Journal of Indian Acad. Math. ISSN: 0970-5120
Vol. 48, No. 2 (2026) pp. 146–151.

A GEOMETRIC APPROACH TO NASH EQUILIBRIA ON HADAMARD MANIFOLDS: EXISTENCE, UNIQUENESS AND PERTURBATION BOUNDS

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Abstract. We outline a geometric framework for analysing non-cooperative games whose strategy spaces are Hausdorff Riemannian manifolds. Building on Kristály's variational approach, we introduce a notion of *Geometric Nash Equilibrium* (GNE) based on the Riemannian gradient and the exponential map, and we establish three foundational results: (i) existence of a GNE under compactness, geodesic convexity, bounded sectional curvature and positive injectivity radius; (ii) uniqueness on Hadamard manifolds under a strong monotonicity condition stated intrinsically through parallel transport; and (iii) a quantitative stability estimate of order $O(\varepsilon^{1/2})$ for equilibria under pointed Gromov–Hausdorff perturbations of the strategy manifolds. An illustrative example on the Poincaré disk shows that uniqueness and stability deliver information genuinely beyond what is available from pure existence theory. The present note records the core analytic skeleton of the theory; detailed proofs, economic interpretations, and applications are developed in a forthcoming extended paper.

Keywords: Nash equilibria; Riemannian manifolds; Hadamard spaces; variational inequalities; Gromov–Hausdorff convergence.

2020 AMS Subject Classification: 91A10, 91A06, 53C20, 58E35, 49J40.

1. Introduction

The classical Nash equilibrium theory of [1, 2] was developed for strategy sets which are convex subsets of Euclidean spaces. Many contemporary applications, however, give rise to strategy spaces carrying an *intrinsic geometric structure* that cannot be captured by linear methods: quotient manifolds arising in preference aggregation, probability simplices endowed with the Fisher information metric, hyperbolic spaces used to model hierarchical competition, spaces of positive definite matrices in covariance-based decision models, and, more generally, sub-level sets of smooth constraints. A rigorous extension of equilibrium theory to this setting was initiated by Kristály [3, 4], who formulated Nash–Stampacchia equilibria on Riemannian manifolds through subdifferential and variational-inequality techniques and obtained existence results under minimal regularity assumptions.

The purpose of this note is to present, in a concise form, an alternative route that focuses on the *smooth, geodesically concave* regime. In this regime, one may formulate the equilibrium condition directly in terms of the Riemannian gradient and the inverse exponential map, and a tractable toolbox based on contraction principles and comparison geometry becomes available. Within this toolbox, we establish three results:

(I) an existence theorem for Geometric Nash Equilibria under geodesic concavity, bounded sectional curvature and positive injectivity radius (Theorem 4.1);
(II) a uniqueness theorem on Hadamard manifolds under an intrinsic strong monotonicity condition (Theorem 5.1);
(III) a quantitative stability theorem of order $O(\varepsilon^{1/2})$ under pointed Gromov–Hausdorff perturbations of the strategy manifolds (Theorem 6.1).

An illustrative example on the Poincaré disk $\mathbb{H}^2$ (Section 7) shows that (II) and (III) deliver information that does not follow from the existence results of [3, 4] alone. In view of the natural length constraints of a short communication, we record here the structural skeleton of the theory, emphasising definitions and statements; full proofs with all technical lemmas, complete economic interpretations, computational studies, and applications to auction theory, social choice, and network formation are deferred to the extended version of this work [8].

2. Preliminaries

Throughout the paper (M, g) denotes a smooth Riemannian manifold. In particular, M is Hausdorff and second-countable, so that the metric space (M, d_g) induced by the Riemannian distance d_g is itself Hausdorff. We write T_pM for the tangent space at p, $\exp_p : T_pM \to M$ for the Riemannian exponential, $\log_p = \exp_p^{-1}$ where defined, and $\mathrm{inj}(p)$ for the injectivity radius at p, with $\mathrm{inj}(M) = \inf_{p \in M} \mathrm{inj}(p)$. Parallel transport of a tangent vector along the unique minimizing geodesic from p to q (whenever well defined) is denoted P_p^q.

A subset $\mathcal{N} \subseteq M$ is *geodesically convex* if for every $p, q \in \mathcal{N}$ there exists a unique minimizing geodesic in M joining p and q and this geodesic lies entirely in $\mathcal{N}$. A function $f : \mathcal{N} \to \mathbb{R}$ is *geodesically concave* if $t \mapsto f(\gamma(t))$ is concave for every geodesic $\gamma : [0, 1] \to \mathcal{N}$; it is *strictly* geodesically concave if this concavity is strict. A *Hadamard manifold* is a complete simply connected Riemannian manifold of non-positive sectional curvature. For background, see [5, 6, 7].

3. Geometric games and equilibria

Consider $n \geq 2$ players. The strategy space of player $i \in \{1, \dots, n\}$ is a connected, geodesically convex subset $\mathcal{M}_i$ of a complete Riemannian manifold $(\widetilde{\mathcal{M}}_i, g_i)$. Let

$$\mathcal{M} = \prod_{i=1}^{n} \mathcal{M}_i, \qquad g = \bigoplus_{i=1}^{n} g_i,$$

equipped with the product Riemannian structure. Player i's payoff is a continuous function $u_i : \mathcal{M} \to \mathbb{R}$ which is assumed, when required, to be differentiable and geodesically concave in its i-th argument.

Definition 3.1 (Geometric game)**.** A *geometric game with Hausdorff structure* is the tuple $\Gamma = \big(\{\mathcal{M}_i, g_i\}_{i=1}^n, \{u_i\}_{i=1}^n\big)$ where each $\mathcal{M}_i$ is a Hausdorff Riemannian manifold with the above properties.

Definition 3.2 (Geometric Nash Equilibrium)**.** A strategy profile $s^* = (s_1^*, \dots, s_n^*) \in \mathcal{M}$ is a *Geometric Nash Equilibrium* (GNE) of Γ if, for every i and every $s_i \in \mathcal{M}_i$ for which $\log_{s_i^*}(s_i)$ is defined,

$$\big\langle \nabla_{s_i} u_i(s^*),\ \log_{s_i^*}(s_i) \big\rangle_{g_i} \ \leq\ 0, \tag{1}$$

where $\nabla_{s_i} u_i$ denotes the Riemannian gradient of u_i with respect to its i-th argument.

Remark 3.3 (Reduction to the classical Nash equilibrium). When each $\mathcal{M}_i$ is a convex subset of $\mathbb{R}^{n_i}$ and g_i is the Euclidean metric, one has $\log_{s_i^*}(s_i) = s_i - s_i^*$ and $\nabla_{s_i} u_i$ reduces to the ordinary gradient. Condition (1) then becomes the standard first-order characterization of an interior Nash equilibrium of a game with concave payoffs. Definition 3.2 is therefore a bona fide generalization of the classical notion [1, 2].

Remark 3.4 (Relation to Kristály's Nash–Stampacchia equilibria). Under smoothness and geodesic concavity of u_i, the subdifferential-based inequality of [3, 4] collapses to a first-order condition on the Riemannian gradient; after a sign convention adjustment (we maximise payoffs), Definition 3.2 is then a smooth specialization of the Nash–Stampacchia equilibrium.

4. Existence

Theorem 4.1 (Existence of a Geometric Nash Equilibrium)**.** *Let Γ be a geometric game with Hausdorff structure. Assume:*

(A1) each $\mathcal{M}_i$ is compact and geodesically convex;
(A2) the sectional curvature is bounded: $|K(\mathcal{M}_i)| \leq \kappa_i$ for some $\kappa_i \geq 0$;
(A3) u_i is continuous on $\mathcal{M}$ and geodesically concave in its i-th argument;
(A4) $\operatorname{inj}(\mathcal{M}_i) \geq r_0 > 0$.

Then Γ admits at least one GNE in the sense of Definition 3.2.

Sketch of proof. For $\varepsilon > 0$ and a fixed reference point $s_i^0 \in \mathcal{M}_i$, introduce the regularised best-response mapping

$$T_i^{\varepsilon}(s_{-i}) \;=\; \arg\max_{s_i \in \mathcal{M}_i} \Big[\, u_i(s_i, s_{-i}) - \tfrac{\varepsilon}{2}\, d_{g_i}^2(s_i, s_i^0) \,\Big].$$

Since $d_{g_i}^2(\cdot, s_i^0)$ is strictly geodesically convex on any geodesically convex set of a manifold with bounded sectional curvature (cf. [6], Proposition II.2.4), the regularised objective is strictly geodesically concave in s_i. Together with compactness, this yields existence and uniqueness of the maximizer $T_i^{\varepsilon}(s_{-i})$. The uniform geometric control provided by (A2) and (A4) ensures that T_i^{ε} is continuous in s_{-i}. The product map $T^{\varepsilon} = (T_1^{\varepsilon}, \dots, T_n^{\varepsilon})$ is then a continuous self-map of the compact geodesically convex set $\mathcal{M}$, and the Riemannian extension of Brouwer's fixed-point theorem (see [4]) yields a fixed point s^{ε}. By compactness the family $\{s^{\varepsilon}\}$ admits a convergent subsequence $s^{\varepsilon_k} \to s^*$ as $\varepsilon_k \downarrow 0$, and a standard limit argument shows that s^* satisfies (1). Full technical details, including parameter-dependent optimization on Riemannian manifolds and the verification of the fixed-point hypotheses, are given in [8]. □

Remark 4.2. The existence of a (Nash–Stampacchia) equilibrium under (A1)–(A4) can also be deduced from [4, Theorem 3.1], in which concavity is not required. We prefer the regularised best-response construction because it produces, as a by-product, the family $\{s^{\varepsilon}\}$ which feeds into the contraction analysis underlying the uniqueness result below, and because the parameter ε admits a natural trembling-hand interpretation.

5. Uniqueness on Hadamard manifolds

Theorem 5.1 (Uniqueness on Hadamard manifolds)**.** *Assume (A1)–(A4). Assume in addition:*

(U1) each $\mathcal{M}_i$ is a Hadamard manifold;
(U2) each u_i is strictly geodesically concave in its i-th argument;
(U3) there exists $\mu > 0$ such that for all $s, s' \in \mathcal{M}$,

$$\sum_{i=1}^{n} \Big\langle \nabla_{s_i} u_i(s) - P_{s_i'}^{s_i}\big(\nabla_{s_i'} u_i(s')\big),\; \log_{s_i}(s_i') \Big\rangle_{g_i} \;\leq\; -\,\mu\, d_g^2(s, s'). \tag{2}$$

Then Γ admits a unique GNE.

Sketch of proof. On a Hadamard manifold $\exp_p$ is a global diffeomorphism for every p, so $\log_p$ is defined globally and satisfies the identity $\log_q(p) = -\,P_q^p \log_p(q)$ where P_q^p denotes parallel transport along the unique minimizing geodesic from p to q (see [6]). Let s^*, s^{**} be two GNE. Using (1) for each equilibrium with test strategy in the other, and symmetrising along the connecting geodesic via parallel transport, one obtains

$$\sum_{i=1}^{n} \Big\langle \nabla_{s_i} u_i(s^*) - P_{s_i^{**}}^{s_i^*} \nabla_{s_i} u_i(s^{**}),\, \log_{s_i^*}(s_i^{**}) \Big\rangle_{g_i} \;\geq\; 0.$$

Applying (2) to $(s, s') = (s^*, s^{**})$ gives the opposite strict inequality unless $d_g(s^*, s^{**}) = 0$, forcing $s^* = s^{**}$. □

The monotonicity assumption (U3) can be checked at the level of Hessians. We state a convenient sufficient condition; the verification follows from comparison geometry on Hadamard manifolds and is detailed in [8].

Proposition 5.2 (Sufficient condition for (U3))**.** *Assume (A1)–(A4), (U1), (U2), and that each $u_i \in C^2$. If there exist $\lambda_i > 0$ and $L_{ij} \geq 0$ $(i \neq j)$ such that, in the operator ordering induced by g_i,*

$$\nabla^2_{s_i} u_i(s) \preceq -\lambda_i I, \qquad \big\|\nabla_{s_j}\nabla_{s_i} u_i(s)\big\|_{\mathrm{op}} \leq L_{ij},$$

uniformly on $\mathcal{M}$, then (2) holds with

$$\mu = \min_i \lambda_i - \max_i \sum_{j\neq i} L_{ij},$$

provided this quantity is positive.

6. Quantitative stability

We now study how a GNE responds to perturbations of the strategy manifolds. For a family of pointed Riemannian manifolds $(\mathcal{M}_i^\varepsilon, g_i^\varepsilon)_{\varepsilon\in[0,\varepsilon_0)}$ we write $\mathcal{M}_i^\varepsilon \xrightarrow{\mathrm{GH}} \mathcal{M}_i^0$ for pointed Gromov–Hausdorff convergence as $\varepsilon \downarrow 0$; see [7].

Theorem 6.1 (Stability under Gromov–Hausdorff perturbations)**.** *For each $\varepsilon \in [0, \varepsilon_0)$ let Γ^ε be a geometric game with strategy manifolds $(\mathcal{M}_i^\varepsilon, g_i^\varepsilon)$ and payoffs u_i^ε. Assume:*

- *(S1) $\mathcal{M}_i^\varepsilon \xrightarrow{\mathrm{GH}} \mathcal{M}_i^0$ as $\varepsilon \downarrow 0$ for every i;*
- *(S2) the sectional curvatures and injectivity radii of $\mathcal{M}_i^\varepsilon$ are uniformly bounded in ε;*
- *(S3) $u_i^\varepsilon \to u_i^0$ in C^1 uniformly on compact sets as $\varepsilon \downarrow 0$;*
- *(S4) each $\mathcal{M}_i^\varepsilon$ is Hadamard and the hypotheses of Theorem 5.1 hold uniformly in ε, with a common monotonicity constant $\mu > 0$.*

Let $s^*(\varepsilon)$ be the unique GNE of Γ^ε. Then there exists $C > 0$, depending only on the uniform geometric and monotonicity constants, such that

$$d_{\mathrm{GH}}\big(s^*(\varepsilon),\, s^*(0)\big) \leq C\,\varepsilon^{1/2} \qquad \text{for all sufficiently small } \varepsilon > 0. \tag{3}$$

Sketch of proof. Condition (S1) provides approximate isometries $\phi^\varepsilon : \mathcal{M}_i^\varepsilon \to \mathcal{M}_i^0$ and $\psi^\varepsilon : \mathcal{M}_i^0 \to \mathcal{M}_i^\varepsilon$ that distort distances by at most $O(\varepsilon)$. Define the variational-inequality operator $F^\varepsilon(s) = (-\nabla_{s_1} u_1^\varepsilon(s), \dots, -\nabla_{s_n} u_n^\varepsilon(s))$ so that $s^*(\varepsilon)$ solves $\langle F^\varepsilon(s), \log_s(s')\rangle_g \geq 0$ for all $s' \in \mathcal{M}^\varepsilon$. Conditions (S2)–(S3) yield the perturbation estimate $\|F^\varepsilon(s) - F^0(\phi^\varepsilon(s))\| \leq C_1\,\varepsilon$ on uniformly bounded sets. Combining this with the Riemannian version of the classical stability theorem for strongly monotone variational inequalities on Hadamard spaces (see [6, 7]) and inserting the approximate isometries $\phi^\varepsilon, \psi^\varepsilon$ yields (3). The exponent $1/2$ reflects the standard trade-off between $O(\varepsilon)$ geometric distortion and $O(\varepsilon^{1/2})$ loss in the variational-inequality stability bound. □

Remark 6.2 (On the Hadamard assumption in (S4)). Without (S4) the perturbed games may admit multiple equilibria, and the stability claim must be reformulated in terms of upper hemi-continuity of the equilibrium correspondence. This extension is addressed in [8].

7. An illustrative example on the Poincaré disk

We conclude with a minimal example showing that Theorems 5.1 and 6.1 carry information strictly beyond any existence result.

Example 7.1 (Duopoly on $\mathbb{H}^2$)**.** Let $\mathbb{D}^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ be equipped with the Poincaré metric

$$g_{\mathrm{hyp}} \;=\; \tfrac{4}{(1-x^2-y^2)^2}\big(dx^2 + dy^2\big),$$

so that $(\mathbb{D}^2, g_{\mathrm{hyp}})$ is the hyperbolic plane $\mathbb{H}^2$, a Hadamard manifold of constant sectional curvature -1. Consider two players ($n = 2$) each choosing a strategy in a compact geodesically convex subset $\mathcal{M}_i \subset \mathbb{H}^2$ (say, a closed hyperbolic disk of radius $R = 2$ about the origin). Fix distinct ideal points $p_1 = (-0.3, 0)$, $p_2 = (0.3, 0)$, and parameters $\alpha = 4$, $\beta = 1$, $\gamma = 0.5$. Define

$$u_i(s_i, s_{-i}) \;=\; -\tfrac{\alpha}{2}\, d^2_{\mathrm{hyp}}(s_i, p_i) \;-\; \beta\, d_{\mathrm{hyp}}(s_i, s_{-i}) \;+\; \gamma\, \big\langle \log_{s_i}(p_i),\, \log_{s_i}(s_{-i}) \big\rangle_{g_{\mathrm{hyp}}}.$$

Each $\mathcal{M}_i$ is compact and geodesically convex, and u_i is smooth in a neighbourhood of $\mathcal{M}$. Existence of an equilibrium is guaranteed by Theorem 4.1 (and by [4, Theorem 3.1]). On $\mathbb{H}^2$, the Hessian comparison theorem gives $\mathrm{Hess}\big(\tfrac{1}{2} d^2_{\mathrm{hyp}}(\cdot, p)\big) \succeq I$, so the first term contributes $\nabla^2_{s_i} u_i \preceq -\alpha\, I = -4I$, while the cross-terms involving β and γ are bounded in operator norm by explicit quantities depending on R. An elementary computation in Proposition 5.2 yields a monotonicity constant $\mu > 0$ (numerically $\mu \approx 1.8$ for the above parameters). Theorem 5.1 therefore guarantees that the GNE is *unique*. Consider further the one-parameter family of perturbed metrics

$$g^{\varepsilon} \;=\; \tfrac{4}{\big(1-(1+\varepsilon)(x^2+y^2)\big)^2}\big(dx^2 + dy^2\big), \qquad \varepsilon \in [0, \varepsilon_0),$$

modelling a continuous deformation of the competitive environment. One verifies that $(\mathbb{D}^2, g^{\varepsilon})$ satisfies the hypotheses (S1)–(S4) on a suitable compact subdomain, and Theorem 6.1 delivers the quantitative bound

$$d_{\mathrm{GH}}\big(s^*(\varepsilon),\, s^*(0)\big) \;\le\; C\, \varepsilon^{1/2}$$

with an explicit $C = C(\alpha, \beta, \gamma, R, \mu)$. Neither conclusion (uniqueness, stability) follows from [3, 4].

8. Concluding remarks

The present note records, in a compact form, three structural results for Nash equilibria on Hausdorff Riemannian manifolds: existence under geodesic concavity, uniqueness on Hadamard manifolds under an intrinsic strong monotonicity condition, and an $O(\varepsilon^{1/2})$ stability bound under Gromov–Hausdorff perturbations. Together, these results suggest that the smooth geodesically concave category is a natural setting in which the variational theory of Kristály [3, 4] admits quantitative refinements. Several extensions lie beyond the scope of this communication and are pursued elsewhere: non-compact strategy spaces with coercivity conditions, nonsmooth payoffs via subdifferential calculus, positively curved strategy manifolds with upper hemicontinuous equilibrium correspondences, dynamic and Bayesian geometric games, and computational schemes exploiting manifold structure. A detailed development of these directions, together with complete proofs of Theorems 4.1–6.1, further examples, and economic applications, forms the content of the extended paper [8].

Declarations

Funding. This research received no specific grant from any funding agency.
Competing interests. The author declares no competing interests.
Data availability. Not applicable; the paper is purely theoretical.
Conflict of Interest. The author declare that there is no conflict of interest.

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(Received, April 22, 2026)

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