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ON MAXIMUM MODULUS THEOREM OF MULTICOMPLEX VALUED FUNCTIONS

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Abstract. In this paper, we consider the multicomplex space $\mathbb{C}_n$ which is a special generalization of bicomplex space. We would like to derive the maximum modulus principle and the minimum modulus principle in terms of multicomplex valued functions. In addition, we also find the analogue of Taylor's series and Schwarz's lemma in terms of multicomplex valued functions.

Keywords: Multicomplex space, multicomplex function, holomorphic function, maximum modulus principle, minimum modulus principle.

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1. Introduction and Preliminaries

The theory of bicomplex numbers is a matter of active research for quite a long time since seminal work as carried in [18] and [1] in search of special algebra. The algebra of bicomplex numbers is widely used in the literature as it becomes a viable commutative alternative [12] to the non-skew field of quaternions introduced by Hamilton [10] (both are four-dimensional and generalizations of complex numbers). Set of all real numbers $\mathbb{R}$ and complex numbers $\mathbb{C}$ are denoted by $\mathbb{C}_0$ and $\mathbb{C}_1$, respectively. The set of all bicomplex numbers is denoted by $\mathbb{C}_2$. Now we will discuss some basic definitions and preliminaries of bicomplex analysis.

A bicomplex number is defined as

$$z \;=\; x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 \;=\; (x_1 + i_1x_2) + i_2\big(x_3 + i_1x_4\big) \;=\; z_1 + i_2z_2, \qquad (1)$$

where $x_i,\ i = 1, 2, 3, 4$, are real numbers with

$$i_1^2 = i_2^2 = -1, \qquad i_1i_2 = i_2i_1, \qquad (i_1i_2)^2 = 1,$$

and z_1, z_2 are complex numbers.

There are four idempotent elements in $\mathbb{C}_2$, which are

$$0, \quad 1, \quad \frac{1 + i_1i_2}{2}, \quad \frac{1 - i_1i_2}{2}.$$

Furthermore, for every $z_1 + i_2z_2 \in \mathbb{C}_2$,

$$z_1 + i_2z_2 = (z_1 - i_1z_2)\left(\frac{1 + i_1i_2}{2}\right) + (z_1 + i_1z_2)\left(\frac{1 - i_1i_2}{2}\right).$$

Here, addition and scalar multiplication are defined in the usual way. $\mathbb{C}_2$ is a linear space with respect to addition and scalar multiplication.

Here, $\|\cdot\|_2$ denotes the norm of elements in $\mathbb{C}_2$ and defined as

$$\|z_1 + i_2 z_2\|_2 = (|z_1|^2 + |z_2|^2)^{1/2} = (x_1^2 + x_2^2 + x_3^2 + x_4^2)^{1/2},$$
$$\|z_1 + i_2 z_2\|_2 = \left(\frac{|z_1 - i_1 z_2|^2 + |z_1 + i_1 z_2|^2}{2}\right)^{1/2}.$$

With this norm, $\mathbb{C}_2$ is a normed linear space.

In addition, $\mathbb{C}_2$ embeds in $\mathbb{C}_0^4$ and $\mathbb{C}_0^4$ is complete. Therefore, $\mathbb{C}_2$ is complete and thus a Banach space. For further study one can see [2], [5], [6], [7], [8] and [9].

2. Definitions

2.1. **The Multicomplex Space $\mathbb{C}_n$[15].** Now we define the space $\mathbb{C}_n$ for $n = 0, 1, 2, \ldots$. $\mathbb{C}_n$ is not just a replacement of 2 by n; it is more than that. Many new relationships come to light in $\mathbb{C}_n$ and many challenges arise in their treatment. An element $\xi_n \in \mathbb{C}_n$ can be represented as a linear combination of elements in $\mathbb{C}_0, \mathbb{C}_1, \ldots, \mathbb{C}_{n-1}$. Corresponding to each such representation of the elements in $\mathbb{C}_{n-1}$ there is a linear space in which $\mathbb{C}_n$ can be embedded.

A multicomplex number is defined by

$$\xi_n \;=\; \xi_{n-1,1} \;+\; i_n\, \xi_{n-1,2},$$

where

$$\xi_{n-1,1},\, \xi_{n-1,2} \in \mathbb{C}_{n-1}, \quad (i_n)^2 = -1.$$

The addition is the operation on $\mathbb{C}_n$ defined by the function

$$\oplus : \mathbb{C}_n \times \mathbb{C}_n \;\longrightarrow\; \mathbb{C}_n$$

given by

$$\xi'_n \oplus \xi''_n = (\xi'_{n-1,1} + i_n\, \xi'_{n-1,2}) + (\xi''_{n-1,1} + i_n\, \xi''_{n-1,2}) = (\xi'_{n-1,1} + \xi''_{n-1,1}) + i_n \left(\xi'_{n-1,2} + \xi''_{n-1,2}\right).$$

Multiplication is the operation on $\mathbb{C}_n$ defined by the function

$$\odot : \mathbb{C}_n \times \mathbb{C}_n \;\longrightarrow\; \mathbb{C}_n$$

given by

$$\xi'_n \odot \xi''_n = \left(\xi'_{n-1,1}\, \xi''_{n-1,1} - \xi'_{n-1,2}\, \xi''_{n-1,2}\right) + i_n \left(\xi'_{n-1,1}\, \xi''_{n-1,2} + \xi'_{n-1,2}\, \xi''_{n-1,1}\right), \quad \text{where } (i_n)^2 = -1.$$

The norm on $\mathbb{C}_n$ is defined by the the function

$$\|\cdot\|_n : \mathbb{C}_n \;\longrightarrow\; \mathbb{R}_{\geq 0}, \qquad \|\xi_n\|_n^2 = \|\xi_{n-1,1}\|_{n-1}^2 \;+\; \|\xi_{n-1,2}\|_{n-1}^2.$$

So, the system $(\mathbb{C}_n, \oplus, \odot)$ is a normed linear space and $\mathbb{C}_n$ is isomorphic and isometric with $\mathbb{C}_0^{2^n}$.

2.2. **Idempotent elements in $\mathbb{C}_n$[15].** We know that in $\mathbb{C}_2$ the four elements

$$0, \quad 1, \quad \frac{1+i_1i_2}{2} \quad \text{and} \quad \frac{1-i_1i_2}{2}$$

are idempotent, and any element of $\mathbb{C}_2$ can be written as a linear combination of these idempotent elements.

Similarly, it is easy to verify that the following are idempotent elements in $\mathbb{C}_n$.

$$0, \quad 1, \quad \frac{1+i_1i_2}{2}, \frac{1-i_1i_2}{2}, \frac{1+i_1i_3}{2}, \frac{1-i_1i_3}{2}, \frac{1+i_2i_3}{2}, \frac{1-i_2i_3}{2}, \ldots, \frac{1+i_{n-1}i_n}{2}, \frac{1-i_{n-1}i_n}{2}.$$

For convenience in notation, define the symbols

$$e(i_ri_s) := \frac{1+i_ri_s}{2}, \qquad e(-i_ri_s) := \frac{1-i_ri_s}{2}, \quad \text{where } r, s \in \mathbb{N}.$$

2.3. **Idempotent representation**[15]**.** Let ξ be an element in $\mathbb{C}_n$ and let

$$\xi = \xi_1 + i_n\xi_2 \quad \text{with} \quad \xi_1, \xi_2 \in \mathbb{C}_{n-1}.$$

Then

$$\xi = \big(\xi_1 - i_{n-1}\,\xi_2\big)\, e\big(i_{n-1}i_n\big) + \big(\xi_1 + i_{n-1}\,\xi_2\big)\, e\big(-i_{n-1}i_n\big).$$

Let $\xi \in C_n$ and let

$$\xi = \xi_1 + i_n\xi_2,$$

where $\xi_1, \xi_2 \in C_{n-1}$. Then

$$\|\xi\|_n = \left(\frac{\|\xi_1 - i_{n-1}\xi_2\|_{n-1}^2 + \|\xi_1 + i_{n-1}\xi_2\|_{n-1}^2}{2}\right)^{1/2}.$$

2.4. **Singular Elements**[15]**.** An element $\xi \in C_n$ is called non-singular if and only if there exists a unique $\eta \in C_n$ such that $\xi\eta = 1$, otherwise ξ is singular. The set of all singular elements in C_n is denoted by

$$\Theta_n, \quad n = 0, 1, 2, \ldots.$$

Another way, the element $\xi = \xi_1 + i_n\xi_2 \in \mathbb{C}_n$ is non-singular if and only if both $\xi_1 - i_{n-1}\xi_2$ and $\xi_1 + i_{n-1}\xi_2$ are non-singular in C_{n-1}. The element ξ is singular if and only if at least one of the elements $\xi_1 - i_{n-1}\xi_2$ or $\xi_1 + i_{n-1}\xi_2$ is singular in C_{n-1}.

2.5. **Neighborhoods in $\mathbb{C}_n$**[15]**.** Suppose that

$$X_1 = \{\xi_1 - i_{n-1}\xi_2 : \xi_1, \xi_2 \in C_{n-1}\}, \quad X_2 = \{\xi_1 + i_{n-1}\xi_2 : \xi_1, \xi_2 \in C_{n-1}\}.$$

Then $X(\subseteq C_n)$ can be written as

$$X = \{\xi_1 + i_n\xi_2 : \xi_1 - i_{n-1}\xi_2 \in X_1,\ \xi_1 + i_{n-1}\xi_2 \in X_2\}.$$

Then neighborhood $N(a + i_nb, \varepsilon) \subseteq X$ is defined as

$$N(a + i_nb, \varepsilon) = \{\xi_1 + i_n\xi_2 \in C_n : \|(\xi_1 + i_n\xi_2) - (a + i_nb)\|_n < \varepsilon\}.$$

Alternatively, it can be written in the form

$$\{\xi_1 + i_n\xi_2 \in C_n : \|(\xi_1 - i_{n-1}\xi_2) - (a - i_{n-1}b)\|_{n-1} < \varepsilon, \|(\xi_1 + i_{n-1}\xi_2) - (a + i_{n-1}b)\|_{n-1} < \varepsilon\}.$$

2.6. **Holomorphic Functions in $\mathbb{C}_n$[15].** If a function $f : X \longrightarrow \mathbb{C}_n$, $X \subseteq \mathbb{C}_n$, $n \geq 1$, satisfies the strong Stolz condition at $\xi_0 \in X$, i.e. for all $n \geq 1$, there exists a constant α (which depends on ξ_0) in $\mathbb{C}_n$ and a function $r(f; \xi_0, \xi)$ defined in a neighborhood of ξ_0 and with values in $\mathbb{C}_n$, such that:

(i) $f(\xi) - f(\xi_0) = \alpha(\xi - \xi_0) + r(f; \xi_0, \xi)(\xi - \xi_0)$
(ii) $\lim\limits_{\xi \to \xi_0} r(f; \xi_0, \xi) = 0$

Then f is differentiable at ξ_0. If f is differentiable in a neighborhood of each point in domain $X(\subseteq \mathbb{C}_n)$, then f is called a holomorphic multicomplex-valued function in X.

2.7. **Idempotent Representation of Holomorphic Multicomplex Valued Functions**[15]**.** Let X be a domain in C_n, $n \geq 1$, and let f be a holomorphic function in $\mathbb{C}_n$ then there exists holomorphic functions

$$f_1 : X_1 \to \mathbb{C}_{n-1} \quad \text{and} \quad f_2 : X_2 \to \mathbb{C}_{n-1} \quad \text{where} \quad X_1, X_2 \subseteq \mathbb{C}_{n-1},$$

such that

$$f(\xi_1 + i_n\xi_2) = f_1(\xi_1 - i_{n-1}\xi_2)e(i_{n-1}i_n) + f_2(\xi_1 + i_{n-1}\xi_2)e(-i_{n-1}i_n).$$

2.8. **Power Series in $\mathbb{C}_n$[15].** Let γ_k, $k = 0, 1, \ldots$, be constants in C_n, and let ζ and ζ_0 denote elements in C_n. Then a power series about ζ_0 in C_n is an infinite series of the form

$$\sum_{k=0}^{\infty} \gamma_k(\zeta - \zeta_0)^k. \tag{2}$$

The study of the convergence of (2) is based on the idempotent representation of the infinite series. In order to state the relevant theorem, it is necessary to represent (2) in the following notation:

$$\sum_{k=0}^{\infty} (c_k + i_n d_k)[(\zeta_1 + i_n\zeta_2) - (a_1 + i_n a_2)]^k. \tag{3}$$

Now by Idempotent representation,

$$c_k + i_n d_k = (c_k - i_{n-1}d_k)e(i_{n-1}i_n) + (c_k + i_{n-1}d_k)e(-i_{n-1}i_n), \tag{4}$$

$$\zeta_1 + i_n\zeta_2 = (\zeta_1 - i_{n-1}\zeta_2)e(i_{n-1}i_n) + (\zeta_1 + i_{n-1}\zeta_2)e(-i_{n-1}i_n), \tag{5}$$

$$a_1 + i_n a_2 = (a_1 - i_{n-1}a_2)e(i_{n-1}i_n) + (a_1 + i_{n-1}a_2)e(-i_{n-1}i_n). \tag{6}$$

Then the infinite series in (2) has the following idempotent representation:

$$\sum_{k=0}^{\infty} (c_k - i_{n-1}d_k)[(\zeta_1 - i_{n-1}\zeta_2) - (a_1 - i_{n-1}a_2)]^k e(i_{n-1}i_n) \tag{7}$$

$$+ \sum_{k=0}^{\infty} (c_k + i_{n-1}d_k)[(\zeta_1 + i_{n-1}\zeta_2) - (a_1 + i_{n-1}a_2)]^k e(-i_{n-1}i_n). \tag{8}$$

If the infinite series (3) converges at $\zeta_1 + i_n\zeta_2$, $n \geq 2$, then the infinite series in (7) and (8) converge at $\zeta_1 - i_{n-1}\zeta_2$ and $\zeta_1 + i_{n-1}\zeta_2$, respectively, and (3) equals the expressions in (7) and (8).

If the series in (7) and (8) converge at $\zeta_1 - i_{n-1}\zeta_2$ and $\zeta_1 + i_{n-1}\zeta_2$, respectively, then the series in (3) converges at $\zeta_1 + i_n\zeta_2$ and equals the expression in (7) and (8).

3. Main Theorems

Theorem 3.1. *Let X be a domain in $\mathbb{C}_n$, $n \geq 2$, and let the function*

$$f\colon X \longrightarrow \mathbb{C}_n$$

be analytic in X. Then for each $\xi_0 \in X$ there is a neighborhood of ξ_0 in which

$$f(\xi) = \sum_{k=0}^{\infty} \frac{D_\xi^k f(\xi_0)}{k!} (\xi - \xi_0)^k.$$

Proof. Since f is analytic in X, there exist holomorphic functions

$$f_1 : X_1 \to \mathbb{C}_{n-1} \quad \text{and} \quad f_2 : X_2 \to \mathbb{C}_{n-1} \quad \text{where} \quad X_1, X_2 \subseteq \mathbb{C}_{n-1},$$

such that

$$f(\xi) = f(\xi_1 + i_n\xi_2) = f_1(\xi_1 - i_{n-1}\xi_2)e(i_{n-1}i_n) + f_2(\xi_1 + i_{n-1}\xi_2)e(-i_{n-1}i_n). \tag{9}$$

Where $\xi_1 + i_n\xi_2$ is an arbitrary point in X and $\xi_1 - i_{n-1}\xi_2$ and $\xi_1 + i_{n-1}\xi_2$ are corresponding points in X_1 and X_2, respectively. We prove this result by induction on n.
For $n = 2$, the result is true for bicomplex valued differentiable functions. Let us assume that the result is true for $n - 1$.
Let $\xi_0 = \xi_1^0 + i_n\xi_2^0$ be a point in X.
Then $\xi_1^0 + i_n\xi_2^0 = (\xi_1^0 - i_{n-1}\xi_2^0)e(i_{n-1}i_n) + (\xi_1^0 + i_{n-1}\xi_2^0)e(-i_{n-1}i_n)$.

Since the functions f_1, f_2 are holomorphic in $\mathbb{C}_{n-1}$, by the induction hypothesis they can be represented by power series in neighborhoods of $\xi_1^0 - i_{n-1}\xi_2^0$ and $\xi_1^0 + i_{n-1}\xi_2^0$. Thus there exist $r_1, r_2 > 0$ such that

$$f_1(\xi_1 - i_{n-1}\xi_2) = \sum_{k=0}^{\infty} \frac{D^k_{\xi_1 - i_{n-1}\xi_2} f_1(\xi_1^0 - i_{n-1}\xi_2^0)}{k!} \{(\xi_1 - i_{n-1}\xi_2) - (\xi_1^0 - i_{n-1}\xi_2^0)\}^k,$$
$$\|(\xi_1 - i_{n-1}\xi_2) - (\xi_1^0 - i_{n-1}\xi_2^0)\|_{n-1} < r_1 \tag{10}$$

and

$$f_2(\xi_1 + i_{n-1}\xi_2) = \sum_{k=0}^{\infty} \frac{D^k_{\xi_1 + i_{n-1}\xi_2} f_2(\xi_1^0 + i_{n-1}\xi_2^0)}{k!} \{(\xi_1 + i_{n-1}\xi_2) - (\xi_1^0 + i_{n-1}\xi_2^0)\}^k,$$
$$\|(\xi_1 + i_{n-1}\xi_2) - (\xi_1^0 + i_{n-1}\xi_2^0)\|_{n-1} < r_2. \tag{11}$$

Substituting (10) and (11) into (9) we get,

$$f(\xi) = \sum_{k=0}^{\infty} \frac{D^k_{\xi_1 - i_{n-1}\xi_2} f_1(\xi_1^0 - i_{n-1}\xi_2^0)}{k!} \{(\xi_1 - i_{n-1}\xi_2) - (\xi_1^0 - i_{n-1}\xi_2^0)\}^k e(i_{n-1}i_n) + \sum_{k=0}^{\infty} \frac{D^k_{\xi_1 + i_{n-1}\xi_2} f_2(\xi_1^0 + i_{n-1}\xi_2^0)}{k!} \{(\xi_1 + i_{n-1}\xi_2) - (\xi_1^0 + i_{n-1}\xi_2^0)\}^k e(-i_{n-1}i_n). \quad (12)$$

We know that [15]

$$D^k_{\xi_1 - i_{n-1}\xi_2} f_1(\xi_1^0 - i_{n-1}\xi_2^0) e(i_{n-1}i_n) + D^k_{\xi_1 + i_{n-1}\xi_2} f_2(\xi_1^0 + i_{n-1}\xi_2^0) e(-i_{n-1}i_n) = D^k_\xi f(\xi_0) \quad (13)$$

$$\{(\xi_1 - i_{n-1}\xi_2) - (\xi_1^0 - i_{n-1}\xi_2^0)\}^k e(i_{n-1}i_n) + \{(\xi_1 + i_{n-1}\xi_2) - (\xi_1^0 + i_{n-1}\xi_2^0)\}^k e(-i_{n-1}i_n) = (\xi - \xi_0)^k \quad (14)$$

Therefore

$$\frac{D^k_{\xi_1 - i_{n-1}\xi_2} f_1(\xi_1^0 - i_{n-1}\xi_2^0)}{k!} \{(\xi_1 - i_{n-1}\xi_2) - (\xi_1^0 - i_{n-1}\xi_2^0)\}^k e(i_{n-1}i_n) + \frac{D^k_{\xi_1 + i_{n-1}\xi_2} f_2(\xi_1^0 + i_{n-1}\xi_2^0)}{k!} \{(\xi_1 + i_{n-1}\xi_2) - (\xi_1^0 + i_{n-1}\xi_2^0)\}^k e(-i_{n-1}i_n) = \frac{D^k_\xi f(\xi_0)}{k!} (\xi - \xi_0)^k. \quad (15)$$

Substituting (13),(14) and (15) into (12) we get,

$$f(\xi) = \sum_{k=0}^{\infty} \frac{D^k_\xi f(\xi_0)}{k!} (\xi - \xi_0)^k,$$

and this will converge for all $\xi = \xi_1 + i_n\xi_2$ such that

$$\|(\xi_1 - i_{n-1}\xi_2) - (\xi_1^0 - i_{n-1}\xi_2^0)\|_{n-1} < r_1,$$
$$\|(\xi_1 + i_{n-1}\xi_2) - (\xi_1^0 + i_{n-1}\xi_2^0)\|_{n-1} < r_2.$$

□

Remark 3.2. *This theorem is known as Taylor's series for a holomorphic multicomplex-valued function.*

Example 3.3. *Let $f\colon \mathbb{C}_n \to \mathbb{C}_n$ be a function such that*

$$\begin{aligned} f(\xi_1 + i_n\xi_2) &= e^{\xi_1 + i_n\xi_2} = e^{(\xi_1 - i_{n-1}\xi_2)} e(i_{n-1}i_n) + e^{(\xi_1 + i_{n-1}\xi_2)} e(-i_{n-1}i_n) \\ &= \sum_{k=0}^{\infty} \frac{(\xi_1 - i_{n-1}\xi_2)^k}{k!} e(i_{n-1}i_n) + \sum_{k=0}^{\infty} \frac{(\xi_1 + i_{n-1}\xi_2)^k}{k!} e(-i_{n-1}i_n) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \{(\xi_1 - i_{n-1}\xi_2)^k e(i_{n-1}i_n) + (\xi_1 + i_{n-1}\xi_2)^k e(-i_{n-1}i_n)\} \\ &= \sum_{k=0}^{\infty} \frac{(\xi_1 + i_n\xi_2)^k}{k!} \end{aligned}$$

This series is convergent for all $(\xi_1 + i_n\xi_2) \in \mathbb{C}_n$. This is a power series of $e^{\xi_1 + i_n\xi_2}$.

Theorem 3.4 (Analogue to Maximum-Modulus Principle)**.** *Let $X \subseteq \mathbb{C}_n$ be a bounded domain and a function $f\colon X \longrightarrow \mathbb{C}_n$ be holomorphic in X. Then $\|f(\xi)\|_n$ can not attain maximum in X unless $f(\xi)$ is constant.*

Proof. We prove it by induction on n.
For $n = 2$, the result is true for bicomplex valued holomorphic functions.
Let the result be true for $n-1$.
Since f is holomorphic i.e. analytic in X then there exists holomorphic functions

$$f_1 : X_1 \to \mathbb{C}_{n-1} \quad \text{and} \quad f_2 : X_2 \to \mathbb{C}_{n-1} \quad \text{where} \quad X_1, X_2 \subseteq \mathbb{C}_{n-1},$$

such that

$$f(\xi) = f(\xi_1 + i_n\xi_2) = f_1(\xi_1 - i_{n-1}\xi_2)e(i_{n-1}i_n) + f_2(\xi_1 + i_{n-1}\xi_2)e(-i_{n-1}i_n).$$

Where $\xi_1 + i_n\xi_2$ is an arbitrary point in X and $\xi_1 - i_{n-1}\xi_2$ and $\xi_1 + i_{n-1}\xi_2$ are corresponding points in X_1 and X_2, respectively.

Let $f(\xi)$ attains maximum at $\xi_0 = \xi_1^0 + i_n\xi_2^0 \in X$.

Now $\xi_1^0 + i_n\xi_2^0 = (\xi_1^0 - i_{n-1}\xi_2^0)e(i_{n-1}i_n) + (\xi_1^0 + i_{n-1}\xi_2^0)e(-i_{n-1}i_n)$.

Therefore for all $\xi_1 + i_n\xi_2 \in X$

$$\begin{aligned}
&\|f(\xi_1 + i_n\xi_2)\|_n \leq \|f(\xi_1^0 + i_n\xi_2^0)\|_n \\
\implies &\left(\frac{\|f_1(\xi_1 - i_{n-1}\xi_2)\|_{n-1}^2 + \|f_2(\xi_1 + i_{n-1}\xi_2)\|_{n-1}^2}{2}\right)^{1/2} \\
&\leq \left(\frac{\|f_1(\xi_1^0 - i_{n-1}\xi_2^0)\|_{n-1}^2 + \|f_2(\xi_1^0 + i_{n-1}\xi_2^0)\|_{n-1}^2}{2}\right)^{1/2} \\
\implies &\|f_1(\xi_1 - i_{n-1}\xi_2)\|_{n-1}^2 + \|f_2(\xi_1 + i_{n-1}\xi_2)\|_{n-1}^2 \\
&\leq \|f_1(\xi_1^0 - i_{n-1}\xi_2^0)\|_{n-1}^2 + \|f_2(\xi_1^0 + i_{n-1}\xi_2^0)\|_{n-1}^2 \\
&\forall\{(\xi_1 - i_{n-1}\xi_2)e(i_{n-1}i_n) + (\xi_1 + i_{n-1}\xi_2)e(-i_{n-1}i_n)\} \in X.
\end{aligned}$$

Hence,

$$\|f_1(\xi_1 - i_{n-1}\xi_2)\|_{n-1} \leq \|f_1(\xi_1^0 - i_{n-1}\xi_2^0)\|_{n-1}$$

for all $\|(\xi_1 - i_{n-1}\xi_2) - (\xi_1^0 - i_{n-1}\xi_2^0)\|_{n-1} < r$ where $(\xi_1 - i_{n-1}\xi_2) \in X_1$.

Similarly,

$$\|f_2(\xi_1 + i_{n-1}\xi_2)\|_{n-1} \leq \|f_2(\xi_1^0 + i_{n-1}\xi_2^0)\|_{n-1}$$

for all $\|(\xi_1 + i_{n-1}\xi_2) - (\xi_1^0 + i_{n-1}\xi_2^0)\|_{n-1} < r$ where $(\xi_1 + i_{n-1}\xi_2) \in X_2$.

So by induction hypothesis we can say that $f_1(\xi_1 - i_{n-1}\xi_2)$ and $f_2(\xi_1 + i_{n-1}\xi_2)$ both are constant functions. Hence, $f(\xi)$ is constant. □

Remark 3.5. *The following example ensures Theorem 3.4.*

Example 3.6. *Let* $f : \mathbb{C}_n \to \mathbb{C}_n$ *be such that*

$$f(\xi) = \xi^2$$

To find $\max_{\xi \in G} \|f(\xi)\|_n$, *with* $G = \{\xi \in \mathbb{C}_n : \|\xi - 1 - i_n\|_n < 1\}$

To find the maximum value of $f(\xi)$, *first we find the maximum value of* f *in* $\mathbb{C}_1$ *(the complex plane), where* $n = 1$.

On $|\xi - 1 - i| = 1$,

$$\xi = 1 + i + e^{i\theta}, \quad (1 + \cos\theta) + i(1 + \sin\theta), \quad 0 \le \theta \le 2\pi$$

$$|\xi^2| = |\xi|^2 = (1 + \cos\theta)^2 + (1 + \sin\theta)^2 = 3 + 2(\cos\theta + \sin\theta)$$

It is clear that the maximum value of $|\xi^2|$ *is* $3 + 2\sqrt{2}$ *in* $\mathbb{C}_1$.

Then we find the maximum value of f *in* $\mathbb{C}_2$ *(the bicomplex plane), where* $n = 2$.

Let $\xi \in \mathbb{C}_2$, *then it can be written in idempotent form as*

$$\xi = (\xi_1 - i_1\xi_2)e_1 + (\xi_1 + i_1\xi_2)e_2 \quad \text{where } \xi_1, \xi_2 \in \mathbb{C}_1$$

$$\text{and} \quad e_1 = \frac{1 + i_1 i_2}{2}, \quad e_2 = \frac{1 - i_1 i_2}{2}.$$

$$(\|\xi\|_2)^2 = \frac{|\xi_1 - i_1\xi_2|^2 + |\xi_1 + i_1\xi_2|^2}{2} \le \frac{(3 + 2\sqrt{2}) + (3 + 2\sqrt{2})}{2} = 3 + 2\sqrt{2}$$

$\therefore$ *Maximum value of* $(\|\xi\|_2)^2$ *is* $3 + 2\sqrt{2}$ *in* $\mathbb{C}_2$.

Next we find the maximum value of f *in* $\mathbb{C}_3$, *i.e. here* $n = 3$.

Let $\xi \in \mathbb{C}_3$, *then it can be written as*

$$\xi = (\xi_1 - i_2\xi_2)e(i_2 i_3) + (\xi_1 + i_2\xi_2)e(-i_2 i_3)$$

$$\text{where} \quad \xi_1 - i_2\xi_2 \in \mathbb{C}_2, \quad \xi_1 + i_2\xi_2 \in \mathbb{C}_2$$

$$\text{and} \quad e(i_2 i_3) = \frac{1 + i_2 i_3}{2}, \quad e(-i_2 i_3) = \frac{1 - i_2 i_3}{2}.$$

$$(\|\xi\|_3)^2 = \frac{(\|\xi_1 - i_2\xi_2\|_2)^2 + (\|\xi_1 + i_2\xi_2\|_2)^2}{2} \le \frac{(3 + 2\sqrt{2}) + (3 + 2\sqrt{2})}{2} = 3 + 2\sqrt{2}$$

$\therefore$ *Maximum value of* $(\|\xi\|_3)^2$ *is* $3 + 2\sqrt{2}$ *in* $\mathbb{C}_3$.

In a similar way, we can say that the maximum value of $(\|\xi\|_n)^2$ *is* $3 + 2\sqrt{2}$ *in* $\mathbb{C}_n$.

Remark 3.7. *The bounded condition of the domain in the Maximum Modulus Theorem is impossible to be dropped, as shown by the following example.*

Let

$$G = \{\xi = \xi_1 + i_n\xi_2 \in \mathbb{C}_n : \text{where } \xi_1, \xi_2 \in \mathbb{C}_n \text{ and } \|\xi_2\| > 0\}.$$

Define the function $f : \mathbb{C}_n \to \mathbb{C}_n$ *by*

$$f(\xi) = \exp(-i_n\xi)$$

and then f *is analytic on* G.

Now, for $\xi \in \partial G$

$$\|f(\xi)\| = \|exp(-i_n\xi_1)\| = 1$$

But for $\xi = \xi_1 + i_n\xi_2 \in G$,

$$\|f(\xi)\| = \|\exp(\xi_2)\| \to \infty \quad as\ \|\xi_2\| \to \infty$$

Thus, the Maximum Modulus Theorem fails on unbounded regions.

Theorem 3.8 (Analogue of the Minimum-Modulus Theorem)**.** *Let* $X \subseteq \mathbb{C}_n$ *be a bounded domain and a function* $f\colon X \longrightarrow \mathbb{C}_n$ *be holomorphic in* X*. Then* $\|f(\xi)\|_n$ *can not attain minimum in* X *unless* $f(\xi)$ *is constant.*

Proof. We prove it by induction on n.
For $n = 2$, the result is true for bicomplex valued holomorphic functions.
Let the result be true for $n-1$.
Since f is holomorphic i.e. analytic in X then there exists holomorphic functions

$$f_1 : X_1 \to \mathbb{C}_{n-1} \quad \text{and} \quad f_2 : X_2 \to \mathbb{C}_{n-1} \quad \text{where} \quad X_1, X_2 \subseteq \mathbb{C}_{n-1},$$

such that

$$f(\xi) = f(\xi_1 + i_n\xi_2) = f_1(\xi_1 - i_{n-1}\xi_2)e(i_{n-1}i_n) + f_2(\xi_1 + i_{n-1}\xi_2)e(-i_{n-1}i_n).$$

Where $\xi_1 + i_n\xi_2$ is an arbitrary point in X and $\xi_1 - i_{n-1}\xi_2$ and $\xi_1 + i_{n-1}\xi_2$ are corresponding points in X_1 and X_2, respectively.

Let $f(\xi)$ attains minimum at $\xi_0 = \xi_1^0 + i_n\xi_2^0 \in X$.

Now $\xi_1^0 + i_n\xi_2^0 = (\xi_1^0 - i_{n-1}\xi_2^0)e(i_{n-1}i_n) + (\xi_1^0 + i_{n-1}\xi_2^0)e(-i_{n-1}i_n)$.

Therefore for all $\xi_1 + i_n\xi_2 \in X$

$$\begin{aligned}
&\|f(\xi_1 + i_n\xi_2)\|_n \le \|f(\xi_1^0 + i_n\xi_2^0)\|_n \\
\implies &\left(\frac{\|f_1(\xi_1 - i_{n-1}\xi_2)\|_{n-1}^2 + \|f_2(\xi_1 + i_{n-1}\xi_2)\|_{n-1}^2}{2}\right)^{1/2} \\
&\ge \left(\frac{\|f_1(\xi_1^0 - i_{n-1}\xi_2^0)\|_{n-1}^2 + \|f_2(\xi_1^0 + i_{n-1}\xi_2^0)\|_{n-1}^2}{2}\right)^{1/2} \\
\implies &\|f_1(\xi_1 - i_{n-1}\xi_2)\|_{n-1}^2 + \|f_2(\xi_1 + i_{n-1}\xi_2)\|_{n-1}^2 \\
&\ge \|f_1(\xi_1^0 - i_{n-1}\xi_2^0)\|_{n-1}^2 + \|f_2(\xi_1^0 + i_{n-1}\xi_2^0)\|_{n-1}^2 \\
&\forall\{(\xi_1 - i_{n-1}\xi_2)e(i_{n-1}i_n) + (\xi_1 + i_{n-1}\xi_2)e(-i_{n-1}i_n)\} \in X.
\end{aligned}$$

Hence,

$$\|f_1(\xi_1 - i_{n-1}\xi_2)\|_{n-1} \ge \|f_1(\xi_1^0 - i_{n-1}\xi_2^0)\|_{n-1}$$

for all $\|(\xi_1 - i_{n-1}\xi_2) - (\xi_1^0 - i_{n-1}\xi_2^0)\|_{n-1} < r$ where $(\xi_1 - i_{n-1}\xi_2) \in X_1$.

Similarly,

$$\|f_2(\xi_1 + i_{n-1}\xi_2)\|_{n-1} \ge \|f_2(\xi_1^0 + i_{n-1}\xi_2^0)\|_{n-1}$$

for all $\|(\xi_1 + i_{n-1}\xi_2) - (\xi_1^0 + i_{n-1}\xi_2^0)\|_{n-1} < r$ where $(\xi_1 + i_{n-1}\xi_2) \in X_2$.

So by induction hypothesis we can say that $f_1(\xi_1 - i_{n-1}\xi_2)$ and $f_2(\xi_1 + i_{n-1}\xi_2)$ both are constant functions. Hence, $f(\xi)$ is constant. □

Theorem 3.9 (Analogue of Schwarz's Lemma)**.** *Let $X \subseteq \mathbb{C}_n$ be a domain and the function $f : X \to \mathbb{C}_n$ be analytic in a disk $\|z\|_n < R(R > 0)$, having zero of order n at the origin and let $\|f(z)\|_n \leq M$ for all $z \in X$. Then*

$$\|f(z)\|_n \leq M\frac{(\|z\|_n)^n}{R^n} \quad \text{for all } z \in X \tag{16}$$

$$\text{and} \quad \|D^n f(0)\|_n \leq \frac{Mn!}{R^n}. \tag{17}$$

Proof. From Taylor's series expansion of f in $\mathbb{C}_n$ and using the fact that f has a zero of multiplicity n at $z = 0$, we have

$$f(z) = \frac{D^n f(0)}{n!}z^n + \frac{D^{n+1}f(0)}{(n+1)!}z^{n+1} + \cdots, \quad \|z\|_n < R.$$

For $z = 0$, (1) is trivially true.

Let $z \neq 0$, then

$$\frac{f(z)}{z^n} = \frac{D^n f(0)}{n!} + \frac{D^{n+1}f(0)}{(n+1)!}z + \cdots, \quad \|z\|_n < R.$$

The infinite series on the right-hand side of the above equation converges for $\|z\| < R$, and if we define

$$\left.\frac{f(z)}{z^n}\right|_{z=0} = \frac{D^n f(0)}{n!} \tag{18}$$

then $\dfrac{f(z)}{z^n}$ is analytic in disk X.

Let C be the circle $\|z\|_n = \rho$, where $\rho < R$. Then by the maximum modulus principle in $\mathbb{C}_n$,

$$\left\|\frac{f(z)}{z^n}\right\|_n \leq \frac{M}{\rho^n}, \quad \forall z \text{ with } \|z\|_n \leq \rho.$$

Since the above inequality is true for all $\rho < R$, letting $\rho \to R$, we get

$$\left\|\frac{f(z)}{z^n}\right\|_n \leq \frac{M}{R^n}$$

i.e.,

$$\|f(z)\|_n \leq M\frac{(\|z\|_n)^n}{R^n}.$$

Now, (17) follows from (16) and in view of (18). □

4. Conclusion

Introduction of quaternions which is Hamilton's innovative approach is aimed to describe physical rotations in four dimensional space using mathematical framework. Building upon the complex number concept, bicomplex numbers extend into four dimensions for finding applications in abstract algebra and theoretical physics. The multicomplex space, which can be viewed as a special generalization of bicomplex space has implication in various fields like topology, functional analysis, is now an active area of researchers.

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