in p as follows

$$u = \sum_{n=0}^{\infty} p^n u_n, v = \sum_{n=0}^{\infty} p^n v_n, w = \sum_{n=0}^{\infty} p^n w_n \tag{12}$$

Now, we substitute (12) in equations (11). Equating like powers of p, we obtain

$$
\begin{aligned}
p^0 &: u_0 = U_0 = U(x,0)\\
&: v_0 = V_0 = V(x,0)\\
&: w_0 = W_0 = W(x,0)\\
p^1 &: u_1 = J_t^\alpha\left[-D_t^\alpha U_0 + \frac{1}{2}\frac{\partial^3 u_0}{\partial x^3} - 3u_0\frac{\partial u_0}{\partial x} + 3w_0\frac{\partial v_0}{\partial x} + 3v_0\frac{\partial w_0}{\partial x}\right]\\
&: v_1 = J_t^\alpha\left[-D_t^\alpha V_0 - \frac{\partial^3 v_0}{\partial x^3} + 3u_0\frac{\partial v_0}{\partial x}\right]\\
&: w_1 = J_t^\alpha\left[-D_t^\alpha W_0 - \frac{\partial^3 w_0}{\partial x^3} + 3u_0\frac{\partial w_0}{\partial x}\right]\\
p^2 &: u_2 = J_t^\alpha\left[\frac{1}{2}\frac{\partial^3 u_1}{\partial x^3} - 3u_1\frac{\partial u_0}{\partial x} - 3u_0\frac{\partial u_1}{\partial x} + 3w_0\frac{\partial v_1}{\partial x} + 3v_1\frac{\partial w_0}{\partial x} + 3w_1\frac{\partial v_0}{\partial x} + 3v_0\frac{\partial w_1}{\partial x}\right]\\
&: v_1 = J_t^\alpha\left[-\frac{\partial^3 v_1}{\partial x^3} + 3u_1\frac{\partial v_0}{\partial x} + 3u_0\frac{\partial v_1}{\partial x}\right]\\
&: w_1 = J_t^\alpha\left[-\frac{\partial^3 w_1}{\partial x^3} + 3u_1\frac{\partial w_0}{\partial x} + 3u_0\frac{\partial w_1}{\partial x}\right]\\
p^3 &: u_3 = J_t^\alpha\left[\frac{1}{2}\frac{\partial^3 u_2}{\partial x^3} - 3u_1\frac{\partial u_1}{\partial x} - 3u_0\frac{\partial u_2}{\partial x} - 3u_2\frac{\partial u_0}{\partial x} + 3w_0\frac{\partial v_2}{\partial x} + 3v_2\frac{\partial w_0}{\partial x} + 3w_1\frac{\partial v_1}{\partial x}\right.\\
&\qquad \left. + 3v_1\frac{\partial w_1}{\partial x} + 3v_0\frac{\partial w_2}{\partial x} + 3w_2\frac{\partial v_0}{\partial x}\right]\\
&: v_3 = J_t^\alpha\left[-\frac{\partial^3 v_2}{\partial x^3} + 3u_2\frac{\partial v_0}{\partial x} + 3u_1\frac{\partial v_1}{\partial x} + 3u_0\frac{\partial v_2}{\partial x}\right]\\
&: w_3 = J_t^\alpha\left[-\frac{\partial^3 w_2}{\partial x^3} + 3u_2\frac{\partial w_0}{\partial x} + 3u_1\frac{\partial w_1}{\partial x} + 3u_0\frac{\partial w_2}{\partial x}\right]\\
&\ \vdots
\end{aligned}
$$

and so on. Now, we solve the above system of equations, starting with initial values

$$
\begin{aligned}
U(x,0) &= \frac{\beta - 2k^2}{3}2k^2\tanh^2(kx)\\
V(x,0) &= -\frac{4k^2c_0\left(\beta + k^2\right)}{3c_1^2} + \frac{4k^2\left(\beta + k^2\right)\tanh(kx)}{3c_1}\\
W(x,0) &= c_0 + c_1\tanh(kx).
\end{aligned}
$$

We get,

$$
\begin{aligned}
U(x,t) =& \frac{\beta - 2k^2}{3}2k^2\tanh^2(kx) + \frac{4\beta k^3 t^\alpha\sinh(kx)}{\cosh(kx)^3\Gamma(\alpha+1)} - \frac{4\left(2\beta^2k^4t^{2\alpha}\sinh(kx)^2 - \beta^2k^4t^{2\alpha}\right)}{\cosh(kx)^4\Gamma(2\alpha+1)}\\
&+ \frac{16}{\cosh(kx)^7\Gamma(3\alpha+1)(\Gamma(\alpha+1))^2}\left(\beta^3k^5t^{3\alpha}\cosh(kx)^4(\Gamma(\alpha+1))^2\right.\\
&+ 9\left(2(\Gamma(\alpha+1))^2 - \Gamma(2\alpha+1)\right)\beta^2k^7t^{3\alpha} - \left(5\left(2(\Gamma(\alpha+1))^2 - \Gamma(2\alpha+1)\right)\beta^2k^7\right.\\
&\left.\left.+ \left((\Gamma(\alpha+1))^2 + \Gamma(2\alpha+1)\right)\beta^3k^5\right)t^{3\alpha}\cosh(kx)^2\sinh(kx) + \dots\right.\\
V(x,t) =& -\frac{4k^2c_0\left(\beta + k^2\right)}{3c_1^2} + \frac{4k^2\left(\beta + k^2\right)\tanh(kx)}{3c_1} + \frac{4\left(\beta k^5 + \beta^2k^3\right)t^\alpha}{3c_1\cosh(kx)^2\Gamma(\alpha+1)}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{8\left(\beta^2k^6+\beta^3k^4\right)t^{2\alpha}\sinh(kx)}{3c_1\cosh(kx)^3\Gamma(2\alpha+1)} + \frac{8}{3c_1\cosh(kx)^6\Gamma(3\alpha+1)(\Gamma(\alpha+1))^2}\times\\
&\left(2\left(\beta^3k^7(\Gamma(\alpha+1))^2+\beta^4k^5(\Gamma(\alpha+1))^2\right)t^{3\alpha}\sinh(kx)^4\right.\\
&+\left(12\left(2(\Gamma(\alpha+1))^2-\Gamma(2\alpha+1)\right)\beta^2k^9+\left(25(\Gamma(\alpha+1))^2-12\Gamma(2\alpha+1)\right)\beta^3k^7\right.\\
&\left.+\beta^4k^5(\Gamma(\alpha+1))^2\right)t^{3\alpha}\sinh(kx)^2\\
&\left.-\left(\beta^3k^7(\Gamma(\alpha+1))^2+\beta^4k^5(\Gamma(\alpha+1))^2\right)t^{3\alpha}\right)+\dots\\
W(x,t) =& c_0+c_1\tanh(kx)+\frac{\beta c_1kt^{\alpha}}{\cosh(kx)^2\Gamma(\alpha+1)}-\frac{2\beta^2c_1k^2t^{2\alpha}\sinh(kx)}{\cosh(kx)^3\Gamma(2\alpha+1)}\\
&+\frac{2}{\cosh(kx)^6\Gamma(3\alpha+1)(\Gamma(\alpha+1))^2}\left(2\beta^3c_1k^3t^{3\alpha}(\Gamma(\alpha+1))^2\sinh(kx)^4\right.\\
&-\beta^3c_1k^3t^{3\alpha}(\Gamma(\alpha+1))^2+\left(12\left(2\alpha t^2-\Gamma(2\alpha+1)\right)\beta^2c_1k^5\right.\\
&\left.\left.+\beta^3c_1k^3(\Gamma(\alpha+1))^2\right)t^{3\alpha}\sinh(kx)^2\right)+\dots
\end{aligned}
$$

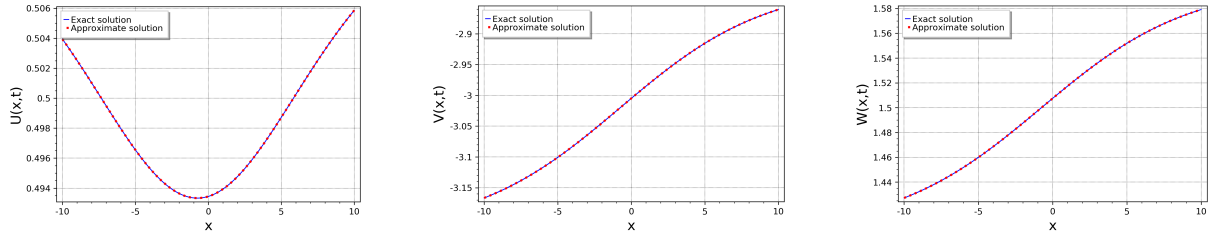
The exact solution are $U(x,t)=\frac{\beta-2k^2}{3}2k^2\tanh^2(k(x+\beta t))$, $V(x,t)=-\frac{4k^2c_0\left(\beta+k^2\right)}{3c_1^2}+\frac{4k^2\left(\beta+k^2\right)\tanh(k(x+\beta t))}{3c_1}$ and $W(x,t)=c_0+c_1\tanh(k(x+\beta t))$ respectively. The absolute error is estimated for different values of x and t in Table 1, which shows that the obtained approximate solution is enormously agreed with the exact solution.

TABLE 2. Absolute error

	$x\to$ $t\downarrow$	1	2	3	4
Error for $U(x,t)$	0.1	6.1564×10^{-10}	4.6085×10^{-10}	2.5131×10^{-10}	3.6424×10^{-11}
	0.2	9.7944×10^{-9}	7.2811×10^{-9}	3.9152×10^{-9}	4.8647×10^{-10}
	0.3	4.9289×10^{-8}	3.6386×10^{-8}	1.9285×10^{-8}	1.9800×10^{-9}
	0.4	1.5481×10^{-7}	1.1348×10^{-7}	5.9260×10^{-8}	4.7490×10^{-9}
	0.5	3.7551×10^{-7}	2.7330×10^{-7}	1.4055×10^{-7}	7.9521×10^{-9}
Error for $V(x,t)$	0.1	6.7913×10^{-10}	1.2275×10^{-9}	1.5885×10^{-9}	1.7318×10^{-9}
	0.2	1.1161×10^{-8}	1.9859×10^{-8}	2.5532×10^{-8}	2.7722×10^{-8}
	0.3	5.7981×10^{-8}	1.0161×10^{-7}	1.2982×10^{-7}	1.4038×10^{-7}
	0.4	1.8787×10^{-7}	3.2451×10^{-7}	4.1200×10^{-7}	4.4375×10^{-7}
	0.5	4.6983×10^{-7}	8.0023×10^{-7}	1.0098×10^{-6}	1.0833×10^{-6}
Error for $W(x,t)$	0.1	3.3731×10^{-10}	6.0970×10^{-10}	7.8899×10^{-10}	8.6017×10^{-10}
	0.2	5.5436×10^{-9}	9.8638×10^{-9}	1.2681×10^{-8}	1.3769×10^{-8}
	0.3	2.8798×10^{-8}	5.0473×10^{-8}	6.4482×10^{-8}	6.9728×10^{-8}
	0.4	9.3314×10^{-8}	1.6118×10^{-7}	2.0463×10^{-7}	2.2040×10^{-7}
	0.5	2.3336×10^{-7}	3.9746×10^{-7}	5.0155×10^{-7}	5.3808×10^{-7}

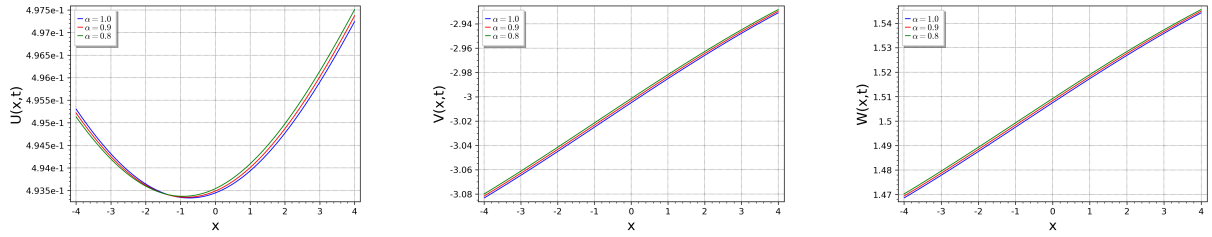
The comparison of the approximate solutions with the exact solutions are represented in Figure 1, which shows that the approximate solution is enormously close to the exact solution. In Figure 2, we examine the behavior of the solution for different values of fractional order α and observed that the amplitude of the wave decreases with time-fractional order α.

In Figure 3, we plot the profiles of the approximate solution $U(x,y,t)$ and $V(x,y,t)$ given above for the parameters $\alpha=0.95, 0\le t\le 4, -40\le x\le 40$, and observe that the obtained solution is a solitary as well as kink-type traveling wave solution.



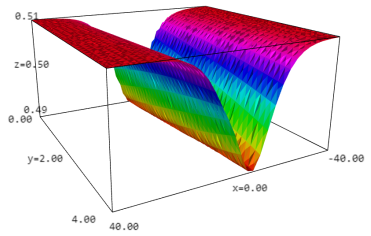
(A) (B) (C)

FIGURE 4. Comparison of the exact solution with the approximate solution at $t = 0.5$

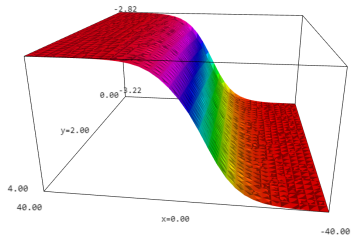


(A) (B) (C)

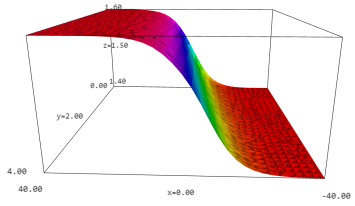
FIGURE 5. Behavior approximate solution for different values of α at $t = 0.5$



(A) Pulse type traveling wave solution of U



(B) Kink-type traveling wave solution of V



(C) Kink-type traveling wave solution of W

FIGURE 6.

5. CONCLUSION

We successfully employed the fractional homotopy perturbation method to estimate the solution of the system of fractional partial differential equations. We assessed the fractional ordered reaction-diffusion Brusselator system, which is extensively used in the modeling of chemical reaction-diffusion processes, and the time-fractional generalized coupled Hirota Satsuma KdV system, which describes the intersection of two long waves with different dispersion relations. We obtain the solutions for different values of time-fractional order and observe that the approximate solutions obtained by the proposed method are enormously close to the exact solutions of the proposed problems. The graphical representation shows that the obtained results converge to the integer order results. Also, we observe the different types of traveling wave solutions, viz., solitary, kink, wave fronts, etc., for the proposed models.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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