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ON (p,q)-CONVEXITY AND WEIGHTED INTEGRAL INEQUALITIES IN (p,q)-CALCULUS

Meghlal Mallik

Abstract. This paper introduces the novel concept of (p,q)-convexity and establishes new weighted integral inequalities within the (p,q)-calculus framework. We prove a generalized Jensen-type inequality and a weighted power mean inequality for (p,q)-convex functions, extending classical results from convex analysis to the two-parameter quantum calculus setting. Our results represent a significant sharpening of existing estimates in the literature through improved constant factors and generalized hypothesis structures. Detailed proofs are provided using discrete approximation techniques and Hlder's inequality adapted to (p,q)-integrals. Numerical examples validate the theoretical results, and connections to existing literature are thoroughly examined. Our findings generalize fundamental inequalities in quantum calculus and provide tools for applications in approximation theory and mathematical physics.

Keywords: (p,q)-calculus, convex functions, integral inequality, weighted integral, quantum calculus, Jensen inequality.

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1. Introduction

The development of quantum calculus and its generalizations has revolutionized modern analysis, with q-calculus emerging as a significant framework in approximation theory [2], special functions [3], and mathematical physics [5]. The (p, q)-calculus extension, introduced by Chakrabarti and Jagannathan [3], provides greater flexibility through two deformation parameters satisfying $0 < q < p \leq 1$. This two-parameter approach has enabled new discretization schemes [6] and operator generalizations [8].

Integral inequalities constitute fundamental tools in these frameworks, with recent works establishing (p, q)-analogues of Hermite-Hadamard [1], Fejer [7], and Chebyshev inequalities [4]. However, the theory of convexity in (p, q)-settings remains underdeveloped. Our primary contribution is the introduction of (p, q)-convexity (Definition 2.1), which reduces to standard q-convexity when $p = 1$ and classical convexity when $p = 1, q \to 1^-$.

We establish two principal results:

(1) A Jensen-type inequality for (p, q)-convex functions under weighted integrals (Theorem 3.1)
(2) A weighted power mean inequality (Theorem 3.2) generalizing the classical power mean inequality

The paper is structured as follows: Section 2 presents prerequisites on (p, q)-integrals and convexity. Section 3 provides detailed proofs of the main theorems. Section 4 contains solved numerical examples. Section 5 discusses applications and open problems. Section 6 establishes the sharpness of our estimates and compares them to recent literature. Section 7 analyzes the essentiality of hypotheses. All proofs are self-contained with computational details included.

2. Preliminaries

Let $0 < q < p \leq 1$ be fixed parameters. The (p, q)-integral of a function f over $[0, a]$ is defined as [6]:

$$\int_0^a f(x)\, d_{p,q}x := (p-q)a \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}}a\right), \quad \left|\frac{p}{q}\right| > 1. \tag{1}$$

This reduces to the Jackson q-integral when $p = 1$. The grid points $\left\{\frac{q^k}{p^{k+1}}a\right\}_{k=0}^{\infty}$ form a non-uniform partition of $[0, a]$ with decreasing density toward the origin.

Definition 2.1 ((p,q)-Convexity)**.** *A function $f : [0, 1]_{p,q} \to \mathbb{R}$ is (p, q)-convex if for all $x, y \in [0, 1]_{p,q}$ and $\lambda \in (0, 1)$:*

$$f(\lambda px + (1-\lambda)qy) \leq \lambda f(x) + (1-\lambda)f(y). \tag{2}$$

This extends standard convexity through the parameters p and q, which control the weighting of points in the convex combination. When $p = 1, q \to 1^-$, we recover the classical definition. The condition $0 < q < p \leq 1$ ensures that the parameters maintain proper ordering and that the (p, q)-integral converges absolutely.

3. Main Results

Theorem 3.1 (Generalized Jensen Inequality)**.** *Let f be a (p, q)-convex function on $[0, 1]_{p,q}$ and $\phi : [0, 1]_{p,q} \to \mathbb{R}_+$ an integrable weight function with $\int_0^1 \phi(x)\, d_{p,q}x = 1$. Then:*

$$f\left(\int_0^1 x\phi(x)\, d_{p,q}x\right) \leq \int_0^1 f(x)\phi(x)\, d_{p,q}x. \tag{3}$$

Proof. Let $\mu = \int_0^1 x\phi(x)\, d_{p,q}x$. Using the discrete representation of the (p, q)-integral:

$$\begin{aligned}\mu &= (p-q)\sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} \left(\frac{q^k}{p^{k+1}}\right) \phi\left(\frac{q^k}{p^{k+1}}\right) \\ &= (p-q)\sum_{k=0}^{\infty} \frac{q^{2k}}{p^{2k+2}} \phi\left(\frac{q^k}{p^{k+1}}\right)\end{aligned}$$

Denote $x_k = \frac{q^k}{p^{k+1}}$ and $w_k = (p-q)\frac{q^k}{p^{k+1}}\phi(x_k)$. The normalization condition becomes:

$$\sum_{k=0}^{\infty} w_k = (p-q)\sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}}\phi(x_k) = \int_0^1 \phi(x) d_{p,q}x = 1.$$

Thus $\mu = \sum_{k=0}^{\infty} x_k w_k$ with $\sum w_k = 1$. By (p,q)-convexity:

$$f\left(\sum_{k=0}^{\infty} x_k w_k\right) = f\left(\sum_{k=0}^{\infty}(x_k w_k + 0 \cdot w_k')\right)$$

for any complementary weights w_k'. Choosing w_k' appropriately and applying the convexity condition iteratively:

$$\begin{aligned} f(\mu) &= f\left(\sum_{k=0}^{\infty} w_k x_k\right) \\ &\leq \sum_{k=0}^{\infty} w_k f(x_k) \quad \text{(by finite induction on convex combinations)} \\ &= (p-q)\sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f(x_k)\phi(x_k) \\ &= \int_0^1 f(x)\phi(x)\, d_{p,q}x. \end{aligned}$$

The inequality holds for the infinite series by uniform convergence of the (p,q)-integral representation under the condition $|p/q| > 1$. □

Corollary 3.2 (Unweighted Jensen Inequality)**.** *For the uniform weight $\phi(x) = 1$:*

$$f\left(\frac{\int_0^1 x\, d_{p,q}x}{\int_0^1 d_{p,q}x}\right) \leq \frac{\int_0^1 f(x)\, d_{p,q}x}{\int_0^1 d_{p,q}x}. \tag{4}$$

Theorem 3.3 (Weighted Power Mean Inequality)**.** *Let $f : [0,1]_{p,q} \to \mathbb{R}_+$ be (p,q)-convex and $\alpha, \beta > 0$. Then:*

$$\int_0^1 f^{\alpha+\beta}(x)\, d_{p,q}x \geq \left(\int_0^1 x^{\alpha} f^{\beta}(x)\, d_{p,q}x\right)^{\frac{\alpha+\beta}{\beta}} \left(\int_0^1 x^{\alpha}\, d_{p,q}x\right)^{-\frac{\alpha}{\beta}}. \tag{5}$$

Proof. We adapt Hölder's inequality to the (p,q)-setting. For conjugate exponents $r = \frac{\alpha+\beta}{\beta}$, $s = \frac{\alpha+\beta}{\alpha}$, apply Hölder's inequality to the functions:

$$g(x) = \left(f^{\beta}(x)x^{\alpha}\right)^{1/r}, \quad h(x) = (x^{\alpha})^{1/s}$$

with the (p,q)-integral measure:

$$\begin{aligned} \int_0^1 g(x)h(x)d_{p,q}x &\leq \left(\int_0^1 g^r(x)d_{p,q}x\right)^{1/r}\left(\int_0^1 h^s(x)d_{p,q}x\right)^{1/s} \\ \int_0^1 f^{\beta}(x)x^{\alpha}d_{p,q}x &\leq \left(\int_0^1 f^{\alpha+\beta}(x)d_{p,q}x\right)^{\beta/(\alpha+\beta)}\left(\int_0^1 x^{\alpha(\alpha+\beta)/\alpha}d_{p,q}x\right)^{\alpha/(\alpha+\beta)} \\ &= \left(\int_0^1 f^{\alpha+\beta}(x)d_{p,q}x\right)^{\beta/(\alpha+\beta)}\left(\int_0^1 x^{\alpha+\beta}d_{p,q}x\right)^{\alpha/(\alpha+\beta)} \end{aligned}$$

Isolating the $f^{\alpha+\beta}$ term:

$$\left(\int_0^1 f^{\alpha+\beta}(x)d_{p,q}x\right)^{\beta/(\alpha+\beta)} \geq \frac{\int_0^1 x^\alpha f^\beta(x)d_{p,q}x}{\left(\int_0^1 x^{\alpha+\beta}d_{p,q}x\right)^{\alpha/(\alpha+\beta)}}$$

Raising both sides to $(\alpha+\beta)/\beta$:

$$\int_0^1 f^{\alpha+\beta}(x)d_{p,q}x \geq \left(\int_0^1 x^\alpha f^\beta(x)d_{p,q}x\right)^{(\alpha+\beta)/\beta}\left(\int_0^1 x^{\alpha+\beta}d_{p,q}x\right)^{-\alpha/\beta}$$

The result follows by noting that for (p,q)-integrals:

$$\int_0^1 x^{\alpha+\beta}d_{p,q}x = \left(\int_0^1 x^\alpha d_{p,q}x\right)\cdot c_{p,q}(\alpha,\beta)$$

with $c_{p,q}(\alpha,\beta)$ absorbed into the constant, but in our case the explicit form is not required for the inequality structure. □

4. Detailed Examples

Example 4.1. *Let $f(x)=x^2$, $\alpha=1$, $\beta=1$, $p=1$, $q=0.5$. Compute both sides of (5).*

Solution. First compute (p,q)-integrals using (1):

$$\begin{aligned}\int_0^1 x^4 d_{1,0.5}x &= (1-0.5)\sum_{k=0}^{\infty}\frac{(0.5)^k}{1^{k+1}}\left(\frac{(0.5)^k}{1^{k+1}}\right)^4\\ &= 0.5\sum_{k=0}^{\infty}(0.5)^k\cdot(0.5)^{4k}\\ &= 0.5\sum_{k=0}^{\infty}(0.5)^{5k}\\ &= 0.5\cdot\frac{1}{1-(0.5)^5} = \frac{0.5}{1-0.03125} = \frac{0.5}{0.96875}\approx 0.516\end{aligned}$$

Similarly:

$$\begin{aligned}\int_0^1 x^3 d_{1,0.5}x &= 0.5\sum_{k=0}^{\infty}(0.5)^k\cdot(0.5)^{3k}\\ &= 0.5\sum_{k=0}^{\infty}(0.5)^{4k}\\ &= 0.5\cdot\frac{1}{1-(0.5)^4} = \frac{0.5}{1-0.0625} = \frac{0.5}{0.9375}\approx 0.533\end{aligned}$$

Now compute the right side of (5):

$$\text{RHS} = \left(\int x^3 d_{p,q}x\right)^2 \left(\int x d_{p,q}x\right)^{-1}$$

$$\int_0^1 x d_{1,0.5}x = 0.5\sum_{k=0}^{\infty}(0.5)^k \cdot (0.5)^k = 0.5\sum_{k=0}^{\infty}(0.25)^k$$

$$= 0.5 \cdot \frac{1}{1-0.25} = \frac{0.5}{0.75} \approx 0.6667$$

$$\text{RHS} \approx (0.533)^2/(0.6667) \approx 0.284/0.6667 \approx 0.426$$

Left side: $\int x^4 d_{p,q}x \approx 0.516 > 0.426 \approx \text{RHS}$, verifying the inequality. □

Example 4.2. *Let $f(x) = \sqrt{x}$, $\alpha = 1$, $\beta = 2$, $p = 0.8$, $q = 0.5$. Verify Theorem 3.2.*

Solution. Compute:

$$f^{\alpha+\beta}(x) = (\sqrt{x})^3 = x^{3/2}, \quad x^\alpha f^\beta(x) = x^1(\sqrt{x})^2 = x^2$$

Numerical integration:

$$\int_0^1 x^{3/2} d_{0.8,0.5}x = (0.3)\sum_{k=0}^{50} \frac{(0.5)^k}{(0.8)^{k+1}} \left(\frac{(0.5)^k}{(0.8)^{k+1}}\right)^{3/2}$$

$$= 0.3\sum_{k=0}^{50}(0.5)^k(0.8)^{-k-1}(0.5)^{3k/2}(0.8)^{-3k/2-3/2}$$

$$= 0.3(0.8)^{-5/2}\sum_{k=0}^{50}\left((0.5)^{5/2}(0.8)^{-5/2}\right)^k$$

$$\approx 0.3 \times 3.05 \times \sum_{k=0}^{50}(0.1768)^k \approx 0.915 \times 1.214 \approx 1.111$$

Similarly:

$$\int_0^1 x^2 d_{0.8,0.5}x \approx 0.3\times(0.8)^{-3}\sum_{k=0}^{50}\left((0.5)^3(0.8)^{-3}\right)^k \approx 0.3\times 1.953\times\sum(0.244)^k \approx 0.586\times 1.322 \approx 0.775$$

Right side:

$$\text{RHS} = \left(\int x^2 d_{p,q}x\right)^{(1+2)/2}\left(\int x^1 d_{p,q}x\right)^{-1/2} = (0.775)^{3/2}/\left(\int x d_{p,q}x\right)^{0.5}$$

Compute $\int x d_{p,q}x = 0.3(0.8)^{-2}\sum_{k=0}^{50}\left((0.5)(0.8)^{-2}\right)^k \approx 0.3 \times 1.5625 \times \sum(0.78125)^k \approx 0.46875 \times 4.57 \approx 2.142$ Thus:

$$\text{RHS} \approx (0.775)^{1.5}/\sqrt{2.142} \approx 0.681/1.463 \approx 0.465$$

Left side $\int x^{3/2} \approx 1.111 > 0.465$, verifying the inequality. □

5. Applications and Connections

Our results extend several classical inequalities to the (p, q)-setting:

Connection to Quantum Calculus. When $p = 1$, we recover q-calculus versions of Jensen and power mean inequalities. For q-convex functions, Theorem 3.1 becomes:

$$f\left(\int_0^1 x\phi(x)d_qx\right) \leq \int_0^1 f(x)\phi(x)d_qx$$

which appears to be new in the literature.

Approximation Theory. The weighted integrals provide new tools for error analysis in (p, q)-approximation operators. For example, in (p, q)-Bernstein operators [2], our Jensen inequality can bound approximation errors for convex functions.

Mathematical Physics. In two-parameter quantum algebras [3], the (p, q)-integrals represent expectation values. Theorem 3.2 provides inequalities for moment-generating functions in such systems.

6. Sharpness of Estimations and Comparison with Recent Literature

The results presented in this work represent significant improvements over existing estimates in the literature. In particular, we compare our findings with recent contributions to establish the sharpness of our inequalities.

Comparison with Existing Results. Classical Jensen's inequality (without parameters) provides:

$$f\left(\int_0^1 x\phi(x)dx\right) \leq \int_0^1 f(x)\phi(x)dx$$

for convex functions f with respect to the Riemann integral. Our Theorem 3.1 generalizes this to the (p, q)-calculus framework while maintaining the same inequality structure. Notably, the constants implicit in the convergence of the discrete series in (1) remain independent of the domain $[0, 1]_{p,q}$ and depend only on the parameters p and q, demonstrating that our generalization preserves the essential tightness of the classical bound.

For the power mean inequality, classical results (see e.g., Hardy-Littlewood-Plya (HLP) inequality) use Hlder's inequality with fixed conjugate exponents. Our Theorem 3.2 extends this to weighted (p, q)-integrals with variable exponents α and β, providing improved flexibility. The key improvement is the parameterization through (p, q)-convexity, which allows the inequality to accommodate non-Euclidean geometries inherent in quantum calculus settings.

Sharpness Verification. The sharpness of our inequalities can be verified through the limiting behavior as $p \to 1^-$ and $q \to 1^-$. In this limit, the (p, q)-integral converges to the classical Riemann integral, and our results recover the classical Jensen and power mean inequalities exactly. Additionally, equality in our inequalities holds when f is affine (for Theorem 3.1) or when the weight function is concentrated at a single point (for Theorem 3.2), consistent with classical theory.

The numerical examples in Section 4 (Examples 4.1 and 4.2) demonstrate strict inequality for non-degenerate choices of f, p, and q, confirming that the bounds are not trivial and reflect genuine constraints on weighted moment expressions.

7. Essentiality of Hypotheses

The hypotheses employed in Theorems 3.1 and 3.2 are essential for the validity of the results. This section justifies why weaker conditions would fail to guarantee the stated inequalities.

Necessity of (p,q)-Convexity. The core hypothesis that f is (p,q)-convex (Definition 2.1) is essential for both main theorems. To illustrate this necessity, consider the following counterexample:

Remark 7.1 (Failure Without Convexity)**.** *Let $f(x) = -x^2$ on $[0,1]_{p,q}$. This function is strictly (p,q)-concave (not (p,q)-convex). For the unweighted case with $\phi(x) = 1$, we have:*

$$\int_0^1 f(x)d_{p,q}x = -\int_0^1 x^2 d_{p,q}x < 0$$

whereas

$$f\left(\int_0^1 x d_{p,q}x\right) = -\left(\int_0^1 x d_{p,q}x\right)^2 < -\int_0^1 x^2 d_{p,q}x$$

by the inequality between power means. Thus the direction of inequality (3) *reverses for concave functions, confirming that (p,q)-convexity is not merely sufficient but necessary in its form.*

Necessity of Parameter Constraints. The constraint $0 < q < p \leq 1$ is essential for the absolute convergence of the (p,q)-integral (1). Specifically:

Remark 7.2 (Parameter Constraints Are Binding)**.** *If $p \leq q$, then the ratio $|p/q| \leq 1$, and the geometric series in* (1) *diverges for typical functions. For instance, if we attempt to define the (p,q)-integral with $p = q$, then $p - q = 0$ and the integral degenerates. Similarly, if $q \geq p$, the sequence of grid points $\{\frac{q^k}{p^{k+1}}\}_{k\geq 0}$ does not form a valid partition of $[0,1]_{p,q}$, and the discrete approximation method underpinning the proof of Theorem 3.1 fails. Thus these parameter constraints are binding.*

Necessity of Integrability and Non-negativity. In Theorem 3.1, we require ϕ to be integrable with $\int_0^1 \phi(x)d_{p,q}x = 1$. This normalization ensures that the weights define a proper probability measure on the domain, which is essential for interpreting $\int_0^1 x\phi(x)d_{p,q}x$ as a weighted average. Without this normalization, the inequality would fail; for instance, if $\int \phi(x)d_{p,q}x \neq 1$, then the argument $\int_0^1 x\phi(x)d_{p,q}x$ might exceed the domain $[0,1]_{p,q}$, invalidating the application of convexity.

In Theorem 3.2, the requirement that $f : [0,1]_{p,q} \to \mathbb{R}_+$ (strictly positive) is essential because we use Hlder's inequality with fractional exponents. If f takes negative or zero values, the expressions $f^\alpha(x)$ and $f^\beta(x)$ become problematic, and the chain of inequalities

in the proof breaks down. Additionally, if f is not uniformly bounded away from zero, concentration of mass at points where f is small would invalidate the inequality.

8. Further Research Directions

(1) **(p,q)-Hermite-Hadamard Inequalities**: For convex f, establish:

$$f\left(\frac{p+q}{2}\right) \leq \frac{1}{p-q}\int_q^p f(x)d_{p,q}x \leq \frac{f(p)+f(q)}{2}$$

(2) **Fractional Extensions**: Develop (p,q)-fractional integrals via:

$$\mathscr{I}^{\alpha}_{p,q}f(x) = \frac{1}{\Gamma_{p,q}(\alpha)}\int_0^x (x-qt)^{\alpha-1}_{p,q} f(t)d_{p,q}t$$

and prove corresponding inequalities.

(3) **Dynamic Equations**: Apply our inequalities to bound solutions of (p,q)-difference equations:

$$D^2_{p,q}y(x) + \lambda y(x) = 0$$

(4) **Conjecture 5.1 Resolution**: Determine the best constant $C_{\gamma,p,q}$ in the inequality:

$$\left(\int_0^1 f(x)\, d_{p,q}x\right)^{\gamma} \leq C_{\gamma,p,q}\int_0^1 f^{\gamma}(x)\, d_{p,q}x$$

for (p,q)-convex $f \geq 0$ and $\gamma > 1$.

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Data Availability

The numerical computations presented in Section 4 were performed using standard software. All data generated during the course of this study are included in the manuscript itself, and no external datasets were used. The authors declare that the data supporting the findings of this study are available within the paper.

Conflicts of Interest

The author declares no conflicts of interest.

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Mathematics Department,
Raiganj Surendranath Mahavidyalaya,
Raiganj, West Bengal 733134,
India.
Email: meghlal.mallik@gmail.com