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DYNAMICS AND VALUE DISTRIBUTION OF COMPLEX VALUED MEROMORPHIC FUNCTIONS OF HIGHER DIMENSIONS

Sanjib Kumar Datta

Abstract. We generalize fundamental results from complex meromorphic dynamics and value-distribution theory to bicomplex meromorphic functions. After introducing bicomplex numbers and bicomplex-holomorphic (meromorphic) functions, we extend the notions of Fatou and Julia sets to the bicomplex plane. Our main results are: (1) a bicomplex analogue of quasi-nested wandering domains and Baker omitted values, including existence and boundedness theorems; (2) the extension of Herman-ring configurations and their classification to the bicomplex case and (3) bicomplex versions of Picard-Montel normality theorems for families omitting values. All proofs proceed by reducing to the idempotent components and applying classical one-variable arguments.

Keywords: Bicomplex analysis, multicomplex functions, bicomplex meromorphic dynamics, normal families, value distribution theory, Nevanlinna theory, Picard theorem, Montel theorem, Julia set, Fatou set, Baker domains, wandering domains, omitted values, higher-dimensional complex dynamics, holomorphic iteration, dynamics of bicomplex-valued functions, quasi-normal families.

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1. Introduction

Classical one-dimensional complex dynamics and Nevanlinna theory have produced deep insights into the iteration and value-distribution of meromorphic functions. In particular, Chakraborty and Datta studied quasi-nested wandering domains and Baker omitted values, configurations of Herman rings and dynamical properties of families omitting values {cf. [8]}. These works were formulated for complex meromorphic functions on $\widehat{\mathbb{C}}$ {cf. [20]}. In this paper, we extend these theories to the bicomplex setting, where functions take values in the algebra of bicomplex numbers $\mathbb{BC}$ {cf. [1],[4]}.

While bicomplex analysis extends the classical theory of complex functions by introducing two commuting imaginary units, the concept can be generalized even further through the framework of *multicomplex analysis*. A multicomplex algebra is generated by more than two commuting imaginary units, typically denoted as $i_1, i_2, \ldots, i_n$, each satisfying $i_k^2 = -1$. The resulting space, known as the multicomplex space $\mathbb{C}_n$, is a finite-dimensional commutative algebra over $\mathbb{R}$, encompassing the complex, bicomplex and tricomplex systems as special cases. Multicomplex numbers allow for the representation of multidimensional analytic structures and naturally decompose into direct sums of complex subspaces using idempotent elements.

This generalization significantly enriches the analytical landscape: fundamental concepts such as differentiability, holomorphicity and singularity theory can all be extended to multicomplex functions, often revealing behaviours not present in the classical or bicomplex settings. Moreover, the higher-dimensional structure of multicomplex spaces enables a unified framework for modelling complex systems involving multiple independent processes. As such, multicomplex analysis is increasingly being explored in areas ranging from advanced function theory to dynamical systems, differential equations and applications in physics and engineering.

We adopt the following plan. In Section 2 we recall the definitions of bicomplex numbers {cf. [1],[4]}, the extended bicomplex plane and bicomplex-holomorphic (meromorphic) functions via idempotent decomposition cf. [1],[4]}. We then define bicomplex Fatou and Julia sets {cf. [7],[12]} analogously to the complex case by using bispherical normality of iterates {cf. [7],[13]}. In Section 3 we introduce bicomplex quasi-nested wandering domains and Baker omitted values and we prove the bicomplex analogues of Chakraborty-Datta's theorems on their existence and properties. In Section 4 we generalize the classification of Herman-ring cycles to bicomplex maps. Finally, in Section 5 we establish bicomplex Montel/Picard-type theorems for families of meromorphic functions with omitted values. We conclude with a discussion of implications and future directions.

2. Preliminaries on Bicomplex Analysis

Let

$$\mathbb{BC} = \{\, z_1 + z_2 i_2 : z_1, z_2 \in \mathbb{C}(i_1) \,\}$$

denote the algebra of *bicomplex* numbers, with independent imaginary units $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1 = j$ so that $j^2 = 1$. Every $w = z_1 + z_2 i_2$ can be uniquely written using the orthogonal idempotents as

$$e_1 = \frac{1+j}{2} \quad \textit{and} \quad e_2 = \frac{1-j}{2},$$

namely

$$w \;=\; (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2 \;=\; P_1(w)e_1 + P_2(w)e_2,$$

where $P_1(w) = z_1 - i_1 z_2$ and $P_2(w) = z_1 + i_1 z_2$ project w onto two copies of the complex plane. The set of zero-divisors (the *null cone*) is

$$N_C \;=\; \{\, w : \|w\| j^2 = 0\},$$

where $|w|j = |P_1(w)|, e_1 + |P_2(w)|, e_2$ is the bicomplex modulus. We also form the *extended bicomplex plane* $\mathbb{BC} \cup I\infty$ by adjoining points at infinity. In particular, one can regard

$$\overline{\mathbb{BC}} \;=\; \mathbb{C}(i_1) \times_e \mathbb{C}(i_1) \;\cup\; I\infty, \qquad \textit{where} \qquad I_\infty = \{\, w : \|w\|_j = \infty \,\},$$

analogously to the Riemann sphere.

A function $f : \Omega \subset \mathbb{BC} \to \mathbb{BC}$ is called *bicomplex-holomorphic* ($\mathbb{BC}$ -holomorphic) if it satisfies the bicomplex Cauchy-Riemann conditions. Equivalently, by Ringleb's Lemma {cf. [2],[3]}, f is $\mathbb{BC}$-holomorphic {cf. [2],[4],[5],[6]} if and only if its projected components

$$f_{e1} :\; P_1(\Omega) \to \mathbb{C}(i_1), \qquad f_{e2} :\; P_2(\Omega) \to \mathbb{C}(i_1)$$

are holomorphic in the usual sense. We say f is *bicomplex meromorphic* on Ω if it can be expressed as the quotient of bicomplex-holomorphic functions, or equivalently if f extends to $\overline{\mathbb{BC}}$ with poles. In fact one has

Theorem 2.1. *Let $\Omega \subset \mathbb{BC}$ be open. A function $f : \Omega \to \mathbb{BC}$ is bicomplex meromorphic if and only if its components $f_{e1} : P_1(\Omega) \to \mathbb{C}(i_1)$ and $f_{e2} : P_2(\Omega) \to \mathbb{C}(i_1)$ are (single-variable) meromorphic and then*

$$f(w) \;=\; f_{e1}(P_1(w))\, e_1 + f_{e2}(P_2(w))\, e_2, \quad w \in \Omega.$$

Accordingly, we define $w_0 \in \Omega$ to be a pole of f if either f_{e1} or f_{e2} has a pole at the corresponding projected point. (Such poles are not isolated in $\mathbb{BC}$.)

We now introduce bicomplex dynamical sets. As usual, the Fatou set *$F(f)$ is the set of points in $\mathbb{BC}$ where the iterates f^n form a normal family with respect to the bicomplex chordal metric. Its complement $J(f) = \mathbb{BC} \setminus F(f)$ is the Julia set {cf.* [18],[20]*}. Equivalently, for a nondegenerate bicomplex polynomial or rational map,*

$$J_2(f) = \{\zeta : \{f^n(\zeta)\} \textit{ is not normal}\}.$$

By construction $J(f)$ is completely invariant under f.

A family $\mathcal{M}(\mathbb{BC})$ of bicomplex meromorphic functions on a domain $\Omega \subset \mathbb{BC}$ is *normal* if every sequence in $\mathcal{M}(\mathbb{BC})$ has a subsequence converging bispherically uniformly on compact subsets to a limit in $\overline{\mathbb{BC}}$. Equivalently, each projection $f_{e1} : f \in \mathcal{M}(\mathbb{BC})$ and $f_{e2} : f \in \mathcal{M}(\mathbb{BC})$ must be a normal family in the usual complex sense.

Remark 2.2. *The bicomplex modulus $\|w\|_j$ does not define a norm in the usual sense because of the presence of zero divisors. Nevertheless, it provides a way to measure "size" component-wise and to classify points as bounded or unbounded in $\mathbb{BC}$.*

3. Bicomplex Quasi-Nested Wandering Domains

Let $f : \mathbb{BC} \to \overline{\mathbb{BC}}$ be bicomplex meromorphic and let $U \subset F(f)$ be a Fatou component. Recall U is a *wandering domain* {cf. [19]} if the iterates $f^n(U)$ are pairwise disjoint and never periodic. We adapt the notion of a quasi-nested wandering domain {cf. [8]} to $\mathbb{BC}$.

Definition 3.1. *A wandering domain $U \subset F(f)$ is called* quasi-nested *if there exists a subsequence of iterates $n_k \to \infty$ such that each $U_{n_k} = f^{n_k}(U)$ is j-bounded and as $k \to \infty$ the domains U_{n_k} cluster onto a single point $a \in \overline{\mathbb{BC}}$. Equivalently, for some $a \in \overline{\mathbb{BC}}$ one has U_{n_k} surrounding a in each projection and $\max_{w \in U_{n_k}} |w - a|_j \to 0$.*

Next we define a bicomplex version of a Baker omitted value.

Definition 3.2. *An omitted value $a \in \overline{\mathbb{BC}}$ of f is a* Baker omitted value *if for all sufficiently small $r > 0$, the preimage $f^{-1}(B(a; r))$ is of the form*

$$\mathbb{BC} \setminus \bigcup_{i=1}^{\infty} D_i,$$

where the D_i are pairwise disjoint bounded domains in $\mathbb{BC}$. In other words, $f^{-1}(B(a;r))$ has bounded complement for each small disk $B(a;r)$.

We can now state the bicomplex analogue of Chakraborty-Datta's result on quasi-nested domains {cf. [8]}.

Theorem 3.3. *Let $a \in \overline{\mathbb{BC}}$ be a bicomplex Baker omitted value of f. Then f has a bicomplex quasi-nested wandering domain if and only if there exists a sequence of iterates U_{n_k} such that each U_{n_k} surrounds a and $U_{n_k} \to a$ as $k \to \infty$.*

Proof. Write $f(w) = f_{e1}(P_1(w))e_1 + f_{e2}(P_2(w))e_2$ by Theorem 2.1. If a is a bicomplex Baker omitted value, then its projections $a_1 = P_1(a)$ and $a_2 = P_2(a)$ are Baker omitted values of the complex functions f_{e1} and f_{e2}. By f_{e1} has a quasi-nested wandering domain if and only if a sequence of iterates surrounds a_1 and tends to a_1 and similarly for f_{e2}.

If $U \subset \mathbb{BC}$ is a bicomplex wandering domain quasi-nested around a, then each projection $P_i(U)$ is quasi-nested around a_i. Hence there exist iterates $P_i(U_{n_k})$ surrounding a_i and tending to a_i. This implies U_{n_k} surrounds a in $\mathbb{BC}$ and $U_{n_k} \to a$. Conversely, if such a sequence of U_{n_k} exists, then projecting shows each f_{ei} has a quasi-nested wandering domain around a_i. Therefore U is bicomplex quasi-nested. □

We can generalize the Theorem 3.1 in multicomplex space as follows.

Theorem 3.4. *Let $a \in \mathbb{C}_n$ be a multicomplex Baker omitted value of a meromorphic function $F : \mathbb{C}_n \to \mathbb{C}_n$. Then F has a multicomplex quasi-nested wandering domain if and only if there exists a sequence of iterates $\{U_{n_k}\}$ such that each U_{n_k} surrounds a and*

$$U_{n_k} \longrightarrow a \quad as \quad k \to \infty.$$

Proof. Write the multicomplex algebra in its idempotent decomposition

$$\mathbb{C}_n \;=\; \bigoplus_{\alpha=1}^{2^{n-1}} \mathbb{C}\, e_\alpha,$$

with canonical projections $P_\alpha : \mathbb{C}_n \to \mathbb{C}$ and mutually orthogonal idempotents e_α. By the multicomplex Ringleb decomposition (the obvious generalisation of Ringleb's Lemma to $\mathbb{C}_n$), the meromorphic map

$$F : \mathbb{C}_n \longrightarrow \mathbb{C}_n$$

can be written on its domain Ω as

$$F(W) = \sum_{\alpha=1}^{2^{n-1}} f_\alpha\big(P_\alpha(W)\big)\, e_\alpha,$$

where each component $f_\alpha : P_\alpha(\Omega) \to \mathbb{C}$ is a (single-variable) meromorphic function {cf. [17]}.

Assume first that $a \in \mathbb{C}_n$ is a multicomplex Baker omitted value of F. Then for each idempotent index α the projection $a_\alpha := P_\alpha(a)$ is a Baker omitted value of the complex meromorphic function f_α. By the classical one-variable theory (the complex version of the Chakraborty-Datta result), for each α the function f_α admits a quasi-nested wandering

domain precisely when there exists a sequence of iterates of f_α whose images are bounded annular neighbourhoods surrounding a_α and which converge to a_α. Consequently, if F is a multicomplex map with Baker omitted value a, then the idempotent components f_α admit corresponding sequences of iterates surrounding a_α. Combining these componentwise sequences yields a sequence of bicomplex (indeed multicomplex) iterates $\{U_{n_k}\}$ for F such that each U_{n_k} surrounds a in $\mathbb{C}_n$ (i.e. each projection $P_\alpha(U_{n_k})$ surrounds a_α) and

$$U_{n_k} \longrightarrow a \quad \text{as } k \to \infty,$$

so F has a multicomplex quasi-nested wandering domain.

Conversely, suppose there exists a sequence of Fatou components (iterates) $\{U_{n_k}\}$ for F with the property that each U_{n_k} surrounds a and $U_{n_k} \to a$ as $k \to \infty$. Projecting onto the idempotent summands shows that for each α the sequence $\{P_\alpha(U_{n_k})\}$ consists of Fatou components (iterates) of f_α which surround a_α and tend to a_α. Again by the classical one-variable characterisation, each f_α therefore possesses a quasi-nested wandering domain around a_α. Reassembling these componentwise quasi-nested domains using the idempotent decomposition produces a multicomplex wandering domain for F which is quasi-nested about a. Finally, the equivalence of the componentwise and multicomplex descriptions of surrounding and convergence (via the projections P_α) yields the claimed if and only if statement.

This completes the proof. □

As in the complex case, the existence of a quasi-nested domain forces all Fatou components to be bounded.

Proposition 3.5. *If f has a quasi-nested wandering domain, then every bicomplex Fatou component is bounded and in particular f has no Baker domain in $\mathbb{BC}$.*

Proof. If U is a bicomplex quasi-nested domain around a, then each projection f_{ei} has a complex quasi-nested domain around a_i. By each f_{ei} has only bounded Fatou components. Since a bicomplex Fatou component is a Cartesian product of components of f_{e1} and f_{e2}, it follows that all bicomplex Fatou components are bounded. In particular, there is no bicomplex Baker domain (which would correspond to an unbounded Fatou component). □

Remark 3.6. *Quasi-nested wandering domains in $\mathbb{BC}$ may appear as "tubes" in one component and "discs" in the other. The idempotent decomposition allows one to visualize these domains as Cartesian products of classical wandering domains in $\mathbb{C}$.*

4. Herman Rings in Bicomplex Dynamics

A *Herman ring* {cf. [9],[10],[11]} for $f : \mathbb{BC} \to \overline{\mathbb{BC}}$ is a doubly-connected periodic Fatou component {cf. [14],[15]} on which f acts as an irrational rotation. Concretely, a p-cycle $U_0, \ldots, U_{p-1}$ of bicomplex Herman rings means each U_k is T-Cartesian (i.e. of the form $U_1e_1 + U_2e_2$ with each U_i an annulus) and f^p is analytically conjugate to an irrational rotation on U.

The possible arrangements of a cycle of Herman rings can be described by maximal nests and nesting, as in the complex theory. We recall: an *outermost ring* in the cycle is one not contained in the closure of any other. A *maximal nest* is a sub-collection of rings consisting of an outermost ring and all rings nested inside it. The cycle is *nested* if it has exactly one maximal nest, *strictly nested* if each ring except the outermost is properly inside another and *strictly non-nested* if no ring lies inside another.

The following lemma relates bicomplex Herman rings to their projections.

Lemma 4.1. *If $U = U_1e_1 + U_2e_2 \subset \mathbb{BC}$ is a p-periodic bicomplex Herman ring of f, then U_1 and U_2 are (period-p) Herman rings for the complex functions f_{e1} and f_{e2}. Conversely, if U_1, U_2 are Herman rings of the same period p for f_{e1}, f_{e2}, then $U = U_1e_1 + U_2e_2$ is a bicomplex Herman ring of f.*

Proof. If $f^p : U \to U$ is conjugate to $w \mapsto e^{2\pi i\theta}w$, then projecting onto each idempotent shows f_{e1}^p is conjugate to rotation by θ on U_1 and similarly f_{e2}^p on U_2. Thus each U_i is an invariant annulus with irrational rotation, i.e. a Herman ring. Conversely, suppose U_1 and U_2 are Herman rings for f_{e1}, f_{e2} with angles θ_1, θ_2. One can construct a bicomplex conjugacy

$$\Phi(w) \;=\; \phi_1(P_1(w))\, e_1 \;+\; \phi_2(P_2(w))\, e_2,$$

using the conjugating maps ϕ_i for each component. Then Φ conjugates f^p on $U_1e_1 + U_2e_2$ to the bicomplex rotation $e^{2\pi i\theta_1}e_1 + e^{2\pi i\theta_2}e_2$, yielding a bicomplex Herman ring of period p. □

Using Lemma 4.1 and known complex results, we derive restrictions on bicomplex Herman rings.

Proposition 4.2. *A bicomplex meromorphic function f cannot have a Herman ring of period 1 or 2. More generally, if either projection f_{e1} or f_{e2} omits an essential singularity (for example, has only one pole), then f has no bicomplex Herman ring.*

Proof. If f had a period-1 Herman ring, then by Lemma 4.1 each f_{ei} would have a period-1 Herman ring. But it is known that no meromorphic function can have a 1-cycle Herman ring. The same argument (and citation) rules out period-2. For the second part, if f_{e1} is entire or has a single pole (omitted value) then by it has no Herman rings. Thus f cannot have a bicomplex Herman ring since one projection would fail to provide a ring. Similarly if f_{e2} omits ∞. □

In summary, the classification of Herman-ring cycles for f follows from the classifications for f_{e1} and f_{e2}. For instance, a bicomplex Herman-cycle is nested only if each projected cycle is nested, etc.

Remark 4.3. *Bicomplex Herman rings can exhibit different rotation speeds in each component. Therefore, the overall orbit in $\mathbb{T}$ might densely fill a torus-like region even if each component rotates along a simple circle in $\mathbb{C}$.*

Example 4.4. *For example, Let* $f(w) = e^{i\pi/3}P_1(w)e_1 + e^{i\pi/4}P_2(w)e_2$. *Then* $U_1 = \{1 < |z| < 2\}$ *and* $U_2 = \{1 < |z| < 2\}$ *form a bicomplex Herman ring where the rotations in* U_1 *and* U_2 *are incommensurate, producing a quasi-periodic orbit in* $\mathbb{BC}$.

5. Dynamics of Bicomplex Families with Omitted Values

We now consider normal families of bicomplex meromorphic functions with omitted values. The bicomplex analogue of Montel's theorem reads as follows.

Theorem 5.1 (Bicomplex Montel Theorem)**.** *Let* $\mathcal{M}(\mathbb{BC})$ *be a family of bicomplex meromorphic functions on a domain* $\Omega \subset \mathbb{BC}$. *Suppose there exist three values* $\alpha, \beta, \gamma \in \overline{\mathbb{BC}}$ *such that* $\alpha-\beta, \beta-\gamma, \gamma-\alpha$ *are all invertible in* $\mathbb{BC}$ *and no* f *attains any of* α, β, γ. *Then* $\mathcal{M}(\mathbb{BC})$ *is a normal family on* Ω.

Proof. Project onto the idempotent components $\mathcal{M}(\mathbb{BC})_{e1} = f_{e1} : f \in \mathcal{M}(\mathbb{BC})$ and $\mathcal{M}(\mathbb{BC})_{e2} = f_{e2} : f \in \mathcal{M}(\mathbb{BC})$. The invertibility conditions imply $P_i(\alpha)$, $P_i(\beta)$, $P_i(\gamma)$ are three distinct complex values. Since no f takes α, β, γ, no $f_{ei} \in \mathbb{F}_{ei}$ takes the corresponding three complex values. By the classical Montel theorem, each $\mathcal{M}(\mathbb{BC})_{ei}$ is normal in $\mathbb{C}(i_1)$. Hence $\mathcal{M}(\mathbb{BC})$ is normal on Ω by Ringleb's decomposition (Theorem 2.1). □

We can generalize the theorem 5.1 in multicomplex space as follows.

Theorem 5.2 (Multicomplex Montel Theorem)**.** *Let* $\mathcal{M}(\mathbb{C}_n)$ *be a family of multicomplex meromorphic functions on a domain* $\Omega \subset \mathbb{C}_n$. *Suppose there exist three distinct values* $\alpha, \beta, \gamma \in \mathbb{C}_n$ *such that* $\alpha-\beta$, $\beta-\gamma$ *and* $\gamma-\alpha$ *are all invertible in* $\mathbb{C}_n$ *and no* $F \in \mathcal{M}(\mathbb{C}_n)$ *attains any of the values* α, β *or* γ. *Then* $\mathcal{M}(\mathbb{C}_n)$ *is a normal family on* Ω.

Proof. Write the multicomplex algebra in its canonical idempotent decomposition

$$\mathbb{C}_n \;=\; \bigoplus_{\alpha=1}^{2^{n-1}} \mathbb{C}\, e_\alpha,$$

and let $P_\alpha : \mathbb{C}_n \to \mathbb{C}$ denote the projection onto the α-th complex summand. By the multicomplex Ringleb decomposition, every

$$F \in \mathcal{M}(\mathbb{C}_n)$$

admits the representation

$$F(W) = \sum_{\alpha=1}^{2^{n-1}} f_\alpha(P_\alpha(W))\, e_\alpha, \qquad W \in \Omega,$$

where for each fixed α the map f_α ranges over a family $\mathcal{F}_\alpha = \{f_\alpha : F \in \mathcal{M}(\mathbb{C}_n)\}$ of single-variable meromorphic functions on the complex domain $P_\alpha(\Omega)$.

By hypothesis there are three distinct values $\alpha, \beta, \gamma \in \mathbb{C}_n$ with $\alpha-\beta, \beta-\gamma, \gamma-\alpha$ invertible in $\mathbb{C}_n$ and no $F \in \mathcal{M}(\mathbb{C}_n)$ attains any of these three values. Fix an idempotent index α. Applying the projection P_α to the three multicomplex values yields the three complex numbers

$$P_\alpha(\alpha), \qquad P_\alpha(\beta) \quad \textit{and} \quad P_\alpha(\gamma).$$

Invertibility of differences in $\mathbb{C}_n$ implies that for each idempotent index α the differences

$$P_\alpha(\alpha) - P_\alpha(\beta), \quad P_\alpha(\beta) - P_\alpha(\gamma) \quad \textit{and} \quad P_\alpha(\gamma) - P_\alpha(\alpha)$$

are nonzero in $\mathbb{C}$; hence $P_\alpha(\alpha), P_\alpha(\beta), P_\alpha(\gamma)$ are three distinct complex values. Moreover, since no multicomplex map $F \in \mathcal{M}(\mathbb{C}_n)$ attains α, β, γ, it follows that no component function $f_\alpha \in \mathcal{F}_\alpha$ attains any of the complex values $P_\alpha(\alpha), P_\alpha(\beta)$ *and* $P_\alpha(\gamma)$.

Therefore, for each fixed idempotent index α the family $\mathcal{F}_\alpha$ of single-variable meromorphic functions omits three distinct complex values in $P_\alpha(\Omega)$. By the classical Montel theorem for meromorphic functions, each family $\mathcal{F}_\alpha$ is normal on $P_\alpha(\Omega)$.

Normality of every component family $\mathcal{F}_\alpha$ implies normality of the original multicomplex family $\mathcal{M}(\mathbb{C}_n)$: indeed, any sequence $\{F_m\} \subset \mathcal{M}(\mathbb{C}_n)$ has, by diagonal extraction on the finitely many (in fact 2^{n-1}) idempotent indices, a subsequence whose component projections $\{f_{\alpha,m}\}$ converge bispherically (uniformly on compacta) in each complex summand to meromorphic limits g_α. Recombining these limits via the idempotent decomposition yields a bispherically convergent subsequence of $\{F_m\}$ in $\overline{\mathbb{C}_n}$. Hence $\mathcal{M}(\mathbb{C}_n)$ is normal on Ω. □

An immediate corollary of Theorem 5.1 is a bicomplex Picard theorem.

Theorem 5.3 (Bicomplex Picard Theorem)**.** *Let $f : \mathbb{BC} \to \overline{\mathbb{BC}}$ be a bicomplex meromorphic function. If there exist three values α, β, γ with $\alpha - \beta, \beta - \gamma, \gamma - \alpha$ invertible, such that $f(w) \neq \alpha, \beta, \gamma$ for all $w \in \mathbb{BC}$, then f is constant.*

Proof. Consider the constant family $\mathcal{M}(\mathbb{BC}) = f$. The Montel theorem implies $\mathcal{M}(\mathbb{BC})$ is normal on $\mathbb{BC}$. Hence each projection f_{ei} omits three values in $\mathbb{C}$ and by Picard's theorem must be constant. Therefore $f(w) = c_1e_1 + c_2e_2$ is constant on $\mathbb{BC}$. □

Similarly, one can obtain bicomplex versions of the Fundamental Normality Test for two values: e.g. a family omitting two bicomplex values (with invertible difference) is normal, by the same reduction.

Remark 5.4. *In bicomplex analysis, the notion of "omitting values" must consider invertibility. Two values may coincide in one idempotent component but differ in the other, so the classical theorems hold only when all three differences are invertible.*

Example 5.5. *For example, Let $f_n(w) = \frac{n}{w-1}e_1 + \frac{1}{w-2}e_2$, $n \in \mathbb{N}$. Then $\{f_n\}$ omits 0 in both components and 1 in the first component. The family is normal on $\mathbb{BC} \setminus \{1, 2\}$ by Theorem 5.1.*

6. Conclusion

We have extended several key aspects of complex meromorphic dynamics into the bicomplex plane. Using the idempotent decomposition, one essentially obtains that bicomplex phenomena mirror two coupled copies of the one-variable case. In particular, bicomplex analogues of quasi-nested wandering domains, Herman rings and Picard-Montel

theorems follow directly by applying known results to each component. Notably, restrictions such as the nonexistence of 1- or 2-cycles of Herman rings, or Montel's normality criterion for families missing three values, remain valid in the bicomplex setting.

This framework suggests many directions for further research. One may study other rotation domains (e.g. bicomplex Siegel disks or Baker domains) {cf. [16]}, bifurcations of bicomplex rational maps, or dynamics in other commutative hypercomplex algebras. The bicomplex Picard-Montel theory developed here could be refined or extended and numerical explorations of bicomplex Julia sets might reveal new structure. We hope this work lays a foundation for a rich bicomplex dynamics theory.

7. Future Prospects

The present work lays a basic foundation for bicomplex dynamical and value distribution theory, but the field remains largely unexplored. Natural extensions include higher-order multicomplex systems, where componentwise dynamics and idempotent structure may produce new behaviours in Fatou-Julia theory; multicomplex Nevanlinna-type results, Picard phenomena and defect relations, where multiple zero divisors suggest richer value distribution patterns and interdisciplinary applications in physics, signal analysis and modelling, where multicomplex representations arise naturally. Computational study and visualization of multicomplex Julia sets and parameter spaces also remain open. Overall, extending these ideas to the full multicomplex hierarchy promises a substantially broader analytical framework with significant theoretical and applied potential.

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Mathematics Department,
University of Kalyani,
P.O.: Kalyani, Dist: Nadia, PIN: 741235,
West Bengal, India.
Email: sanjibdatta05@gmail.com