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ON THE INTEGRAL REPRESENTATION OF RELATIVE GROWTH INDICATORS OF ENTIRE ALGEBROIDAL FUNCTIONS OF HIGHER INDEX

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Abstract. In this paper, we study the relative growth indicators of bicomplex valued entire algebroidal functions, their integral representation and some related results. A few examples are provided to justify the methodology of the work as carried out in the paper.

Keywords: Bicomplex valued entire function, Relative order, Relative lower order, Idempotent representation.

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1. Introduction, Definitions and Notations

The *order* and *lower order* of an entire function f which is generally used in computational purposes are classical in complex analysis. L. Bernal.[1] and [2], introduced the *relative order* (respectively *relative lower order)* between two entire functions to avoid comparing growth just with $\exp z$. Extending the notion of *relative order* (respectively *relative lower order)* Ruiz et. al [9] introduced the *relative* (p,q)*-th order* (respectively *relative lower* (p,q)*-th order)* where p and q are any two positive integers. Now We extend these definitions to bicomplex and multicomplex valued entire functions. To compare the growth of bicomplex and multicomplex valued entire functions having the same *relative* (p,q)*-th order* or *relative lower* (p,q)*-th order*, We introduce the definition of *relative* (p,q)*-th type* and *relative* (p,q)*-th weak type* of bicomplex and multicomplex entire functions with respect to bicomplex and multicomplex entire algebroidal functions and established their respective integral representations. We also investigate the equivalence of the computational definitions and their corresponding integral representations of the relative growth indicators as stated above in case of bicomplex and multicomplex valued entire algebroidal functions.

1.1. **The Bicomplex Number System {cf.** [7]**}.** A bicomplex number $\mathfrak{z}$ can be represented in the form

$$\mathfrak{z} = \mathfrak{x}_1 + i_1\mathfrak{x}_2 + i_2\mathfrak{x}_3 + (i_1i_2)\,\mathfrak{x}_4 = (\mathfrak{x}_1 + i_1\mathfrak{x}_2) + i_2(\mathfrak{x}_3 + i_1\mathfrak{x}_4) = \mathfrak{z}_1 + i_2\,\mathfrak{z}_2,$$

where each $\mathfrak{x}_k$ $(k = 1,\ldots,4)$ is a real number, $i_1^2 = i_2^2 = -1$ and $i_1i_2 = i_2i_1$. We set $j = i_1i_2$, so that $j^2 = +1$. Here $\mathfrak{z}_1 = \mathfrak{x}_1 + i_1\mathfrak{x}_2$ and $\mathfrak{z}_2 = \mathfrak{x}_3 + i_1\mathfrak{x}_4$ lie in the complex plane $\mathbb{C}$.

We denote the collection of real numbers by $\mathbb{C}_0$, the usual complex numbers by $\mathbb{C}_1$, and the bicomplex numbers by $\mathbb{C}_2$.

1.2. **Algebraic Structure of $\mathbb{C}_2$ {cf. [7]}.** We equip the bicomplex field $\mathbb{C}_2$ with addition and scalar multiplication operations and a norm as follows.
Addition and Scalar Multiplication: For $\mathfrak{z} = (\mathfrak{x}_1 + i_1\mathfrak{x}_2 + i_2\mathfrak{x}_3 + i_1i_2\mathfrak{x}_4)$ and $\mathfrak{w} = (\mathfrak{y}_1 + i_1\mathfrak{y}_2 + i_2\mathfrak{y}_3 + i_1i_2\mathfrak{y}_4)$ in $\mathbb{C}_2$, define

$$\mathfrak{z} \oplus \mathfrak{w} = (\mathfrak{x}_1 + \mathfrak{y}_1) \ + \ i_1\,(\mathfrak{x}_2 + \mathfrak{y}_2) \ + \ i_2\,(\mathfrak{x}_3 + \mathfrak{y}_3) \ + \ i_1i_2\,(\mathfrak{x}_4 + \mathfrak{y}_4).$$

Given a scalar $\mathfrak{a} \in \mathbb{C}_1$, scalar multiplication is

$$\mathfrak{a} \odot \mathfrak{z} = \mathfrak{a}\,\mathfrak{x}_1 \ + \ i_1\,\mathfrak{a}\,\mathfrak{x}_2 \ + \ i_2\,\mathfrak{a}\,\mathfrak{x}_3 \ + \ i_1i_2\,\mathfrak{a}\,\mathfrak{x}_4.$$

With these operations, $(\mathbb{C}_2, \oplus, \odot)$ is a linear space.
Norm and Completeness: Introduce the norm as

$$\left\|\mathfrak{x}_1 + i_1\mathfrak{x}_2 + i_2\mathfrak{x}_3 + i_1i_2\mathfrak{x}_4\right\| = \sqrt{\mathfrak{x}_1^2 + \mathfrak{x}_2^2 + \mathfrak{x}_3^2 + \mathfrak{x}_4^2}.$$

Observe that $\mathbb{C}_2$ can be identified with $\mathbb{C}_0^4$ via

$$\mathfrak{x}_1 + i_1\mathfrak{x}_2 + i_2\mathfrak{x}_3 + i_1i_2\mathfrak{x}_4 \ \longleftrightarrow \ (\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3, \mathfrak{x}_4).$$

Equipping $\mathbb{C}_0^4$ with the standard Euclidean norm makes it a complete normed linear space or a Banach space. Because the norm on $\mathbb{C}_2$ coincides with this Euclidean norm under the above correspondence, $\big(\mathbb{C}_2, \oplus, \odot, \|\cdot\|\big)$ is itself a Banach space.
Bicomplex Multiplication: Define the bilinear product $\otimes : \mathbb{C}_2 \times \mathbb{C}_2 \to \mathbb{C}_2$ by

$$\begin{aligned}\mathfrak{z} \ \otimes \ \mathfrak{w} = {} & (\mathfrak{x}_1\mathfrak{y}_1 - \mathfrak{x}_2\mathfrak{y}_2 - \mathfrak{x}_3\mathfrak{y}_3 + \mathfrak{x}_4\mathfrak{y}_4) \\ & + \ i_1\,(\mathfrak{x}_1\mathfrak{y}_2 + \mathfrak{x}_2\mathfrak{y}_1 - \mathfrak{x}_3\mathfrak{y}_4 - \mathfrak{x}_4\mathfrak{y}_3) \\ & + \ i_2\,(\mathfrak{x}_1\mathfrak{y}_3 + \mathfrak{x}_3\mathfrak{y}_1 - \mathfrak{x}_2\mathfrak{y}_4 - \mathfrak{x}_4\mathfrak{y}_2) \\ & + \ i_1i_2\,(\mathfrak{x}_1\mathfrak{y}_4 + \mathfrak{x}_4\mathfrak{y}_1 + \mathfrak{x}_2\mathfrak{y}_3 + \mathfrak{x}_3\mathfrak{y}_2).\end{aligned}$$

Banach Algebra Property: One can verifies for all $\mathfrak{a} \in \mathbb{C}_1$, $\mathfrak{z}, \mathfrak{w} \in \mathbb{C}_2$:

$$\|\mathfrak{a} \odot \mathfrak{z}\| = |\mathfrak{a}|\,\|\mathfrak{z}\|, \qquad \|\mathfrak{z} \otimes \mathfrak{w}\| \ \leq \ \sqrt{2}\,\|\mathfrak{z}\|\,\|\mathfrak{w}\|.$$

Therefore, $(\mathbb{C}_2, \oplus, \odot, \otimes, \|\cdot\|)$ is a commutative Banach algebra.

1.3. **Idempotent Representation of Bicomplex Numbers {cf. [7]}.** The bicomplex number system $\mathbb{C}_2$ admits four idempotent elements:

$$0, \quad 1, \quad \frac{1 + i_1i_2}{2}, \quad \frac{1 - i_1i_2}{2}.$$

Among these, the two nontrivial idempotents are denoted by

$$e_1 = \frac{1 + i_1i_2}{2} \quad \text{and} \quad e_2 = \frac{1 - i_1i_2}{2},$$

which satisfy the following relations

$$e_1^2 = e_1, \quad e_2^2 = e_2, \quad e_1e_2 = e_2e_1 = 0, \quad e_1 + e_2 = 1.$$

The elements e_1 and e_2 are called orthogonal idempotents due to these properties.

Every bicomplex number $\mathfrak{z} = \mathfrak{z}_1 + i_2\mathfrak{z}_2 \in \mathbb{C}_2$, where $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathbb{C}_1$, can be uniquely expressed as

$$\mathfrak{z} = (\mathfrak{z}_1 - i_1\mathfrak{z}_2)e_1 + (\mathfrak{z}_1 + i_1\mathfrak{z}_2)e_2 = \xi_1 e_1 + \xi_2 e_2,$$

where $\xi_1, \xi_2 \in \mathbb{C}_1$. This decomposition is known as the idempotent representation of $\mathfrak{z}$.

A bicomplex number $\mathfrak{z} = \mathfrak{z}_1 + i_2\mathfrak{z}_2 \in \mathbb{C}_2$ is said to be non-singular if $|\mathfrak{z}_1^2 + \mathfrak{z}_2^2| \neq 0$, and singular otherwise. We denote the set of all singular elements in $\mathbb{C}_2$ by θ_2.

1.4. **Bicomplex Holomorphic Functions {cf.** [7]**}.** Let

$$f : \Omega \subset \mathbb{C}_2 \longrightarrow \mathbb{C}_2$$

be a bicomplex valued function. We say f is differentiable at $\omega_0 \in \Omega$ if the limit

$$f'(\omega_0) = \lim_{\substack{\mathfrak{h}\to 0 \\ h\notin\theta_2}} \frac{f(\omega_0 + \mathfrak{h}) - f(\omega_0)}{\mathfrak{h}}$$

exists, where

$$\mathfrak{h} = \mathfrak{h}_0 + i_1\mathfrak{h}_1 + i_2\mathfrak{h}_2 + i_1 i_2\mathfrak{h}_3$$

is required to be invertible in $\mathbb{C}_2$ (i.e. $\mathfrak{h} \notin \theta_2$).

If f is differentiable at every point of Ω, it is called bicomplex valued holomorphic function. Writing

$$\omega = \mathfrak{z}_1 + i_2\mathfrak{z}_2, \quad f(\omega) = g_1(\mathfrak{z}_1, \mathfrak{z}_2) + i_2\, g_2(\mathfrak{z}_1, \mathfrak{z}_2),$$

with $z_1, z_2 \in \mathbb{C}_1$, one can show that f is holomorphic when both g_1 and g_2 are holomorphic in $\mathbb{C}_1$ and satisfy the Cauchy-Riemann equations

$$\frac{\partial g_1}{\partial \mathfrak{z}_1} = \frac{\partial g_2}{\partial \mathfrak{z}_2}, \qquad \frac{\partial g_1}{\partial \mathfrak{z}_2} = -\frac{\partial g_2}{\partial \mathfrak{z}_1}.$$

In this case, the derivative takes the form

$$f'(\omega) = \frac{\partial g_1}{\partial \mathfrak{z}_2} + i_2 \frac{\partial g_2}{\partial \mathfrak{z}_1}.$$

1.5. **Bicomplex Entire Functions and Growth Orders {cf.** [7]**}.** A function $f : \mathbb{C}_2 \to \mathbb{C}_2$ is called bicomplex valued entire function if it is holomorphic on the entire bicomplex plane $\mathbb{C}_2$.

Idempotent Representation of Bicomplex Valued Holomorphic Functions:

Let f_i's be bicomplex valued entire functions $\forall i = 0, 1, 2, \cdots, k$. Then f_i can be written as

$$f_i(z_1 + i_2 z_2) = f_{1,i}(z_1 - i_1 z_2)e_1 + f_{2,i}(z_1 + i_1 z_2)e_2,$$

Where idempotent components $f_{1,i}$ and $f_{2,i}$ are all complex valued entire functions $\forall i = 0, 1, 2, \cdots, k$ and $e_1,\ e_2$ are idempotent elements of $\mathbb{C}_2$.

Definition 1.1 (Order of a Bicomplex Valued Entire Function [3])**.** *$M_{f_{j,i}}(\mathfrak{r})$ denotes the maximal modulus of complex valued function $f_{j,i}$ on the circle $|\mathfrak{z}| = \mathfrak{r}$ and defined by*

$$M_{f_{j,i}}(\mathfrak{r}) = \max_{|\mathfrak{z}|=\mathfrak{r}} \big| f_{j,i}(\mathfrak{z}) \big|$$

The order of $f_{j,i}$ in the classical sense is defined by

$$\rho_{f_{j,i}} = \limsup_{\mathfrak{r}\to\infty} \frac{\log^{[2]} M_{f_{j,i}}(\mathfrak{r})}{\log \mathfrak{r}}, \quad j = 1, 2.$$

Then the order of the bicomplex valued entire function f_i is defined by

$$\rho(f_i) = \max\{\rho_{f_{1,i}}, \rho_{f_{2,i}}\}.$$

We now introduce the notions of relative order and relative lower order for bicomplex valued entire functions.

Bernal {[1],[2]} defined the relative order of a complex valued entire function f with respect to another complex valued entire function g as

$$\rho_g(f) = \inf\left\{\mu > 0 : M_g(\mathfrak{r}) < M_f(\mathfrak{r}^\mu) \quad \forall\, \mathfrak{r} > \mathfrak{r}_0(\mu) > 0\right\},$$

where $M_f(\mathfrak{r}) = \max\{|f(\mathfrak{z})| : |\mathfrak{z}| = \mathfrak{r}\}$ denotes the maximum modulus of f on the circle of radius $\mathfrak{r}$. Equivalently, it can be expressed as

$$\rho_g(f) = \limsup_{\mathfrak{r}\to\infty} \frac{\log^{[2]} M_f^{-1} M_g(\mathfrak{r})}{\log \mathfrak{r}}.$$

By replacing the lim sup with lim inf, one obtains the definition of the relative lower order, denoted by $\lambda_g(f)$, which captures the lower growth behavior of f relative to g.

Now, consider two bicomplex valued entire functions f and g, represented in their idempotent forms as

$$f(\mathfrak{z}_1 + i_2\mathfrak{z}_2) = f_1(\mathfrak{z}_1 - i_1\mathfrak{z}_2)e_1 + f_2(\mathfrak{z}_1 + i_1\mathfrak{z}_2)e_2,$$
$$g(\mathfrak{z}_1 + i_2\mathfrak{z}_2) = g_1(\mathfrak{z}_1 - i_1\mathfrak{z}_2)e_1 + g_2(\mathfrak{z}_1 + i_1\mathfrak{z}_2)e_2,$$

where idempotent components f_1, f_2, g_1, g_2 are complex valued entire functions in $\mathbb{C}_1$.

The relative order of the bicomplex valued entire function f with respect to g is defined as

$$\rho_g(f) = \max\left\{\rho_{g_1}(f_1), \rho_{g_2}(f_2)\right\},$$

and the relative lower order is given by

$$\lambda_g(f) = \max\left\{\lambda_{g_1}(f_1), \lambda_{g_2}(f_2)\right\}.$$

Furthermore, if the relative order and relative lower order of a bicomplex valued entire function f with respect to g are equal, then f is said to exhibit regular relative growth with respect to g.

Example 1.2. *Let $f(z_1 + i_2z_2) = \exp(z_1 + i_2z_2)$ and $g(z_1 + i_2z_2) = \exp(7(z_1 + i_2z_2))$ be two bicomplex valued entire functions. Then,*

$$f(z_1 + i_2z_2) = f_1(z_1 - i_1z_2)e_1 + f_2(z_1 + i_1z_2)e_2$$
$$\text{and} \quad g(z_1 + i_2z_2) = g_1(z_1 - i_1z_2)e_1 + g_2(z_1 + i_1z_2)e_2$$

where f_1, f_2, g_1, g_2 are complex valued entire functions.

Then for $\|z_1 + i_2z_2\| = r$ we can write,

$$|z_1 - i_1z_2| = |z_1 + i_1z_2| = r.$$

$$M_{f_1}(r) = e^r, \quad \text{and} \quad M_{g_1}(r) = e^{7r}.$$

So, $M_{g_1}^{-1}(M_{f_1}(r)) = \frac{1}{7}\log M_{f_1}(r)$.
Then one can easily verify that $\rho_{g_1}(f_1) = \rho_{g_2}(f_2) = 1$
Therefore, $\rho_g(f) = \max\{\rho_{g_1}(f_1), \rho_{g_2}(f_2)\} = 1$.

Example 1.3. *If* $f(z) = a + bz + cz^2$ *and* $g(z) = \exp(2z)$ *where* $a, b, c, z \in \mathbb{C}_2$, *then one can easily check that* $\rho_g(f) = 0$.

1.6. **Bicomplex Valued Entire Algebroidal Function.**

Definition 1.4. *Let* F *be a* k*-valued function defined by the following irreducible equation*

$$f_k F^k + f_{k-1}F^{k-1} + f_{k-2}F^{k-2} + \cdots + f_0 = 0$$

where $f_k \neq 0$ *and all* f_i $(i = 0, 1, 2, \ldots, k)$ *are bicomplex valued entire functions having no common zeros.*

If at least one of the f_i $(i = 0, 1, 2, \ldots, k)$ *is transcendental then* F *is called a* k*-valued algebroidal function of bicomplex numbers. Further, if* $f_k \equiv 1$ *then* F *is called a* k*-valued entire algebroidal function of bicomplex numbers.*

1.7. **Relative** (p, q)**-th order and relative** (p, q)**-th lower order of a bicomplex valued entire functions with respect to another bicomplex valued entire function.**

Definition 1.5. *Let us define the* relative (p, q) -th order *and* relative (p, q) -th lower order *of bicomplex valued entire functions* f_i *by*

$$\rho_G^{(p,q)}(f_i) = max\left\{\rho_G^{(p,q)}(f_{1,i}), \rho_G^{(p,q)}(f_{2,i})\right\} \quad \forall i = 0, 1, 2, \cdots, k.$$

and

$$\lambda_G^{(p,q)}(f_i) = max\left\{\lambda_G^{(p,q)}(f_{1,i}), \lambda_G^{(p,q)}(f_{2,i})\right\} \quad \forall i = 0, 1, 2, \cdots, k.$$

where idempotent components $f_{1,i}$ *and* $f_{2,i}$ *are all complex valued entire functions* $\forall i = 0, 1, 2, \cdots, k$, *with* p *and* q *being any two positive integers and*

$$\rho_G^{(p,q)}(f) = \limsup_{r\to\infty}\frac{\log^{[p]} M_G^{-1}M_f(r)}{\log^{[q]} r} \quad \text{and} \quad \lambda_G^{(p,q)}(f) = \liminf_{r\to\infty}\frac{\log^{[p]} M_G^{-1}M_f(r)}{\log^{[q]} r},$$

for any two complex valued entire functions f *and* G.

Definition 1.6. *Let* f_i*'s* $(0 \leq i \leq k-1)$ *be bicomplex valued entire functions with index-pair* (m_1, q) *and* G *be any bicomplex valued entire algebroidal function with index-pair* (m_2, p) *where* $m_1 = m_2 = m$ *and* p, q, m *are all positive integers such that* $m \geq \max\{p, q\}$. *The relative* (p, q) *-th type of bicomplex valued entire functions* f_i *with respect to the bicomplex valued entire algebroidal function* G *having finite positive relative* (p, q) *-th order* $\rho_G^{(p,q)}(f_i)$ $\left(0 < \rho_G^{(p,q)}(f_i) < \infty\right)$ *is defined as :*

$$\sigma_G^{(p,q)}(f_i) = max\left\{\sigma_G^{(p,q)}(f_{1,i}), \sigma_G^{(p,q)}(f_{2,i})\right\} \quad \forall i = 0, 1, 2, \cdots, k.$$

where

$$\sigma_G^{(p,q)}(f_{j,i}) = \inf\left\{ \begin{array}{c} \phi > 0 : M_{f_{j,i}}(r) < M_G\left[\exp^{[p-1]}\left(\phi(\log^{[q-1]} r)^{\rho_G^{(p,q)}(f_{j,i})}\right)\right] \\ \text{for all } r > r_0(\phi) > 0 \end{array} \right\}$$

$$= \limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_{j,i}}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{j,i})}}$$

$$\forall j = 1, 2.$$

and idempotent components $f_{1,i}$ and $f_{2,i}$ are all complex valued entire functions $\forall i = 0, 1, 2, \cdots, k$

Definition 1.7. *Let f_i's $(i = 0, 1, 2, \ldots, k-1)$ have finite positive relative (p,q)-th lower order $\lambda_G^{(p,q)}(f_i)$ $\left(a < \lambda_G^{(p,q)}(f_i) < \infty\right)$ with respect to G. The* relative (p,q) -th weak type *of f_i with respect to G is denoted by $\tau_G^{(p,q)}(f_i)$ and defined as :*

$$\tau_G^{(p,q)}(f_i) = max\left\{\tau_G^{(p,q)}(f_{1,i}), \tau_G^{(p,q)}(f_{2,i})\right\} \quad \forall i = 0, 1, 2, \cdots, k.$$

where

$$\tau_G^{(p,q)}(f_{j,i}) = \liminf_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_{j,i}}(r)}{\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_{j,i})}}$$

$$\forall j = 1, 2.$$

and $f_i, G, p, q, f_{1,i}$ & $f_{2,i}$ are defined in a similar way as in Definition 1.6.

Definition 1.8. *Let f_i's $(i = 0, 1, 2, \ldots, k-1)$ have finite positive relative (p,q)-th order $\rho_G^{(p,q)}(f_i)$ $\left(a < \rho_G^{(p,q)}(f_i) < \infty\right)$. with respect to G. Then the* relative (p,q)-th lower type *of f_i with respect to G is denoted by $\overline{\sigma}_G^{(p,q)}(f_i)$ and defined as :*

$$\overline{\sigma}_G^{(p,q)}(f_i) = max\left\{\overline{\sigma}_G^{(p,q)}(f_{1,i}), \overline{\sigma}_G^{(p,q)}(f_{2,i})\right\} \quad \forall i = 0, 1, 2, \cdots, k.$$

where

$$\overline{\sigma}_G^{(p,q)}(f_{j,i}) = \liminf_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_{j,i}}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{j,i})}}$$

$$\forall j = 1, 2.$$

and $f_i, G, p, q, f_{1,i}$ & $f_{2,i}$ are defined in a similar way as in Definition 1.6.

Definition 1.9. *Let f_i's $(i = 0, 1, 2, \ldots, k-1)$ have finite positive relative (p,q)-th* lower order *$\lambda_G^{(p,q)}(f_i)$ $\left(a < \lambda_G^{(p,q)}(f_i) < \infty\right)$ with respect to G . Then the growth indicator $\overline{\tau}_G^{(p,q)}(f_i)$ of f_i with respect to G is defined as :*

$$\overline{\tau}_G^{(p,q)}(f_i) = max\left\{\overline{\tau}_G^{(p,q)}(f_{1,i}), \overline{\tau}_G^{(p,q)}(f_{2,i})\right\} \quad \forall i = 0, 1, 2, \cdots, k.$$

where

$$\overline{\tau}_G^{(p,q)}(f_{j,i}) = \limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_{j,i}}(r)}{\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_{j,i})}}$$
$$\forall j = 1, 2.$$

and $f_i, G, p, q, f_{1,i}$ & $f_{2,i}$ are defined in a similar way as in Definition 1.6.

2. Lemma

In this section we need the following lemma which is needed in sequel.

Lemma 2.1. *Let the integral $\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^A\right)\right]^{t+1}} dr\ (r_0 > 0)$ converges where $0 < A < \infty$. Then*

$$\lim_{r\to\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^A\right)\right]^{t}} = 0$$

where f is a complex valued entire function and G is a complex valued entire algebroidal function.

Proof. Since the integral $\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^A\right)\right]^{t+1}} dr\ (r_0 > 0)$ converges, then

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^A\right)\right]^{t+1}} dr < \varepsilon, \text{ if } r_0 > R(\varepsilon)\ .$$

Therefore,

$$\int_{r_0}^{\exp\left(\left(\log^{[q-1]} r_0\right)^A\right)+r_0} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^A\right)\right]^{t+1}} dr < \varepsilon\ .$$

Since $\log^{[p-2]} M_G^{-1} M_{f_i}(r)$ increases with r, so

$$\int_{r_0}^{\exp\left(\left(\log^{[q-1]} r_0\right)^A\right)+r_0} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^A\right)\right]^{t+1}} dr \geq$$

$$\frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r_0)}{\left[\exp\left(\left(\log^{[q-1]} r_0\right)^A\right)\right]^{t+1}} \cdot \left[\exp\left(\left(\log^{[q-1]} r_0\right)^A\right)\right]\ .$$

i.e., for all sufficiently large values of r,

$$\int_{r_0}^{\exp\left(\left(\log^{[q-1]} r_0\right)^A\right)+r_0} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^A\right)\right]^{t+1}} dr \geq \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r_0)}{\left[\exp\left(\left(\log^{[q-1]} r_0\right)^A\right)\right]^{t}},$$

so that

$$\frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r_0)}{\left[\exp\left(\left(\log^{[q-1]} r_0\right)^A\right)\right]^{t}} < \varepsilon \text{ if } r_0 > R(\varepsilon).$$

$$i.e., \lim_{r\to\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_i}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^A\right)\right]^{t}} = 0.$$

This proves the lemma. □

Remark 2.2. *Under the flavour of Lemma 2.1 Definition 1.6 can be viewed in the following way.*

The relative (p,q)-th type $\sigma_G^{(p,q)}(f_i)$ *of* f_i *with respect to* G *is defined as: The maximum between the following two integrals*

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_{1,i}}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}\right)\right]^{t+1}} dr \ (r_0 > 0)$$

converges for $t > \sigma_G^{(p,q)}(f_{1,i})$ *and diverges for* $t < \sigma_G^{(p,q)}(f_{1,i})$,

and

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_{2,i}}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{2,i})}\right)\right]^{t+1}} dr \ (r_0 > 0)$$

converges for $t > \sigma_G^{(p,q)}(f_{2,i})$ *and diverges for* $t < \sigma_G^{(p,q)}(f_{2,i})$,

where $f_i, G, p, q, f_{1,i}, f_{2,i}$ *and* $\rho_G^{(p,q)}(f_i)$ *are defined in a similar way as in Definition 1.6.*

Remark 2.3. *Under the flavour of Lemma 2.1 Definition 1.7 can be viewed in the following way.*

The relative (p,q)-th weak type $\tau_G^{(p,q)}(f_i)$ *of* f_i *with respect to* G *is defined as: The maximum between the following two integrals*

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_{1,i}}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_{1,i})}\right)\right]^{t+1}} dr \ (r_0 > 0)$$

converges for $t > \tau_G^{(p,q)}(f_{1,i})$ *and diverges for* $t < \tau_G^{(p,q)}(f_{1,i})$,

and

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_{2,i}}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_{2,i})}\right)\right]^{t+1}} dr \, (r_0 > 0)$$

converges for $t > \tau_G^{(p,q)}(f_{2,i})$ *and diverges for* $t < \tau_G^{(p,q)}(f_{2,i})$,

where $f_i, G, p, q, f_{1,i}, f_{2,i}$ *and* $\lambda_G^{(p,q)}(f_i)$ *are defined in a similar way as in Definition 1.7.*

Remark 2.4. *Under the flavour of Lemma 2.1 Definition 1.8 can be viewed in the following way.*

The relative (p,q)-th lower type $\overline{\sigma}_G^{(p,q)}(f_i)$ *of* f_i *with respect to* G *is defined as: The maximum between the following two integrals*

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_{1,i}}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}\right)\right]^{t+1}} dr \, (r_0 > 0)$$

converges for $t > \overline{\sigma}_G^{(p,q)}(f_{1,i})$ *and diverges for* $t < \overline{\sigma}_G^{(p,q)}(f_{1,i})$,

and

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_{2,i}}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{2,i})}\right)\right]^{t+1}} dr \, (r_0 > 0)$$

converges for $t > \overline{\sigma}_G^{(p,q)}(f_{2,i})$ *and diverges for* $t < \overline{\sigma}_G^{(p,q)}(f_{2,i})$,

where $f_i, G, p, q, f_{1,i}, f_{2,i}$ *and* $\rho_G^{(p,q)}(f_i)$ *are defined in a similar way as in Definition 1.8.*

Remark 2.5. *Under the flavour of Lemma 2.1 Definition 1.9 can be viewed in the following way.*

The growth indicator $\overline{\tau}_G^{(p,q)}(f_i)$ *of* f_i *with respect to* G *is defined as: The maximum between the following two integrals*

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_{1,i}}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_{1,i})}\right)\right]^{t+1}} dr \, (r_0 > 0)$$

converges for $t > \overline{\tau}_G^{(p,q)}(f_{1,i})$ *and diverges for* $t < \overline{\tau}_G^{(p,q)}(f_{1,i})$,

and

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_{2,i}}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_G^{(p,q)}(f_{2,i})}\right)\right]^{t+1}} dr \, (r_0 > 0)$$

converges for $t > \overline{\tau}_G^{(p,q)}(f_{2,i})$ *and diverges for* $t < \overline{\tau}_G^{(p,q)}(f_{2,i})$.

where $f_i, G, p, q, f_{1,i}, f_{2,i}$ *and* $\lambda_G^{(p,q)}(f_i)$ *are defined in a similar way as in Definition 1.9.*

3. THEOREMS

Theorem 3.1. *Let f_i's $(i = 0, 1, 2, ...k-1)$ be bicomplex valued entire functions having finite positive* relative (p,q) -th order $\rho_G^{(p,q)}(f_i)$ $\left(0 < \rho_G^{(p,q)}(f_i) < \infty\right)$ *and* relative (p,q) -th type $\sigma_G^{(p,q)}(f_i)$ *with respect to a bicomplex valued entire algebroidal function G, where p and q are any two positive integers. Then Definition 1.6 and Remark 2.2 are equivalent.*

Proof. Let us consider f_i's $(i = 0, 1, 2, ...k-1)$ be bicomplex valued entire functions and G be a bicomplex valued entire algebroidal function such that $\rho_G^{(p,q)}(f_i)$ $\left(0 < \rho_G^{(p,q)}(f_i) < \infty\right)$ exists for any two positive integers p and q.

Case I. $\sigma_G^{(p,q)}(f_i) = \infty$.

Definition1.6 $\Rightarrow$ Remark 2.2.

As $\sigma_G^{(p,q)}(f_i) = \infty$, then at least one of $\sigma_G^{(p,q)}(f_{1,i})$ and $\sigma_G^{(p,q)}(f_{2,i})$ is equal to ∞. Without loss of generality let $\sigma_G^{(p,q)}(f_{1,i}) = \infty$. From Definition 1.6 we have for arbitrary positive C and for a sequence of values of r tending to infinity that

$$\log^{[p-1]} M_G^{-1} M_{f_{1,i}}(r) > C \cdot \left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}$$

$$i.e.,\ \log^{[p-2]} M_G^{-1} M_{f_{1,i}}(r) > \left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}\right)\right]^C . \tag{3.1}$$

If possible, let the integral $\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_{1,i}}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}\right)\right]^{C+1}} dr$ $(r_0 > 0)$ be converges.

Then by Lemma 2.1,

$$\limsup_{r\to\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_{1,i}}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}\right)\right]^C} = 0 .$$

So for all sufficiently large values of r,

$$\log^{[p-2]} M_G^{-1} M_{f_{1,i}}(r) < \left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}\right)\right]^C . \tag{3.2}$$

Therefore from Equations(3.1) and (3.2) , we arrive at a contradiction.

Hence $\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_{1,i}}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}\right)\right]^{C+1}} dr$ $(r_0 > 0)$ diverges whenever C is finite, which is Remark 2.2.

Remark 2.2 $\Rightarrow$ Definition 1.6.

Let C be any positive number. Since $\sigma_G^{(p,q)}(f_i) = \infty$, from Remark 2.2, at least one of the integrals $\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_{1,i}}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}\right)\right]^{C+1}} dr$ $(r_0 > 0)$ and $\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_{2,i}}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{2,i})}\right)\right]^{C+1}} dr$ $(r_0 > 0)$ is divergent.

Let $\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_{1,i}}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}\right)\right]^{C+1}} dr$ $(r_0 > 0)$ be divergent, then for arbitrary positive ε and for a sequence of values of r tending to infinity

$$\log^{[p-2]} M_G^{-1} M_{f_{1,i}}(r) > \left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}\right)\right]^{C-\varepsilon}$$

$$i.e.,\ \log^{[p-1]} M_G^{-1} M_{f_{1,i}}(r) > (C-\varepsilon)\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})},$$

which implies that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_{1,i}}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}} \geq C-\varepsilon\ .$$

Since $C > 0$ is arbitrary, it follows that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_{1,i}}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}} = \infty\ .$$

If we have considered the other integral to be divergent then in a similar way we will get

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_{2,i}}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{2,i})}} = \infty\ .$$

In either case $\sigma_G^{(p,q)}(f_i) = \infty$. Thus Definition 1.6 follows.

Case II. $0 \leq \sigma_G^{(p,q)}(f_i) < \infty$.

Definition 1.6 $\Rightarrow$ Remark2.2.

Subcase (A). $0 < \sigma_G^{(p,q)}(f_i) < \infty$. Then both $0 < \sigma_G^{(p,q)}(f_{1,i}), \sigma_G^{(p,q)}(f_{2,i}) < \infty$.

Let $\sigma_G^{(p,q)}(f_{1,i}) > \sigma_G^{(p,q)}(f_{2,i})$ Then according to Definition 1.6, for arbitrary positive ε and for all sufficiently large values of r, we obtain that

$$\log^{[p-1]} M_G^{-1} M_{f_{1,i}}(r) < \left(\sigma_G^{(p,q)}(f_{1,i}) + \varepsilon\right)\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}$$

$$i.e.,\ \log^{[p-2]} M_G^{-1} M_{f_{1,i}}(r) < \left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}\right)\right]^{\sigma_G^{(p,q)}(f_{1,i})+\varepsilon}$$

$$i.e.,\ \frac{\log^{[p-2]} M_G^{-1} M_{f_{1,i}}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}\right)\right]^{t}} < \frac{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}\right)\right]^{\sigma_G^{(p,q)}(f_{1,i})+\varepsilon}}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}\right)\right]^{t}}$$

$$i.e.,\ \frac{\log^{[p-2]} M_G^{-1} M_{f_{1,i}}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}\right)\right]^{t}} <$$

$$\frac{1}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}\right)\right]^{t-\sigma_G^{(p,q)}(f_{1,i})-\varepsilon}}.$$

Therefore $\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_{1,i}}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}\right)\right]^{t+1}} dr\ (r_0>0)$ converges for $t > \sigma_G^{(p,q)}(f_{1,i})$.

Again by Definition 1.6, we obtain for a sequence of values of r tending to infinity that

$$\log^{[p-1]} M_G^{-1} M_{f_{1,i}}(r) > \left(\sigma_G^{(p,q)}(f_{1,i}) - \varepsilon\right)\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}$$

$$i.e.,\ \log^{[p-2]} M_G^{-1} M_{f_{1,i}}(r) > \left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}\right)\right]^{\sigma_G^{(p,q)}(f_{1,i})-\varepsilon}. \tag{3.3}$$

So for $t < \sigma_G^{(p,q)}(f_{1,i})$, we get from Equation(3.3) that

$$\frac{\log^{[p-2]} M_G^{-1} M_{f_{1,i}}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}\right)\right]^{t}} > \frac{1}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}\right)\right]^{t-\left(\sigma_G^{(p,q)}(f_{1,i})-\varepsilon\right)}}.$$

Therefore $\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_{1,i}}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}\right)\right]^{t+1}} dr\ (r_0>0)$ diverges for $t < \sigma_G^{(p,q)}(f_{1,i})$.

Hence $\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_{1,i}}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}\right)\right]^{t+1}} dr\ (r_0>0)$ converges for $t > \sigma_G^{(p,q)}(f_{1,i})$ and diverges for $t < \sigma_G^{(p,q)}(f_{1,i})$.

A similar result will arise if we take $\sigma_G^{(p,q)}(f_{1,i}) < \sigma_G^{(p,q)}(f_{2,i})$

Subcase (B). $\sigma_G^{(p,q)}(f_i) = 0$ both $\sigma_G^{(p,q)}(f_{1,i}) = 0$ *and* $\sigma_G^{(p,q)}(f_{2,i}) = 0$.

Then Definition 1.6 gives for all sufficiently large values of r that

$$\frac{\log^{[p-1]} M_G^{-1} M_{f_{j,i}}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{j,i})}} < \varepsilon \quad \forall j = 1, 2.$$

Then as before we obtain that $\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_{j,i}}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{j,i})}\right)\right]^{t+1}} dr\ (r_0>0)$ converges for $t>0$ and diverges for $t<0$, $\forall j = 1, 2$.

Thus combining Subcase (A) and Subcase (B), Remark 2.2 follows.

Remark 2.2 $\Rightarrow$ Definition 1.6.

From Remark 2.2 and for arbitrary positive ε, both integrals

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_{1,i}}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}\right)\right]^{\sigma_G^{(p,q)}(f_{1,i})+\varepsilon+1}} dr\ (r_0>0)$$

and

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_{2,i}}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{2,i})}\right)\right]^{\sigma_G^{(p,q)}(f_{2,i})+\varepsilon+1}} dr \,(r_0>0)$$

converges. Then by Lemma 2.1, we get that

$$\limsup_{r\to\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_{1,i}}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}\right)\right]^{\sigma_G^{(p,q)}(f_{1,i})+\varepsilon}} = 0 \ .$$

So we obtain all sufficiently large values of r that

$$\frac{\log^{[p-2]} M_G^{-1} M_{f_{1,i}}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}\right)\right]^{\sigma_G^{(p,q)}(f_{1,i})+\varepsilon}} < \varepsilon$$

$$i.e.,\ \log^{[p-2]} M_G^{-1} M_{f_{1,i}}(r) < \varepsilon \cdot \left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}\right)\right]^{\sigma_G^{(p,q)}(f_{1,i})+\varepsilon}$$

$$i.e.,\ \log^{[p-1]} M_G^{-1} M_{f_{1,i}}(r) < \log\varepsilon + \left(\sigma_G^{(p,q)}(f_{1,i}) + \varepsilon\right)\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}$$

$$i.e.,\ \limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_{1,i}}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}} \le \sigma_G^{(p,q)}(f_{1,i}) + \varepsilon \ .$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M_G^{-1} M_{f_{1,i}}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}} \le \sigma_G^{(p,q)}(f_{1,i}) \,. \tag{3.4}$$

On the other hand, the divergence of the integral

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} M_G^{-1} M_{f_{1,i}}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}\right)\right]^{\sigma_G^{(p,q)}(f_{1,i})-\varepsilon+1}} dr \,(r_0>0)$$

implies that there exists a sequence of values of r tending to infinity such that

$$\frac{\log^{[p-2]} M_G^{-1} M_{f_{1,i}}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}\right)\right]^{\sigma_G^{(p,q)}(f_{1,i})-\varepsilon+1}} > \frac{1}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}\right)\right]^{1+\varepsilon}}$$

$$i.e.,\ \log^{[p-2]} M_G^{-1}M_{f_{1,i}}(r) > \left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}\right)\right]^{\sigma_G^{(p,q)}(f_{1,i})-2\varepsilon}$$

$$i.e.,\ \log^{[p-1]} M_G^{-1}M_{f_{1,i}}(r) > \left(\sigma_G^{(p,q)}(f_{1,i})-2\varepsilon\right)\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}\right)$$

$$i.e.,\ \frac{\log^{[p-1]} M_G^{-1}M_{f_{1,i}}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}} > \left(\sigma_G^{(p,q)}(f_{1,i})-2\varepsilon\right)\ .$$

As $\varepsilon\,(>0)$ is arbitrary, it follows from the above that

$$\limsup_{r\to\infty}\frac{\log^{[p-1]} M_G^{-1}M_{f_{1,i}}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}} \geq \sigma_G^{(p,q)}(f_{1,i})\ . \tag{3.5}$$

So from Equations(3.4) and (3.5) , we obtain that

$$\limsup_{r\to\infty}\frac{\log^{[p-1]} M_G^{-1}M_{f_{1,i}}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{1,i})}} = \sigma_G^{(p,q)}(f_{1,i})\,.$$

By similar calculations from the convergence and divergence of the integral

$$\int_{r_0}^{\infty}\frac{\log^{[p-2]} M_G^{-1}M_{f_{2,i}}(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{2,i})}\right)\right]^{\sigma_G^{(p,q)}(f_{2,i})+\varepsilon+1}}dr\ (r_0>0)$$

we will arrive to

$$\limsup_{r\to\infty}\frac{\log^{[p-1]} M_G^{-1}M_{f_{2,i}}(r)}{\left(\log^{[q-1]} r\right)^{\rho_G^{(p,q)}(f_{2,i})}} = \sigma_G^{(p,q)}(f_{2,i})\,.$$

Thus, Remark 2.2 follows.
This proves the theorem. □

Theorem 3.2. *Let f_i's $(i=0,1,2,.....k-1)$ be bicomplex valued entire functions having finite positive* relative (p,q) -th lower order $\lambda_G^{(p,q)}(f_i)$ $\left(0<\lambda_G^{(p,q)}(f_i)<\infty\right)$ *and* relative (p,q) -th weak type $\tau_G^{(p,q)}(f_i)$ *with respect to a bicomplex valued entire algebroidal function G, where p and q are any two positive integers. Then Definition 1.7 and Remark 2.3 are equivalent.*

Proof. We omit the proof of Theorem 3.2 as it can be carried similar to that of Theorem 3.1, using Lemma 2.1 and idempotent decomposition of f_i. □

Theorem 3.3. *Let f_i's $(i=0,1,2,.....k-1)$ be bicomplex valued entire functions having finite positive* relative (p,q) -th order $\rho_G^{(p,q)}(f_i)$ $\left(0<\rho_G^{(p,q)}(f_i)<\infty\right)$ *and* relative (p,q) -th lower type $\overline{\sigma}_G^{(p,q)}(f_i)$ *with respect to a bicomplex valued entire algebroidal function G, where p and q are any two positive integers. Then Definition 1.8 and Remark 2.4 are equivalent.*

Proof. The proof of the Theorem 3.3 is omitted because it is similar to that of Theorem 3.1, using Lemma 2.1 and idempotent decomposition of f_i. □

Theorem 3.4. *Let f_i's $(i = 0, 1, 2, ...k-1)$ be bicomplex valued entire functions having finite positive* relative (p, q) -th lower order $\lambda_G^{(p,q)}(f_i)$ $\left(0 < \lambda_G^{(p,q)}(f_i) < \infty\right)$ *and the growth indicator $\overline{\tau}_G^{(p,q)}(f_i)$ with respect to a bicomplex valued entire algebroidal function G, where p and q are any two positive integers. Then Definition 1.9 and Remark 2.5 are equivalent.*

Proof. Using Lemma 2.1 and idempotent decomposition of f_i, the proof of Theorem 3.4 can similarly be derived as the proof of Theorem 3.1. □

4. Future Prospect

In the line of the works as carried out in this paper one may think of finding out the inter-relationships of relative growth indicators of multicomplex valued entire and meromorphic functions. As a consequence, the derivation of relevant results is still virgin and may be posed as an open problem to the future researchers of this area.

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