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ON WEAKLY NIL CLEAN AND WEAKLY NIL NEAT RINGS

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Abstract. In this paper we obtain some significant results on weakly nil clean and weakly nil neat rings. In addition this work extends some results on commutative weakly nil clean and weakly nil neat rings appeared in [4] to noncommutative weakly nil clean rings and weakly nil neat rings. More interestingly we provide certain results on noncommutative weakly nil neat rings having more than two idempotents.

Keywords: Nil clean ring, weakly nil-clean ring, weakly nil neat ring, weakly UU ring.

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1. Introduction

Throughout this paper, R is an associative ring with identity element. We denote the set of all units, set of all idempotents and the set of all nilpotents in R by $U(R)$, $Id(R)$ and $N(R)$ respectively. The Jacobson radical of R is denoted by $J(R)$ and the set of all unipotent elements in R is denoted by $Unip(R)$.

McGovern [1] has introduced the notion of neat rings which is defined as a ring whose every proper homomorphic images is clean. M. Samiei has studied nil neat rings in [2] and these rings are defined as rings whose every proper homomorphic images are nil clean. We have studied and extended some results presented in [2] on nil neat rings in [3]. Danchev and Samiei have studied the notion of weakly nil neat rings in [4]. A ring R is called a weakly nil neat ring if every proper homomorphic image of R is weakly nil clean [4].

In [4] commutative weakly nil clean rings and weakly nil neat rings have been studied. Following [4] here we obtain some significant results on weakly nil clean and weakly nil neat rings and we extend some results of [4]. This should be emphasized that in [4] each ring is a commutative ring however in the present paper a ring R is not necessarily a commutative ring.

For the sake of convenience we shall provide some useful definitions needed to understand this paper as follows. A ring R is called a clean ring if every element in R is the sum of a unit and an idempotent [10-11]. A ring R is called a nil clean ring if every element of R is the sum of a nilpotent and an idempotent [6]. A ring R is called a weakly nil clean ring if for each $a \in R$ we have $a = u + v$ or $a = u - v$ for some $u \in N(R)$ and $v \in Id(R)$ [9]. A ring R is called weakly UU ring if for each $a \in U(R)$ we have $a = n + 1$ or $a = n - 1$ [4]. Here $n \in N(R)$. In other words a ring R is called weakly UU ring if for each $a \in U(R)$ we have either $a \in Unip(R)$ or $a \in -Unip(R)$.

In the next section we provide main results of this paper.

2. Main Results

We obtain the following results on weakly nil clean and weakly nil neat rings. It may be emphasized that these results hold for commutative as well as noncommutative rings under the given condition however in [4], all results have been obtained for commutative rings only.

Proposition 2.1. *Let R is a ring without non-trivial idempotents. Then R is a weakly nil-clean ring iff for each $a \in R$, either $a \in N(R)$, or $a \in Unip(R)$ or $a \in -Unip(R)$.*

Proof. Let R is a ring without non-trivial idempotents. Let R is a weakly nil-clean ring and a is any element of R. Clearly $a = b+c$ or $a = b-c$ such that $b \in Id(R)$ and $c \in N(R)$. Clearly if $b = 0$, then $a \in N(R)$ and if $b = 1$, then $a \in Unip(R)$ or $a \in -Unip(R)$. Hence for each $a \in R$, either $a \in N(R)$, or $a \in Unip(R)$ or $a \in -Unip(R)$. Conversely let for each $a \in R$, either $a \in N(R)$, or $a \in Unip(R)$ or $a \in -Unip(R)$. Let $a \in N(R)$, then $a = 0 + a$. If $a \in Unip(R)$, then $a = 1 + c$ for some $c \in N(R)$. If $a \in -Unip(R)$, then $a = c - 1$ for some $c \in N(R)$. Hence R is a weakly nil clean ring.

Proposition 2.2. *Let R is a ring without non-trivial idempotents. Then the following results are equivalent.*

(i) R is a weakly nil clean ring such that $Unip(R) \neq -Unip(R)$.
(ii) $J(R)$ is nil and $\frac{R}{J(R)}$ is the field of order three.

Proof. (i) $\Leftrightarrow$ (ii). Let R is a weakly nil clean ring such that $Unip(R) \neq -Unip(R)$. Clearly for each $a \in R$, either $a \in N(R)$, or $a \in Unip(R)$ or $a \in -Unip(R)$. This implies that if $a \in R$, then either a is nilpotent or a is a unit. Thus all non-unit is a nilpotent element. This leads that $u + v \in N(R)$ and $uv \in N(R)$ for each $u, v \in N(R)$. Also $au \in N(R), ua \in N(R)$ for each $a \in R$ and $u \in N(R)$. Thus $N(R)$ a nil ideal of R and so $J(R) = N(R)$ as $J(R)$ cannot contain any element $a \in R$ such that $a \in Unip(R)$ or $a \in -Unip(R)$.

Further let $a \in R$ is any element of R. If $a \in J(R)$, then $a + J(R) = J(R)$. Similarly if $a \in Unip(R)$, then we have $a + J(R) = 1 + J(R)$ since $1 + J(R) = Unip(R)$. Next if $a \in -Unip(R)$, then we have $a + J(R) = -1 + J(R)$ since $-1 + J(R) = -Unip(R)$. Hence $\frac{R}{J(R)} = \{J(R), 1 + J(R), -1 + J(R)\}$ is the field of order three. The converse easily follows.

Corollary 2.3. *Let R is a ring without non-trivial idempotents. Then the following results are equivalent.*

(i) R is a weakly nil clean ring such that $Unip(R) = -Unip(R)$.
(ii) $J(R)$ is nil and $\frac{R}{J(R)}$ is the field of order two.

Remark 2.4. *Let R is a weakly nil clean ring such that $Unip(R) = -Unip(R)$. Then for each $a \in N(R)$, we have some $b \in N(R)$ such that $1 + a = b - 1$. This implies that $2 = b - a$. As already noted $N(R)$ is an ideal, so $2 = b - a \in N(R)$. Thus 2 is nilpotent.*

Proposition 2.5. *Let R is a ring without non-trivial idempotents. If R is a weakly nil clean ring, then R has a unique maximal ideal.*

Proof. Let R is a ring without non-trivial idempotents. Let R is a weakly nil clean ring. We have noted above (we refer the proof of the Proposition 2.4) that $J(R) = N(R)$ is a nil ideal of R. Let I is any ideal of R. Since for each $a \in R$, either $a \in N(R)$, or $a \in Unip(R)$ or $a \in -Unip(R)$ and so each element of I must be nilpotent. Hence I must be contained in $J(R)$. Hence $J(R)$ is the unique maximal ideal of R.

Proposition 2.6. *Let R is a ring without non-trivial idempotents. Then the following results are equivalent.*

(i) R is a local weakly nil clean ring.
(ii) R is an indecomposable weakly nil clean ring.

(i) $\Leftrightarrow$ (ii). Let R is a ring without non-trivial idempotents. Let R is a local weakly nil clean ring. This immediately follows that R is an indecomposable weakly nil clean ring. Conversely let R is an indecomposable weakly nil clean ring. It suffices to prove that R is a local ring. It is known that a weakly nil clean ring is a clean ring ([9, Corollary 8]). Therefore R is a clean ring. Further a clean ring R without non-trivial idempotents is a local ring [10]. Hence R is a local weakly nil clean ring as desired.

Proposition 2.7. *Let R is a ring without non-trivial idempotents. Then the following results are equivalent.*

(i) R is a weakly nil clean ring.
(ii) R is a Weakly UU ring and R has exactly one prime ideal.

Proof. (i) $\Rightarrow$ (ii) Let R is a weakly nil clean ring without non-trivial idempotents. Therefore for each $a \in R$, either $a \in N(R)$, or $a \in Unip(R)$ or $a \in -Unip(R)$ (using Proposition 2.1). Thus R is a weakly UU ring by the definition of UU ring. Now we shall prove that R has exactly one prime ideal. Let I is a prime ideal of R. It is easy to note that for each $a \in R$ there exists a positive integer n such that a^nR is a two sided ideal (since $a^n = 1$ or $a^n = 0$ for some positive integer n). This implies that R is weakly right duo ring (we refer [5]). Since R is without nontrivial idempotents, therefore R is an abelian weakly nil clean ring and every abelian weakly nil clean ring is strongly π-regular (Proposition 15, [9]). Now using Theorem 3.10 [5] we obtain that I is a maximal ideal. But we have already proved that R has a unique maximal ideal (see Proposition 2.5). Hence $I = N(R)$ is the only prime ideal of R. (ii) $\Rightarrow$ (i) easily follows as if R is a UU ring then for each non-nilpotent (unit) element $a \in R$ we have $a \in Unip(R)$ or $a \in -Unip(R)$ and if $a \in R$ is non-unit then $a \in N(R)$. Hence for each $a \in R$, either $a \in N(R)$ or, $a \in Unip(R)$ or $a \in -Unip(R)$. Therefore by Proposition 2.1R is a weakly nil clean ring.

Corollary 2.8. *Let R is a ring without non-trivial idempotents. Then the following statements are equivalent.*

(i) For each $a \in R$, either $a \in N(R)$ or, $a \in Unip(R)$ or $a \in -Unip(R)$.

(ii) R is a weakly UU ring and R has exactly one prime ideal.

Proposition 2.9. *Let R is a non-reduced ring such that $K(R) = N(R)$. Here $K(R)$ is the set of all elements $a \in R$ satisfying $a = kb$ for some fixed positive integer k and some $b \in R$. Then R is weakly nil neat iff R is weakly nil clean.*

Proof. Let R is a non-reduced ring such that $K(R) = N(R)$. Here $K(R)$ is the set of all elements $a \in R$ satisfying $a = kb$ for some fixed positive integer k and some $b \in R$. Further let R is a weakly nil neat ring. Since $K(R) = N(R)$ and therefore it follows from Proposition 2.7[7] that $N(R)$ is an ideal of R. Clearly $\frac{R}{N(R)}$ is a weakly nil clean ring because every proper homomorphic image of R is a weakly nil clean ring. Hence R is a weakly nil clean ring (we refer Lemma 1, [9]). Conversely let R is a weakly nil clean ring. Then each homomorphic image of R is weakly nil clean (we refer Lemma 1, [9]). Therefore it follows that R is weakly nil neat ring.

Proposition 2.10. *Let R is a non-reduced ring such that $K(R) = I$. Here $K(R)$ is the set of all elements $a \in R$ satisfying $a = kb$ for some fixed positive integer k and some $b \in R$ and I is a non-zero subset of $N(R)$. Then R is weakly nil neat iff R is weakly nil clean.*

Proof. The proof of this Proposition is similar to the proof of above Proposition 2.9. More explicitly like above in this case $\frac{R}{I}$ is a weakly nil clean ring and therefore it follows that R is a weakly nil clean ring. Conversely if R is a weakly nil clean ring, then every homomorphic image of R is a weakly nil clean ring and hence R is a weakly nil neat ring.

Proposition 2.11. *Let R is a non-reduced ring such that $E(R) = N(R)$, where $E(R)$ is the set of all even elements of R. Then R is weakly nil neat iff R is weakly nil clean.*

Proof. The proof can be produced using arguments given in the proof of Proposition 2.9 and noting that $N(R)$ is an ideal (we refer Proposition 2.6, [7]). More explicitly like above $\frac{R}{N(R)}$ is a weakly nil clean ring and therefore it follows that R is a weakly nil clean ring. Conversely if R is a weakly nil clean ring, then every homomorphic image of R is a weakly nil clean ring and hence R is a weakly nil neat ring.

Proposition 2.12. *Let R is a non-reduced ring such that $E(R) = I$, where $E(R)$ is the set of all even elements of R and I is a non-zero subset of $N(R)$. Then R is weakly nil neat iff R is weakly nil clean.*

Proof. The proof is similar to the Proof of proposition 2.11. For even elements one may refer [8].

Remark 2.13. *A reduced weakly nil clean ring is always commutative (Corollary 14, [9]).*

Conclusion

Unlike [4] our results presented here hold for commutative as well as noncommutative weakly nil clean rings and weakly nil neat rings. Thus we have improved some results

on weakly nil clean rings and weakly nil neat rings presented in [4]. In addition we have given some interesting results on noncommutative weakly nil neat rings having nontrivial idempotents.

Though we have given certain results on noncommutative weakly nil neat rings having non-trivial idempotents however in future a general theory of noncommutative weakly nil neat rings may be developed and such a theory appears to be highly desirous and the results presented here may motivate for the same.

3. Statements and Declaration

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