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RELATION-THEORETIC NONUNIQUE FIXED POINT THEOREMS

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Abstract. In this paper, we establish a couple of nonunique fixed point results for the orbital relationally continuous self maps of a orbital relationally complete metric space which include some interesting fixed point results as the special cases. Our abstract results are also illustrated with the help of a numerical example.

Keywords: Relational complete metric space; Relational contraction mapping; Nonunique fixed point theorem.

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1. Introduction

Ciric [3] proved some nonunique fixed point results for a certain class of contraction mappings on a metric space into itself under weaker version of continuity and completeness of a metric space. Thereafter, several authors extended and generalized the above results to other classes of contraction mappings in a metric space which is not necessarily complete. See for example, Achari [1], Dhage [4, 5, 6], Pachpatte [12] and references therein. Similarly, Alam and Imdad [2] proved a basic relation theoretic fixed point theorem for relational contraction mappings in relational complete metric spaces. Later, this result is generalized to a wider classes of mappings by Dubey *et al.* [11] and Samet and Turinici [13] with fixed point. All the above fixed point results are about the existence of a unique fixed point and to the best of our knowledge there is no result for nonunique fixed point. In this paper we blend the above two approaches and prove a couple of relation theoretic nonunique fixed point theorems for the selfmappings of a metric space under weaker suitable conditions. Before going to the main results we give some preliminary definitions.

Let (X, d) be a metric space and let $\mathcal{T} : X \to X$ be a mapping. Then $\mathcal{T}$-orbit of $\mathcal{T}$ at a point $x \in X$ is a set $O(x, \mathcal{T})$ in X defined by

$$O(x, \mathcal{T}) = \{x, \mathcal{T}x, \mathcal{T}^2x, \cdots, \mathcal{T}^nx, \cdots\} = \{\mathcal{T}^nx\}_{n=0}^{\infty}. \tag{1}$$

The mapping $\mathcal{T}$ is called $\mathcal{T}$-orbitally continuous if and only if a sequence $\{x_n\} \subseteq O(x, \mathcal{T})$ converges to x implies $\mathcal{T}x_n \to \mathcal{T}x$. X is called $\mathcal{T}$-orbitally complete if every Cauchy sequence $\{x_n\}$ in $O(x, \mathcal{T})$ converges to a point in X. It is known that every continuous mapping is $\mathcal{T}$-orbitally continuous and every complete metric space X is $\mathcal{T}$-orbitally complete, but the reverse implication need not hold. See Dhage *et al.* [10] and references therein.

A binary relation or simply relation $\mathbf{R}$ in X is a subset of the Cartesian product $X \times X$ which is full if $\mathbf{R} = X \times X$ and which is empty if $\mathbf{R} = \emptyset$. $\mathbf{R}$ is called partial relation or proper relation if $\emptyset \neq \mathbf{R} \neq X \times X$. We introduce a partial relation $\mathbf{R}$ in X and two elements x and y in X are said to be relatable if they are related to each other under the relation $\mathbf{R}$. In this case we write $x\,\mathbf{R}\,y$ or $(x, y) \in \mathbf{R}$. A metric space X together wth a partial relation $\mathbf{R}$ is called a relational metric space and it is denoted by $(X, d, \mathbf{R})$. A sequence $\{x_n\}$ of points in X is called **relatonic** or **term-wise related** if $(x_n, x_{n+1}) \in \mathbf{R}$ and **inverse relatonic** if $(x_{n+1}, x_n) \in \mathbf{R}$ for each $n \in \mathbb{N}$. We say the metric space X is $\mathbf{R}$-regular or PR-closed or simply closed with respect to the relation $\mathbf{R}$, if $\{x_n\}$ is a relatonic (resp. inverse relatonic) sequence of points in X converging to a point x^*, then $(x_n, x^*) \in \mathbf{R}$ (resp. $(x^*, x_n) \in \mathbf{R}$) for all $n \in \mathbb{N}$. A relatonic sequence $\{x_n\}$ in $(X, d, \mathbf{R})$ is called $\mathbf{R}$-Cauchy if $\lim_{m,n\to\infty} d(x_m, x_n) = 0$. Again, $(X, d, \mathbf{R})$ is called $\mathbf{R}$-complete if every $\mathbf{R}$-Cauchy sequence converges to a point in X. A mapping $\mathcal{T} : X \to X$ is called is called relational $\mathbf{R}$-continuous at a point $x \in \mathbb{R}^n$ if and only if $x_n \to x$ implies $\mathcal{T}x_n \to \mathcal{T}x$ for every sequence $\{x_n\}$ of points relatable to x in $\mathbb{R}^n$. A few details of above notions appear in Anam and Imdad [2] and Dhage [8].

2. Main Results

Before going to the main nonunique fixed point theorems we state a couple of mathematical concepts whih we need in what follows.

Definition 2.1. *A mapping $\mathcal{T} : X \to X$ is called is called $\mathcal{T}$-orbital $\mathbf{R}$-continuous at a point $x \in X$ if and only if it is $\mathbf{R}$-continuous on $O(x, \mathcal{T})$ for each $x \in X$. Again, $(X, d, \mathbf{R})$ is called $\mathcal{T}$-orbital $\mathbf{R}$-complete if $O(x, \mathcal{T})$ is $\mathbf{R}$-complete for each $x \in X$.*

Remark 2.2. *It is clear that*

Completeness of $X \Rightarrow$ $\mathcal{T}$-orbital completeness of $X \Rightarrow$ $\mathcal{T}$-orbital $\mathbf{R}$-completeness of X

and

Continuity of $\mathcal{T} \Rightarrow$ $\mathcal{T}$-orbital continuity of $\mathcal{T} \Rightarrow$ $\mathcal{T}$-orbital $\mathbf{R}$-continuity of $\mathcal{T}$,

however the reverse implications may not hold.

Definition 2.3. *A mapping $\mathcal{T} : X \to X$ is called a* **relational R-relatone** *if it preserves the binary relation $\mathbf{R}$ in X, that is, if $x, y \in X$ with $(x, y) \in \mathbf{R}$, then $(\mathcal{T}x, \mathcal{T}y) \in \mathbf{R}$.*

We consider the following notations in what follows. We denote

(i) $F_{\mathcal{T}}(X) = \{x \in X : \mathcal{T}x = x\}$,
(ii) $L_{\mathcal{T}}(X) = \{x \in X : (x, \mathcal{T}x) \in \mathbf{R}\}$, and
(iii) $U_{\mathcal{T}}(X) = \{x \in X : (\mathcal{T}x, x) \in \mathbf{R}\}$.

Theorem 2.4. *Let $\mathcal{T}$ be a $\mathcal{T}$-orbital $\mathbf{R}$-continuous and $\mathbf{R}$-relatone self-mapping of a $\mathbf{R}$-regular and $\mathcal{T}$-orbital $\mathbf{R}$-complete metric space X. Suppose that there exists a real*

number b such that

$$0 \leq \min\{d(\mathcal{T}x, \mathcal{T}y), d(x, \mathcal{T}x), d(x, \mathcal{T}x)\} + b\min\{d(x, \mathcal{T}y), d(y, \mathcal{T}x)\} \leq q\, d(x, y) \tag{2}$$

for all relatable elements $x, y \in X$, where $0 \leq q < 1$. Furthermore, if $L_{\mathcal{T}}(X) \neq \emptyset$ or $U_{\mathcal{T}}(X) \neq \emptyset$, then $F_{\mathcal{T}}(X) \neq \emptyset$. Moreover, $\mathcal{T}$ has a unique relatable fixed point provided $b > q$.

Proof. Suppose first that $L_{\mathcal{T}}(S) \neq \emptyset$, then there exists an element $x_0 \in S$ such that $(x_0, \mathcal{T}x_0) \in \mathbf{R}$. Define a sequence $\{x_n\}_{n=0}^{\infty} = \{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of iterations of $\mathcal{T}$ at $x_0 \in X$. Since $\mathcal{T}$ is relatone, we have $(x_n, x_{n+1}) \in \mathbf{R}$ for each n, $n = 0, 1, \ldots$. If $x_r = x_{r+1}$ for some $r \in \mathbb{N}$, then the conclusion of the theorem follows immediately. Therefore, we assume that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}$. We show that $\{x_n\}$ is a R-Cauchy sequence. Let $x = x_{n-1}$ and $y = x_n$ in the inequality (2), we obtain

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq qd(x_{n-1}, x_n) \tag{3}$$

for all $n \in \mathbb{N}$. By repeated application of the above inequality yields

$$d(x_n, x_{n+1}) \leq q^n d(x_0, x_1) \tag{4}$$

for each $n \in \mathbb{N}$. Next for any positive integer p, by triangle inequality, we set

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + ... + d(x_{n+p-1}, x_{n+p}) \\ &\leq (q^n + ... + q^{n+p-1}) d(x_0, x_1) \\ &= q^n \left(\frac{1 - q^{p-1}}{1 - q}\right) d(x_0, x_1) \\ &\leq \left(\frac{q^n}{1 - q}\right) d(x_0, x_1) \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{5}$$

This shows that $\{x_n\}$ is a $\mathbf{R}$-Cauchy sequence in $O(x, \mathcal{T})$. Since X is $\mathcal{T}$-orbital $\mathbf{R}$-complete, $\{x_n\}$ converges to a point u in X. From $\mathcal{T}$-orbital $\mathbf{R}$-continuity of $\mathcal{T}$, it follows that

$$u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \mathcal{T}x_n = \mathcal{T}\left(\lim_{n\to\infty} x_n\right) = \mathcal{T}u.$$

Thus u is a fixed point of $\mathcal{T}$. Smilarly, if $U_{\mathcal{T}}(X) \neq \emptyset$, then following similar arguments, it can be shown that $\mathcal{T}$ has a fixed point. Further assume that $b > q$ and let $v(\neq u)$ be another relatable fixed point of $\mathcal{T}$. Then form condition (2) we get a contradiction whence $\mathcal{T}$ has a unique relatable fixed point. This completes the proof. □

On taking $b = 0$ in contraction condition (2) we obtain the following fixed point result.

Corollary 2.5. *Let $\mathcal{T}$ be a $\mathcal{T}$-orbital $\mathbf{R}$-continuous and $\mathbf{R}$-relatone self-mapping of a $\mathbf{R}$-regular and $\mathcal{T}$-orbital $\mathbf{R}$-complete metric space X. Suppose that $\mathcal{T}$ satisfies the contraction*

condition

$$\min\{d(\mathcal{T}x, \mathcal{T}y), d(x, \mathcal{T}x), d(x, \mathcal{T}x)\} \leq q\, d(x, y) \tag{6}$$

for all relatable elements $x, y \in X$*, where* $0 \leq q < 1$*. Furthermore, if* $L_{\mathcal{T}}(X) \neq \emptyset$ *or* $U_{\mathcal{T}}(X) \neq \emptyset$*, then* $\mathcal{T}$ *has a fixed point.*

Again taking $b = -1$ in contraction condition (2), we obtain the following fixed point result Ciric [3] type mappings in metric spaces.

Corollary 2.6. *Let* $\mathcal{T}$ *be a* $\mathcal{T}$*-orbital* $\mathbf{R}$*-continuous and* $\mathbf{R}$*-relatone self-mapping of a* $\mathbf{R}$*-regular and* $\mathcal{T}$*-orbital* $\mathbf{R}$*-complete metric space* X*. Suppose that* $\mathcal{T}$ *satisfies the contraction condition*

$$\begin{aligned} 0 \leq \min\{d(\mathcal{T}x, \mathcal{T}y), d(x, \mathcal{T}x), d(x, \mathcal{T}x)\} \\ - \min\{d(x, \mathcal{T}y), d(y, \mathcal{T}x)\} \leq q\, d(x, y) \end{aligned} \tag{7}$$

for all relatable elements $x, y \in X$*, where* $0 \leq q < 1$*. Furthermore, if* $L_{\mathcal{T}}(X) \neq \emptyset$ *or* $U_{\mathcal{T}}(X) \neq \emptyset$*, then* $\mathcal{T}$ *has a fixed point.*

Remark 2.7. *The Ciric's sufficient contraction condition for the existence of nonunique fixed point is*

$$\begin{aligned} \min\{d(\mathcal{T}x, \mathcal{T}y), d(x, \mathcal{T}x), d(x, \mathcal{T}x)\} \\ - \min\{d(x, \mathcal{T}y), d(y, \mathcal{T}x)\} \leq q\, d(x, y) \end{aligned}$$

for all $x, y \in X$*, where* $0 \leq q < 1$*. Note that the term* $\min\{d(x, \mathcal{T}y), d(y, \mathcal{T}x)\}$ *is not always equal to* 0 *which otherwise is assumed in the proof of the fixed point theorem. If we assume it never be zero, then the mapping* $\mathcal{T} : \mathbb{R}_+ \to \mathbb{R}_+$ *defined by* $\mathcal{T}x = x + 1$ *satisfies the Ciric's contraction condition with no fixed point in* $\mathbb{R}_+$*. Therefore, we have slightly modified it into the form (7) to have greater applicability to nonlinear equations.*

We introduce a partial order $\preceq$ in X which is a reflexive, anti-symmetric and transitive relation, so that non-empty set X now becomes a **partially ordered metric space** and it is denoted by $(X, d, \preceq)$. A mapping $\mathcal{T}$ is called **partially continuous** at a point $x \in X$ if and only if $x_n \to x$ implies $\mathcal{T}x_n \to \mathcal{T}x$ for every sequence $\{x_n\}$ of points comparable to x in X. $\mathcal{T}$ is called partial continuous on X if it is partial continuous at each point $x \in X$. A **partially ordered complete metric space** X is one in which every monotonic Cauchy sequence converges to a point in X. The details of these partial notions appear in Dhage [7] and Dhage *et al.* [8] and references therein. We consider the following definitions in what follows.

Definition 2.8. *A mapping* $\mathcal{T} : X \to X$ *is called is called* $\mathcal{T}$*-orbital partially continuous at a point* $x \in X$ *if and only if it is partial continuous on* $O(x, \mathcal{T})$ *for each* $x \in X$*. Again,* $(X, d, \preceq)$ *is called* $\mathcal{T}$*-orbital partially complete if* $O(x, \mathcal{T})$ *is partial complete for each* $x \in X$*.*

Definition 2.9. *A mapping* $\mathcal{T} : X \to X$ *is called a* **monotone nondecreasing** *if it preserves the partial order* $\preceq$ *in* X*, that is, if* $x, y \in X$ *with* $x \preceq y$*, then* $(\mathcal{T}x \preceq \mathcal{T}y$*.*

Now as a consequence of Theorem 2.4 we obtain the following corollary which is also new to the literature.

Corollary 2.10. *Let $\mathcal{T}$ be a $\mathcal{T}$-orbital partial continuous and monotone nondecreasing self-mapping of a regular and $\mathcal{T}$-orbital partial complete metric space X. Suppose that there exists a real number b such that*

$$\begin{aligned} 0 \leq \min\{d(\mathcal{T}x, \mathcal{T}y), d(x, \mathcal{T}x), d(x, \mathcal{T}x)\} \\ + b \min\{d(x, \mathcal{T}y), d(y, \mathcal{T}x)\} \leq q\ d(x, y) \end{aligned} \tag{8}$$

for all comparable elements $x, y \in X$, where $0 \leq q < 1$. Furthermore, if $L_{\mathcal{T}}(X) \neq \emptyset$ or $U_{\mathcal{T}}(X) \neq \emptyset$, then $F_{\mathcal{T}}(X) \neq \emptyset$. Moreover, $\mathcal{T}$ has a unique comparable fixed point provided $b > q$.

Remark 2.11. *We note that Corollary 2.10 includes two more nonunique fixed point results for the mapping satisfying the contraction conditions (6) and (7) in the partially ordered metric spaces.*

Next, we generalize Corollary 2.5 by the improvement of the contraction condition (6).

Theorem 2.12. *Let $\mathcal{T}$ be a $\mathcal{T}$-orbital $\mathbf{R}$-continuous and $\mathbf{R}$-relatone self-mapping of a $\mathbf{R}$-regular and $\mathcal{T}$-orbital $\mathbf{R}$-complete metric space X. Suppose that $\mathcal{T}$ satisfies the contraction condition*

$$\begin{aligned} \min\big\{d(\mathcal{T}x, \mathcal{T}y), & d(x, \mathcal{T}x), d(x, \mathcal{T}x)\big\} \\ & \leq q\ \max\big\{d(x, y) \bigcup \big[\min\{d(x, \mathcal{T}y), d(y, \mathcal{T}x)\}\big]\big\} \end{aligned} \tag{9}$$

for all relatable elements $x, y \in X$, where $0 \leq q < 1$. Furthermore, if $L_{\mathcal{T}}(X) \neq \emptyset$ or $U_{\mathcal{T}}(X) \neq \emptyset$, then $\mathcal{T}$ has a fixed point.

Proof. Suppose first that $L_{\mathcal{T}}(S) \neq \emptyset$, then there exists an element $x_0 \in S$ such that $(x_0, \mathcal{T}x_0) \in \mathbf{R}$. Define a sequence $\{x_n\}_{n=0}^{\infty} = \{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of iterations of $\mathcal{T}$ at $x_0 \in S$. Since $\mathcal{T}$ is relatone, we have $(x_n, x_{n+1}) \in \mathbf{R}$ for each n, $n = 0, 1, \ldots$. If $x_r = x_{r+1}$ for some $r \in \mathbb{N}$, then the conclusion of the theorem follows immediately. Therefore, we assume that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}$. We show that $\{x_n\}$ is a R-Cauchy sequence. Let $x = x_{n-1}$ and $y = x_n$ in the inequality (2), we obtain

$$\begin{aligned} \min\big\{d(\mathcal{T}x_{n-1}, \mathcal{T}x_n), & d(x_{n-1}, \mathcal{T}x_{n-1}), d(x_n, \mathcal{T}x_n)\big\} \\ & \leq q\ \max\big\{d(x_{n-1}, x_n) \bigcup \big[\min\{d(x_{n-1}, \mathcal{T}x_n), d(x_n, \mathcal{T}x_{n-1})\}\big]\big\}, \end{aligned}$$

$$\begin{aligned} \text{i.e.} \quad \min\big\{d(x_n, x_{n+}), & d(x_{n-1}, x_n), d(x_n, x_{n+1})\big\} \\ & \leq q\ \max\big\{d(x_{n-1}, x_n) \bigcup \big[\min\{d(x_{n-1}, x_{n+1}), d(x_n, x_n)\}\big]\big\}, \end{aligned}$$

which implies

$$\min\big\{d(x_n, x_{n+}), d(x_{n-1}, x_n)\big\} \leq q\ d(x_{n-1}, x_n).$$

Since $d(x_{n-1}, x_n)\} \leq q\ d(x_{n-1}, x_n)$ is not possible because $q < 1$, we have that

$$d(x_n, x_{n+1}) \leq q\ d(x_{n-1}, x_n)$$

for all $n \in \mathbb{N}$. The rest of the proof is similar to the proof of Theorem 2.4. Hence we omit the details. □

As a consequence of Theorem 2.12, we obtain the following result which is also new the metric fixed point theory.

Corollary 2.13. *Let $\mathcal{T}$ be a $\mathcal{T}$-orbital partial continuous and monotone nondecreasing self-mapping of a regular and $\mathcal{T}$-orbital partial complete metric space X. Suppose that $\mathcal{T}$ satisfies the contraction condition*

$$\begin{aligned} \min\big\{d(\mathcal{T}x,\mathcal{T}y),&d(x,\mathcal{T}x),d(x,\mathcal{T}x)\big\} \\ &\leq q\ \max\big\{d(x,y)\bigcup\big[\min\{d(x,\mathcal{T}y),d(y,\mathcal{T}x)\}\big]\big\} \end{aligned} \tag{10}$$

for all comparable elements $x, y \in X$, where $0 \leq q < 1$. Furthermore, if $L_{\mathcal{T}}(X) \neq \emptyset$ or $U_{\mathcal{T}}(X) \neq \emptyset$, then $\mathcal{T}$ has a fixed point.

Example 2.14. *Let $X = [0,1)$ and let $\leq$ be the partial order defined in $[0,1)$. Then the triplet $\big([0,1), |\cdot|, \leq\big)$ becomes a partially ordered metric space. Clearly, $[0,1)$ is not a complete metric space. Define a mapping $\mathcal{T} : [0,1) \to [0,1)$ by*

$$\mathcal{T}x = \begin{cases} \dfrac{x}{2} & \text{if} \quad x \in \big[0, \frac{3}{4}\big), \\ \dfrac{1}{2} & \text{if} \quad x \in \big[\frac{3}{4}, 1\big). \end{cases}$$

Note that $\mathcal{T}$ is not continuous on $[0,1)$. However, $\mathcal{T}$ is $\mathcal{T}$-orbital partial continuous on $[0,1)$ and $[0,1)$ is $\mathcal{T}$-orbital partially complete metric space. Moreover the condition (8) is satisfied with $b = 0$ and $q = \frac{1}{4}$. Also we have $U_{\mathcal{T}}(X) \neq \emptyset$. Therefore, by Corollary 2.10, $\mathcal{T}$ has a fixed point $u = 0$.

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REFERENCES

[1] J. Achari, On the generalization of Pachpatte's nonunique fixed point theorem, *Indian J. Pure Appl. Math.*, **13** (3) (1982), 299-302.
[2] A. Alam, M. Imdad, Relation-theoretic contraction principle, *J. Fixed Point Theory Appl.*, **17** (4) (2015), 693-702.
[3] Lj.B. Ciric, On some maps with nonunique fixed point, *Publ. Inst. Math.*, **17** (1974), 52-58.
[4] B.C. Dhage, *A study of fixed some fixed theorems* Ph. D. thesis, Marath. Univ. Aurangabad, India, 1984,
[5] B.C. Dhage, Some results for the maps with a nonunique fixed point, *Indain J. Pure Appl. Math.*, **16** (1985), 245-256.
[6] B.C. Dhage, Basic hybrid fixed point theorems for contractive mappings in partially ordered metric spaces, *Thai J. Math.*, **15** (3) (2017), 721–732.
[7] B.C. Dhage, Nonlinear partial completely continuous operators in a partially ordered Banach space and nonlinear hyperbolic partial differential equations, *Malaya J. Mat.*, **12** (4) (2024), 330-338.
[8] J.B. Dhage, B.C. Dhage, Some relation theoretic fixed point theorems for relational **R**-contraction mappings in the Euclidean space $\mathbb{R}^n$, *Facta Univ. Ser. Math. Inform.*, (2025) (to appear)
[9] J.B. Dhage, S.B. Dhage, B.C. Dhage, An algorithmic approach to local solution of the nonlinear higher order ordinary hybrid differential equations, *Jñānābha*, **54** (1) (2024), 29-40.

[10] J.B. Dhage, B.C. Dhage, L.P. Aware, On some fixed point theorems for partial contraction mappings in a partially ordered multi-metric space, *Malaya J. Mat.*, **13** (3) (2025), 1-13.
[11] N. Dubey, S. Rai, S. Shukla, Hardy-Rogers type mappings in relational metric spaces and fixed point theorems, *Journal of Indian Acad. Math.*, **46** (2) (2024), 207–223.
[12] B. G. Pachpatte, On Ciric type maps with a nonunique fixed point, *Indian J. Pure Appl. Math.*, **10** (8) (1979), 1039-1043.
[13] B. Samet, M. Turinici, Fixed point theorems on a metric space endowed with an arbitrary binary relation and applications, *Commun. Math. Anal.*, **13** (2012), 82-97.

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