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ON MINIMAL n-PRIMES SUBGROUPS OF ℓ-GROUPS

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Abstract. This paper explores the notion of n-prime ℓ-subgroups in ℓ-groups. It derives a characterization for minimal n-prime ℓ-subgroups and defines a topology on the set $\mathcal{P}_n(G)$ of minimal n-prime ℓ-subgroups of an ℓ-group. The paper studies various properties of this topology and establishes a necessary and sufficient condition for $\mathcal{P}_n(G)$ to be compact. It proves that $\mathcal{P}_n(G)$ to be compact if and only if it is finite.

Keywords: prime ℓ-subgroup, minimal prime ℓ-subgroup, n-prime ℓ-subgroup, ℓ-group.

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1. Introduction

In this paper we introduce the concept of n-prime ℓ-subgroups in ℓ-groups. The concept of n-points in a topological space was introduced and studied by Pierce [10]. Hindman [5] extended this idea and studied n-prime ideals in a commutative ring. Research on the space of minimal n-prime ideals in commutative rings with unity has been done by Hindman [5]. An ideal I in a commutative ring R is called *n-prime*$(n \geq 2)$ if $S \cap I \neq \emptyset$, whenever S is a set of n elements of R with each pairwise product lying in I. Building on this, Anderson and Badawi [1] explored n- absorbing ideals in commutative rings, which bear similarities to Hindman's n-prime ideals (see Hindman[5]). Additionally, Halaš [4] introduced n-prime ideals in posets, while Mundlik and Joshi [7] extended the concept of n-prime ideals introduced in Hindman[5] for 0-distributive lattices.

In the study of lattice-ordered groups (ℓ-groups), the notion of primality arises in two distinct contexts. On one hand, when an ℓ-group is viewed purely as a lattice, we consider prime ideals in the lattice-theoretic sense. On the other hand, when we consider the full algebraic structure of the ℓ-group, we encounter prime ℓ-subgroups. It is important to note that these two notions of primality do not coincide in general.

To illustrate this distinction, consider the group $\mathbb{Z}$ with its usual order. Treated as a lattice, the set $(x] = \{y \in \mathbb{Z} \mid y \leq x\}$ forms a prime ideal for any $x \in \mathbb{Z}$. However, when $\mathbb{Z}$ is regarded as an ℓ-group, it possesses only two prime ℓ-subgroups: $\mathbb{Z}$ itself and the set $(0]$. For a detailed discussion of this, one may refer to Darnel[3]. This example clearly demonstrates that the set of all prime ideals in a lattice and the set of all prime ℓ-subgroups in an ℓ-group are, in general, different. Consequently, it becomes essential to study the structure of prime ℓ-subgroups, particularly focusing on minimal and n-prime ℓ-subgroups, in the broader context of ℓ-groups.

P. Bhattacharjee and W. Wm. McGovern [2] have studied space of all minimal prime subgroups of G endowed with the inverse topology. In this paper, we generalize this concept to n-prime ℓ-subgroup in an ℓ-group G.

In section 1, we describe basic fundamental concepts and terminology related to ℓ-groups. In section 2, we define the notion of n-prime ℓ-subgroup and minimal n-prime ℓ-subgroup and give its characterization. In section 3, we define a toplogy on the set $\mathcal{P}_n(G)$ of minimal n-prime ℓ-subgroups of an ℓ-group.
We begin with basic notions and terminologies related to ℓ-groups.

Definition 1.1 (Darnel [3]). *A group $(G,+)$ is said to be a partially ordered group (po-group) if it is equipped with a reflexive, antisymmetric and transitive relation $\leq$ (a partial order) which is compatible with $+$, that is if $g \leq h$ and $x, y \in G$ then $x+g+y \leq x+h+y$.*

Definition 1.2 (Darnel [3]). *Let $(G,+)$ be a po-group. Then the subset $G^+ = \{g \in G : g \geq 0\}$ is called the positive cone of G. Here 0 is identity of group G.*

Observe that, the partial order is determined by G^+, in the sense that $g \leq h$ if and only if $h - g \in G^+$.

Definition 1.3 (Darnel [3]). *A po-group G is called lattice-ordered group abbreviated as ℓ-group, if the underlying structure is lattice.*

Definition 1.4 (Darnel [3]). *Let L be a lattice and A be a subset of L. Then A is a sublattice of L if for any $a,\ b \in A$, $a \vee b \in A$ and $a \wedge b \in A$.*

Definition 1.5 (Darnel [3]). *Let S be a subgroup of an ℓ-group G. Then S is called an ℓ-subgroup of G if it is a sublattice of G.*

Definition 1.6 (Darnel [3]). *A subset S of a po-group G is called convex if whenever $s, t \in S$ and $g \in G$ with $s \leq g \leq t$ then $g \in S$.*

Definition 1.7 (Darnel [3]). *Let G be an ℓ-group with identity 0 and let $g \in G$.*

1) *The positive part of g is $g_+ = g \vee 0$.*
2) *The negative part of g is $g_- = -g \vee 0$.*
3) *The absolute value of g is $|g| = (g_+) + (g_-)$.*

Denote by $\mathfrak{C}(G)$ the set of all convex ℓ-subgroups of G. Observe that if S is a convex ℓ-subgroup of G then $a \in S$ if and only if $|a| \in S$.

Definition 1.8 (Darnel [3]). *A convex ℓ-subgroup P of an ℓ-group G is called prime ℓ-subgroup if whenever $0 \leq a, 0 \leq b$, and $a \wedge b \in P$, then either $a \in P$ or $b \in P$.*

Let S be a subgroup of group G. Denote by $\mathcal{R}(S)$ the set of all right cosets of S in G.

Proposition 1.9 (Darnel [3]). *Let G be a po-group and let S be a convex subgroup of G. Define the relation " $\geq$ " on $\mathcal{R}(S)$ by $S + x \geq S + y$ if and only if there exists $s \in S$ such that $x \geq s + y$. Then " $\geq$ " is a partial order on $\mathcal{R}(S)$.*

Proposition 1.10 (Darnel [3]). *Let P be a convex ℓ-subgroup of an ℓ-group G. Then the following conditions are equivalent:*

1) *P is prime ℓ-subgroup*
2) *If $g \wedge h = 0$ then $g \in P$ or $h \in P$*
3) *If $g, h \in G^+ \setminus P$ then $g \wedge h > 0$*
4) *$\mathcal{R}(P)$ is totally ordered.*
5) *If $A, B \in \mathfrak{C}(G)$ such that $P \subseteq A$ and $P \subseteq B$, then either $A \subseteq B$ or $B \subseteq A$.*
6) *If $P \subset A$ and $P \subset B$, then $P \subset A \cap B$*
7) *If $g, h \in G^+ \setminus P$, then $g \wedge h \in G^+ \setminus P$*

Definition 1.11 (Darnel [3]). *A prime ℓ-subgroup P of an ℓ-group G is called a minimal prime ℓ-subgroup if there does not exist another prime ℓ-subgroup Q such that $Q \subsetneqq P$.*

Lemma 1.12 (Darnel [3]). *Let G be an ℓ-group. Then there exist a minimal prime ℓ-subgroup of G and each prime ℓ-subgroup of G contains atleast one minimal prime ℓ-subgroup.*

Lemma 1.13 (Darnel [3]). *Let G be an ℓ-group and let $A, B, P \in \mathfrak{C}(G)$. Then P is a prime ℓ-subgroup of G if and only if $A \cap B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.*

Denote by $Spec(G)$, the set of all prime ℓ-subgroups of an ℓ-group G. It is known that for a prime ℓ-subgroup P, the complement $G^+ \setminus P$, is a filter on the lattice G^+. Furthermore, the intersection of a chain of prime ℓ-subgroups yields another prime ℓ-subgroup. Consequently, by Zorn's lemma, minimal prime ℓ-subgroups exist, and every prime ℓ-subgroup contains a minimal prime ℓ-subgroup. $Min(G)$ denotes the set of all minimal prime ℓ-subgroups of G. For a subset $A \subseteq G$, the polar of A is denoted by $A^\perp = \{g \in G : |g| \wedge |a| = 0 \text{ for all } a \in A\}$. For $A = \{a\}$, we write it's polar as $a^\perp$. Observe that, polars are convex ℓ-subgroups.

Definition 1.14 (Bhattacharjee and McGovern [2]). *Let G be an ℓ-group. A subset F of G^+, maximal with respect to the property that $a \wedge b > 0$ for all $a, b \in F$ is called an ultrafilter on G^+.*

Theorem 1.15 (Bhattacharjee and McGovern [2]). *Let G be an ℓ-group. For $P \in Spec(G)$, the following statements are equivalent:*

1) *P is a minimal prime ℓ-subgroup.*
2) *$G^+ \setminus P$ is an ultrafilter on G^+.*
3) *$P = \bigcup \{g^\perp : g \notin G^+ \setminus P\}$.*
4) *P is prime and for all $g \in P$, $g^\perp \not\subseteq P$.*

We now obtain a characterization of minimal prime ℓ-subgroup. This characterization is frequently used in the development of the paper.

Lemma 1.16. *Let P be a prime ℓ-subgroup of an ℓ-group G. Then the following statements are equivalent.*

1) *P is a minimal prime ℓ-subgroup.*
2) *For every $x \in P$, there exists $y \notin P$ such that $|x| \wedge |y| = 0$.*
3) *For any $x \in P$ and I is any convex ℓ-subgroup of G not contained in P, then there exists $a \in I \backslash P$ such that $|x| \wedge |a| = 0$.*

Proof. 1) $\Leftrightarrow$ 2) : follows from the Theorem 1.15.
2) $\Rightarrow$ 3) : Let P be a prime ℓ-subgroup of an ℓ-group G. Suppose that, $x \in P$ and I is a convex ℓ-subgroup of G such that $I \not\subseteq P$. By the hypothesis, there exists $y \notin P$ such that $|x| \wedge |y| = 0$. Choose $t \in I \setminus P$. Setting $a = |t| \wedge |y|$, we have $0 \leq |t| \wedge |y| \leq |t|$. As I is a convex ℓ-subgroup, $|t| \wedge |y| \in I$ but $|t| \wedge |y| \notin P$. Note that, $|x| \wedge |a| = |x| \wedge |\,(|y| \wedge |t|)\,| = |x| \wedge |y| \wedge |t| = 0 \wedge |t| = 0$.
3) $\Rightarrow$ 1) : Let P be a prime ℓ-subgroup of an ℓ-group G and let $x \in P$. In particular, if we choose $I = G$ then by the hypothesis there exists $y \in G \setminus P$ such that $|x| \wedge |y| = 0$. $\square$

2. Minimal n-prime ℓ-subgroups

In this section, we define the notion of n-prime ℓ-subgroup and minimal n-prime ℓ-subgroup and give its characterization.

Definition 2.1. *A proper ℓ-subgroup P of an ℓ-group G is said to be an n-prime ℓ-subgroup($n \geq 2$) if it can be represented as an intersection of at most $(n-1)$ distinct prime ℓ-subgroups.*

In the following example, we show that n-prime ℓ-subgroup need not be a prime ℓ-subgroup.

Example 2.2. *Consider the ℓ-group $G = \mathbb{Z} \times \mathbb{Z}$ with the component wise order and addition. Observe that, $A = \{0\} \times \mathbb{Z}$ and $B = \mathbb{Z} \times \{0\}$ are both prime ℓ-subgroups of G but $A \cap B = \{(0,0)\}$ is not a prime ℓ-subgroup. However, $\{(0,0)\}$ is a 3-prime ℓ-subgroup of G.*

It can be easily observed that every n-prime ℓ-subgroup becomes an $(n+1)$-prime ℓ-subgroup and the notion 2-prime ℓ-subgroup is equivalent to the standard concept of prime ℓ-subgroup.

Definition 2.3. *An n-prime ℓ-subgroup P of an ℓ-group G is called a minimal n-prime ℓ-subgroup if there does not exist another n-prime ℓ-subgroup Q such that $Q \subsetneqq P$.*

Example 2.4. *Consider the ℓ-group $G = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ with the component wise order and addition. Observe that, $A = \{0\} \times \mathbb{Z} \times \mathbb{Z}$, $B = \mathbb{Z} \times \{0\} \times \mathbb{Z}$, $B = \mathbb{Z} \times \mathbb{Z} \times \{0\}$ are three prime ℓ-subgroups of G. Here $A \cap B = \{0\} \times \{0\} \times \mathbb{Z}$, $A \cap C = \{0\} \times \mathbb{Z} \times \{0\}$, $B \cap C = \mathbb{Z} \times \{0\} \times \{0\}$. Thus $A \cap B$, $A \cap C$, $B \cap C$ are minimal 3-prime ℓ-subgroups.*

Denote by $\mathcal{P}_n(G)$ the set of all minimal n-prime ℓ-subgroups of an ℓ-group G. If $n = 2$, we observe that $\mathcal{P}_2(G)$ is equivalent to $Min(G)$.
We now prove the following Lemma which is cruical as it implies that for fixed n, where $n \geq 2$, if an ℓ-subgroup is minimal n-prime, then it cannot be minimal $(n-1)$-prime.

Lemma 2.5. *Let G be an ℓ-subgroup with $|\mathcal{P}_2(G)| = |Min(G)| > n-2$. Then no element of $\mathcal{P}_n(G)$ is an $(n-1)$-prime ℓ-subgroup. Consequently, every minimal n-prime ℓ-subgroup can be represented as an intersection of exactly $(n-1)$ distinct prime ℓ-subgroups of G.*

Proof. Let G be an ℓ-subgroup with $|\mathcal{P}_2(G)| = |Min(G)| > n-2$. Suppose on the contrary that, $Q \in \mathcal{P}_n(G)$ and it is minimal $(n-1)$-prime ℓ-subgroup. Then $Q = \bigcap_{i=1}^{m} P_i$, where $P_i's$ are distinct prime ℓ-subgroups of G and $m \leq (n-2)$. Since $|Min(G)| > (n-2)$, there exists a minimal prime ℓ-subgroup of G say P' such that $P' \neq P_i, 1 \leq i \leq m$. Let $R = Q \bigcap P' = \left(\bigcap_{i=1}^{m} P_i\right) \bigcap P'$. If $R = Q$ then $\left(\bigcap_{i=1}^{m} P_i\right) \subseteq P'$. As $P_i \in \mathfrak{C}(G)$, by the Lemma 1.13 and minimality of P', we have $P_k = P'$ for some k, a contradiction. Thus $R \subsetneqq Q$ implies that R is an $(n-1)$- prime ℓ-subgroup. As every $(n-1)$- prime ℓ-subgroup of G is an n-prime ℓ-subgroup, R is an n-prime ℓ-subgroup. But $R \subsetneqq Q$, a contradiction to the fact that Q is a minimal n-prime ℓ-subgroup. Consequently, every minimal n-prime ℓ-subgroup can be represented as an intersection of exactly $(n-1)$ distinct prime ℓ-subgroups of G. □

Lemma 2.6. *Let G be an ℓ-group with $|\mathcal{P}_2(G)| = |Min(G)| > n-2$ and let P be a minimal n-prime ℓ-subgroup of G. If $x_1, x_2, \ldots, x_n$ are elements of G satisfying $|x_i| \wedge |x_j| \in P, i \neq j$ then there exists $x_i (1 \leq i \leq n)$ such that $|x_i| \in P$.*

Proof. Let P be a minimal n-prime ℓ-subgroup of G such that $P = P_1 \cap P_2 \cap \cdots \cap P_{(n-1)}$, for $P_1, P_2, \ldots, P_{(n-1)} \in Spec(G)$. Let $x_1, x_2, \ldots, x_n$ be elements of G such that $|x_i| \wedge |x_j| \in P$ for all $i \neq j$ and $1 \leq i, j \leq n$. Then $|x_i| \wedge |x_j| \in P_k$ for every $k \in \{1, 2, \ldots, (n-1)\}$. By primeness of each P_k, we have $|x_i| \in P_k$ or $|x_j| \in P_k$. So there exists at most one x_i such that $|x_i| \notin P_k$ for every k. Since $k < n$, there exists some x_i such that $|x_i| \in P_1 \cap P_2 \cap \cdots \cap P_{(n-1)}$ and so, $|x_i| \in P$. □

We now characterize the minimal n-prime ℓ-subgroups in an ℓ-group G.

Theorem 2.7. *Let P be an n-prime ℓ-subgroup of an ℓ-group G with $|\mathcal{P}_2(G)| = |Min(G)| > n-2$. Then the following statements are equivalent:*

1) *P is a minimal n-prime ℓ-subgroup.*
2) *P is an intersection of exactly $n-1$ minimal prime ℓ-subgroups.*
3) *For each $x \in P$, there exist $a_1, a_2, \ldots, a_{n-1} \notin P$ with $|x| \wedge |a_i| = 0, 1 \leq i \leq n-1$ and $|a_i| \wedge |a_j| = 0$ for $i \neq j$.*

Proof. 1) ⇒ 2) Let P be a minimal n-prime ℓ-subgroup of an ℓ-group G with $|\mathcal{P}_2(G)| = |Min(G)| > n-2$. By the Lemma 2.5, $P = \bigcap_{i=1}^{n-1} P_i$, where each P_i is prime ℓ-subgroup. Again, by the Lemma 1.12, each prime ℓ-subgroup P_i contains a minimal prime ℓ-subgroup say P_i'. Now, if $R = \bigcap_{i=1}^{m} P_i'$, where $m \leq n-1$ then R is an n-prime ℓ-subgroup and clearly $R \subseteq P$. As P is a minimal n-prime ℓ-subgroup, we have $R = P$. If $m \leq n-2$ then P

is an $(n-1)$-prime ℓ-subgroup, a contradiction to the Lemma 2.5 and so, $m = n-1$. Therefore, $P = \bigcap_{i=1}^{n-1} P_i'$ is an intersection of exactly $n-1$ minimal prime ℓ-subgroups.

2) $\Rightarrow$ 3) Suppose that, every n-prime ℓ-subgroup of an ℓ-group G is an intersection of exactly $n-1$ minimal prime ℓ-subgroups. Let P be an n-prime ℓ-subgroup of an ℓ-group G.

We prove the result by the induction on n. By the Lemma 1.16, 3) holds for $n = 2$. Assume that, if P is an intersection of exactly $n-2$ minimal prime ℓ-subgroups then for each $x \in P$ there exists $a_1, a_2, \ldots, a_{n-2} \notin P$ with $|x| \wedge |a_i| = 0$ and $|a_i| \wedge |a_j| = 0$, for $i \neq j$. Now, suppose that P is an intersection of exactly $(n-1)$ minimal prime ℓ-subgroups. Then $P = \bigcap_{i=1}^{n-1} P_i$, where each P_i is a minimal prime ℓ-subgroups. Let $x \in P$. Consider, $Q = \bigcap_{i=1}^{n-2} P_i$, implies that $x \in Q$. By the induction assumption, there exist $a_1, a_2, \ldots, a_{n-2} \notin Q$ such that $|x| \wedge |a_i| = 0$ and $|a_i| \wedge |a_j| = 0$, for $i \neq j$. Now, for fix $k \in \{1, 2, \ldots, n-2\}$, $|a_i| \wedge |a_j| \in P_k$, for all $i \neq j$. Therefore, there exists at most one a_i such that $|a_i| \notin P_k$. Otherwise, we have $|a_{i_1}|, |a_{i_2}| \notin P_k$ implies that $|a_{i_1}| \wedge |a_{i_2}| = 0 \notin Q$, a contradiction. Without loss of generality, we may assume that $|a_i| \in \left(\bigcap_{\substack{j=1 \\ j \neq i}}^{n-2} P_j \right) \setminus P_i$. Now we have following two cases:

(A) : $|a_i| \in P_{n-1}, i = 1, 2, \ldots, (n-2)$. Since P_{n-1} in a minimal prime, by the Lemma 1.16 there exists $c_1, c_2, \ldots, c_{n-2} \notin P_{n-1}$ such that $|a_i| \wedge |c_i| = 0$. Also, as $x \in P_{n-1}$ there exists $f \notin P_{n-1}$ such that $|x| \wedge |f| = 0$ follows from the case $n = 2$. Consider $a_{n-1} = |f| \wedge |c_1| \wedge |c_2| \wedge \ldots \wedge |c_{n-2}|$. Observe that $|x| \wedge |a_{n-1}| = 0$ and $|a_i| \wedge |a_{n-1}| = 0$ for each i, $i = 1, 2, \ldots, (n-2)$. Thus in this case 3) holds.

(B) : Now, assume that P_{n-1} does not contain at most one $|a_i|$, say $|a_k| \notin P_{n-1}$. Otherwise, $|a_k| \wedge |a_j| = 0 \in P_{n-1}$ for $i \neq j$, but $|a_j|, |a_k| \notin P_{n-1}$, a contradiction. As $P_k \neq P_{n-1}$, there exists d such that $d \in P_{n-1} \backslash P_k$. Consider, $a_k' = |a_k| \wedge |d|$. Clearly, $|a_k'| = |a_k| \wedge |d| \leq |d| \in P_{n-1}$ and hence $|a_k'| \in P_{n-1}$. Also, $|a_k| \notin P_k$ and $|d| \notin P_k$ implies that $|a_k'| \notin P_k$. Thus $|a_k'| \in P_{n-1} \backslash P_k$. Replacing a_k by a_k' in the set $\{a_1, a_2, \ldots, a_{n-2}\}$, this case follows from (A).

3) $\Rightarrow$ 1) : Suppose on the contrary that, Q is an n-prime ℓ-subgroup with $Q \subsetneqq P$. Choose $x \in P \backslash Q$. By the hypothesis, there exist $a_1, a_2, \ldots, a_{n-1} \notin P$ such that $|x| \wedge |a_i| = 0$ and $|a_i| \wedge |a_j| = 0$ for $i \neq j$, and $i, j \in \{1, 2, \ldots, n-1\}$. But Q is an n-prime ℓ-subgroup. Therefore by the Lemma 2.6, $|x| \in Q$ or $|a_i| \in Q \subseteq P$, a contradiction. Thus, P is a minimal n-prime ℓ-subgroup of G. $\square$

3. Space of minimal n-prime ℓ-subgroups

In this section we define a toplogy on the set $\mathcal{P}_n(G)$ of all minimal n-prime ℓ-subgroups of an ℓ-group. Here we study the open and closed subsets in $\mathcal{P}_n(G)$. We prove few results related to compactness, Hausdorff property and regularity of this topological space. We further consider the product of distinct minimal prime ℓ-subgroups and study its properties. Finally we prove the characterization that $\mathcal{P}_n(G)$ is compact if and only if $\mathcal{P}_n(G)$ is finite.

For $a \in G$, we define the subset $h_n(a) = \{P \in \mathcal{P}_n(G) \ : \ a \in P\}$ of $\mathcal{P}_n(G)$.

Lemma 3.1. *Let G be an ℓ-group and let $a \in G$. Then $h_n(a) = h_n(|a|)$.*

Proof. Let P be a minimal prime n-subgroup of an ℓ-group G such that $P \in h_n(a)$. P is a convex ℓ-subgroup of G and thus $0 \in P$. Then $a \vee 0 = a_+ \in P$ and $-a \vee 0 = a_- \in P$. This implies that, $a_+ \vee a_- = |a| \in P$ and so, $P \in h_n(|a|)$. Conversely, let $P \in h_n(|a|)$. Then $|a| \in P$. Since $0 \leq a \leq |a| \in P$ or $0 \leq -a \leq |a| \in P$. By using convexity of P, we have $a \in P$ or $-a \in P$. Consequently, we have $a \in P$ implies that $P \in h_n(a)$. Thus, $h_n(a) = h_n(|a|)$. □

Proposition 3.2. *Let G be an ℓ-group and let $a, b \in G$. Then $h_n(|a|) \cap h_n(|b|) = h_n(|a| \vee |b|)$.*

Proof. Let P be a minimal prime n-subgroup of an ℓ-group G. Suppose that, $P \in h_n(|a|) \cap h_n(|b|)$. Then $|a| \in P$ and $|b| \in P$, implies that $|a| \vee |b| \in P$. Thus $P \in h_n(|a| \vee |b|)$. Conversely, if $P \in h_n(|a| \vee |b|)$ then $|a| \vee |b| \in P$. As $|a| \leq |a| \vee |b|$ and $|b| \leq |a| \vee |b|$ implies that $|a| \in P$ and $|b| \in P$. Thus $P \in h_n(|a|) \cap h_n(|b|)$ and so, $h_n(|a|) \cap h_n(|b|) = h_n(|a| \vee |b|)$. □

Definition 3.3 (Munkres [8])**.** *A subbasis S for a topology on X is a collection of subsets of X whose union equals X. The topology generated by the subbasis S is defined to be the collection τ of all unions of finite intersections of elements of S.*

We define $S = \{\mathcal{P}_n(G) \setminus h_n(|a|) \ : \ a \in G\}$ the collection of a set complement of $h_n(|a|)$, for $a \in G$. S is subset of $\mathcal{P}_n(G)$. Note that, if $P \in \mathcal{P}_n(G)$ then there exists some $a \in G$ such that $|a| \notin P$. Thus $P \in \mathcal{P}_n(G) \setminus h_n(|a|)$. Therefore the collection $S = \{\mathcal{P}_n(G) \setminus h_n(|a|) \ : \ a \in G\}$ forms a subbase for a topology on $\mathcal{P}_n(G)$. Thus any basic open set in this topology is a finite intersections of members of S. Also, every open set of this topology is a union of finite intersections of members of S.

The following lemma provides a detailed description of the basis elements for this topology.

Lemma 3.4. *Let G be an ℓ-group with $|\mathcal{P}_2(G)| = |Min(G)| > n-2$. A basis for the open sets in $\mathcal{P}_n(G)$ is formed by the collection $\left\{\bigcap_{i=1}^{n-1} \mathcal{P}_n(G) \setminus h_n(|a_i|) : a_i \in G\right\}$.*

Proof. Let $P \in \bigcap_{i=1}^{m} (\mathcal{P}_n(G) \setminus h_n(|b_i|))$ for some m. As P is a minimal n-prime ℓ-subgroup, it can be expressed as the intersection of $n-1$ minimal prime ℓ-subgroups, say, $P_1, P_2, \ldots, P_{(n-1)}$.

Thus $P = \bigcap_{j=1}^{n-1} P_j$. Since $|b_i| \notin P$ for each $i = 1, 2, \ldots, m$, there exists some minimal prime ℓ-subgroup P_j such that $|b_i| \notin P_j$. Now, for each $i \in \{1, 2, \ldots, n-1\}$, define the set $B_i = \{|b_j| \ : \ |b_j| \notin P_i\}$. Clearly, each B_i is finite and nonempty for some i. Without loss of generality, assume that $B_1 \neq \emptyset$ and let $a_1 = \wedge B_1$.

Now let us define $a_i = \begin{cases} \wedge B_i, & \text{if } B_i \neq \emptyset; \\ a_1, & \text{if } B_i = \emptyset, \end{cases}$ where $2 \leq i \leq n-1$.

We show that, $\bigcup_{j=1}^{m} h_n(|b_j|) \subseteq \bigcup_{i=1}^{n-1} h_n(|a_i|)$. Let $Q \in \bigcup_{j=1}^{m} h_n(|b_j|)$ be a minimal n-prime ℓ-subgroup. Then $|b_j| \in Q$ for some $1 \leq j \leq m$. By the definition of a_i's, the expression of some $|a_k|$ must contains the element $|b_j|$. Hence $Q \in h_n(|a_k|)$ for $1 \leq k \leq n-1$.

Now, we show that $P \notin \bigcup_{i=1}^{n-1} h_n(|a_i|)$. Suppose on the contrary that, $P \in \bigcup_{i=1}^{n-1} h_n(|a_i|)$, then $P \in h_n(|a_i|)$ for some $i \in \{1, 2, \ldots, n-1\}$. Let $B_i = \{|b_1|, |b_2|, \ldots, |b_k|\}$ then P_i does not contain $|b_1|, |b_2|, \ldots, |b_k|$. As P_i is prime, we have $|a_i| = \bigwedge B_i \notin P_i$. Thus $P \notin h_n(|a_i|)$, a contradiction. Therefore, $P \in \bigcap_{i=1}^{n-1} \mathcal{P}_n(G) \setminus h_n(|a_i|)$. Thus $\left\{ \bigcap_{i=1}^{n-1} \mathcal{P}_n(G) \setminus h_n(|a_i|) \ : \ a_i \in G \right\}$ forms a basis for the topology generated by S on $\mathcal{P}_n(G)$. □

From Lemma 3.4, the closed sets in $P_n(G)$ are of the form $\bigcup_{i=1}^{n-1} h_n(|a_i|), \ a_i \in G$.

Lemma 3.5. *Let G be an ℓ-group with $|\mathcal{P}_2(G)| = |Min(G)| > n-2$. The space $\mathcal{P}_n(G)$ has a subbasis of clopen sets.*

Proof. Since $S = \{\mathcal{P}_n(G) \setminus h_n(|a|) \ : \ a \in G\}$ forms a subbase of topology on $\mathcal{P}_n(G)$, the set $\mathcal{P}_n(G) \setminus h_n(|a|)$ is open. Now, we prove that $h_n(|a|)$ is open for any $a \in G$. Clearly, if $h_n(|a|) = \emptyset$ or $h_n(|a|) = \mathcal{P}_n(G)$ then $h_n(|a|)$ is open. Let $P \in h_n(|a|)$, for $a \in G$. Then by the Theorem 2.7, there exist $a_1, a_2, \ldots, a_{n-1} \notin P$ such that $|a| \wedge |a_i| = 0$ and $|a_i| \wedge |a_j| = 0, i \neq j$. Clearly, $P \notin h_n(|a_i|)$ for each $i = 1, \ 2, \ \ldots, n-1$. Thus $P \notin \bigcup_{i=1}^{n-1} h_n(|a_i|)$. Now, if R is any minimal n-prime ℓ-subgroup then $|a| \wedge |a_i| = 0 = |a_i| \wedge |a_j| \in R$. Therefore, by the Lemma 2.6, $|a| \in R$ or $|a_i| \in R$ for some i. Thus $h_n(|a|) \bigcup \left(\bigcup_{i=1}^{n-1} h_n(|a_i|) \right) = \mathcal{P}_n(G)$ implies that $h_n(|a|)$ is open. Therefore, $\mathcal{P}_n(G)$ has a subbasis of clopen sets. □

Corollary 3.6. *Let G be an ℓ-group with $|\mathcal{P}_2(G)| = |Min(G)| > n-2$. The collection $\left\{ \bigcap_{i=1}^{n-1} \mathcal{P}_n(G) \setminus h_n(|a_i|) \ : \ a_i \in G \right\}$ forms a basis of clopen sets for $\mathcal{P}_n(G)$.*

Definition 3.7. *(Munkres [8]) A topological space is said to be zero-dimensional if it is a non-empty T_1 space with a base consisting of clopen sets.*

Definition 3.8. *(Munkres [8]) A topological space X is said to be a completely regular space if every closed set A in X and a point $x_0 \in X$, $x_0 \notin A$, then there exists a continuous function $f : X \to [0,1]$, such that $f(x_0) = 0$ and $f(A) = 1$.*

Corollary 3.9. *Let G be an ℓ-group with $|\mathcal{P}_2(G)| = |Min(G)| > n-2$. $\mathcal{P}_n(G)$ is zero-dimensional, completely regular and Hausdorff.*

Proof. It is well known that, every Hausdorff space is T_1 and every zero-dimensional space is completely regular. By the Lemma 3.5, $\mathcal{P}_n(G)$ has a basis of clopen sets. Thus it is enough to prove that $\mathcal{P}_n(G)$ is Hausdorff space. Let $P,\ Q$ be distinct elements of $\mathcal{P}_n(G)$ and let $a \in P \setminus Q$. Then $P \in h_n(|a|)$. By Theorem 2.7, there exist $a_1, a_2, \ldots, a_{n-1} \notin P$ such that $|a| \wedge |a_i| = 0$ and $|a_i| \wedge |a_j| = 0$. Then $P \notin \bigcup_{i=1}^{n-1} h_n(|a_i|)$ and $Q \notin h_n(|a|)$. Since $|a_i| \wedge |a_j| = 0 \in Q$, by Lemma 2.6, $|a_k| \in Q$ for some k and so, $Q \in \bigcup_{i=1}^{n-1} h_n(|a_i|)$. As $h_n(|a|) \bigcup \left(\bigcup_{i=1}^{n-1} h_n(|a_i|) \right) = \mathcal{P}_n(G)$ implies that $\mathcal{P}_n(G)$ is Hausdorff. Therefore, $\mathcal{P}_n(G)$ is zero-dimensional and completely regular. □

Definition 3.10. *Denote* $\pi_n(G) = \underbrace{\mathcal{P}_2(G) \times \cdots \times \mathcal{P}_2(G)}_{(n-1)\text{-times}} \setminus \Delta_n(G)$ *where*

$$\Delta_n(G) = \left\{ (P_1, P_2, \ldots, P_{n-1}) \in \underbrace{\mathcal{P}_2(G) \times \cdots \times \mathcal{P}_2(G)}_{(n-1)\text{-times}} \ : \ P_i = P_j \text{ for some } i \neq j \right\}$$

From Pawar and Thakare [9] (see Mundlik, Joshi and Halas [6]), the collection $\mathcal{B} = \{\mathcal{P}_2(G) \setminus h(|x|) \ : \ x \in G\}$ forms a basis for open sets in $\mathcal{P}_2(G)$. Thus $\underbrace{\mathcal{B} \times \cdots \times \mathcal{B}}_{(n-1)\text{-times}}$ forms a basis for open sets for the product topology on $\underbrace{\mathcal{P}_2(G) \times \cdots \times \mathcal{P}_2(G)}_{(n-1)\text{-times}}$. The subsequent Lemma gives us the basis for open sets for a subspace topology on $\pi_n(G)$.

Lemma 3.11. *Let G be an ℓ-group with $|\mathcal{P}_2(G)| = |Min(G)| > n-2$. The collection* $\left\{ \prod_{i=1}^{n-1} (\mathcal{P}_2(G) \setminus h_2(|a_i|)) \ : \ |a_i| \wedge |a_j| = 0 \text{ for } i \neq j \right\}$ *forms a basis for open sets in $\pi_n(G)$.*

Proof. Let $(P_1, P_2, \ldots, P_{n-1}) \in \prod_{i=1}^{n-1} (\mathcal{P}_2(G) \setminus h_2(|b_i|))$. Since $P_1,\ P_2,\ \ldots,\ P_{n-1}$ are minimal prime ℓ-subgroups, therefore for each $i,\ j \in \{1,\ 2,\ \ldots,\ n-1\}$, there exists $x_{ij} \in P_i \setminus P_j$, for $i \neq j$. By Theorem 2.6, there exists $y_i \notin P_i$ such that $|x_{ij}| \wedge |y_i| = 0$. Let $r_j \in P_j \setminus P_i$ and $|y_i| \wedge |r_j| = |x_{ji}|$(say) for $i \neq j$. Clearly, $|x_{ji}| \in P_j \setminus P_i$. This gives that $|x_{ij}| \wedge |x_{ji}| = |x_{ij}| \wedge |y_i| \wedge |r_j| = 0$. Now, setting $|a_i| = |b_i| \wedge |x_{1i}| \wedge |x_{2i}| \wedge \ldots \wedge |x_{(i-1)i}| \wedge |x_{(i+1)i}| \wedge \ldots \wedge |x_{(n-1)i}|$ for each $i \in \{1,\ 2,\ \ldots,\ n-1\}$, gives that each $|a_i| \notin P_i$

and so $P_i \in \mathcal{P}_2(G) \setminus h_2(|a_i|)$. Note that $(P_1, P_2, \ldots, P_{n-1}) \in \prod_{i=1}^{n-1} (\mathcal{P}_2(G) \setminus h_2(|a_i|)) \subseteq \prod_{i=1}^{n-1} (\mathcal{P}_2(G) \setminus h_2(|b_i|))$, where $|a_i| \wedge |a_j| = 0, i \neq j$. □

Definition 3.12. *(Munkres [8]) Let X and Y be topological spaces. A function $f : X \to Y$ is a local homeomorphism, if for every point $x \in X$, there exists an open set U containing x such that the image $f(U)$ is open in Y and the restriction $f|_U : U \to f(U)$ is a homeomorphism.*

Theorem 3.13. *Let G be an ℓ-group with $|\mathcal{P}_2(G)| = |Min(G)| > n-2$. The map $\phi : \pi_n(G) \to \mathcal{P}_n(G)$ defined by $\phi(P_1, P_2, \ldots, P_{n-1}) = \bigcap_{i=1}^{n-1} P_i$ is a local homeomorphism.*

Proof. Clearly, ϕ is well-defined. Now, we prove that ϕ is continuous.
By the Corollary 3.6, let $\bigcup_{i=1}^{n-1} h_n(|a_i|)$ be an open set in $\mathcal{P}_n(G)$.
We claim that $\phi^{-1}\left(\bigcup_{i=1}^{n-1} h_n(|a_i|)\right) = \bigcup_{1\leq r\leq n-1} \underbrace{h_2(|a_r|) \times h_2(|a_r|) \times \cdots \times h_2(|a_r|)}_{(n-1)-times}$.
Let $(P_1, P_2, \ldots, P_{n-1}) \in \phi^{-1}\left(\bigcup_{i=1}^{n-1} h_n(|a_i|)\right)$. Then $\phi(P_1, P_2, \ldots, P_{n-1}) = \bigcap_{i=1}^{n-1} P_i \in \bigcup_{i=1}^{n-1} h_n(|a_i|)$.
Thus $\bigcap_{i=1}^{n-1} P_i \in h_n(|a_r|)$ for some $r \in \{1, 2, \ldots, n-1\}$. Hence $|a_r| \in P_i$ for $i = \{1, 2, \ldots, n-1\}$,
implies that $|a_r| \in \bigcap_{i=1}^{n-1} P_i \subseteq P_i$.
Therefore $(P_1, P_2, \ldots, P_{n-1}) \in h_2(|a_r|) \times h_2(|a_r|) \times \cdots \times h_2(|a_r|) \subseteq$
$\bigcup_{1\leq r\leq n-1} \underbrace{h_2(|a_r|) \times h_2(|a_r|) \times \cdots \times h_2(|a_r|)}_{(n-1)-times}$. Thus $|a_r| \in P_i$ for $i \in \{1, 2, \ldots, n-1\}$, implies
that $|a_r| \in \bigcap_{i=1}^{n-1} P_i$. This means that $\bigcap_{i=1}^{n-1} P_i \in h_n(|a_r|) \subseteq \bigcup_{i=1}^{n-1} h_n(|a_i|)$. Thus $(P_1, P_2, \ldots, P_{n-1}) \in$
$\phi^{-1}\left(\bigcup_{i=1}^{n-1} h_n(|a_i|)\right)$ and so, ϕ is continuous.

Now, we prove that ϕ is an open map. Let $\prod_{i=1}^{n-1} \mathcal{P}_2(G) \setminus h_2(|a_i|)$ be a basic open set in $\pi_n(G)$, where $|a_i| \wedge |a_j| = 0$ for $i \neq j \in \{1, 2, \ldots, n-1\}$. We claim that $\phi\left(\prod_{i=1}^{n} \mathcal{P}_2(G) \setminus h_2(|a_i|)\right) = \mathcal{P}_n(G) \setminus \bigcup_{i=1}^{n-1} h_n(|a_i|)$. Let $Q \in \phi\left(\prod_{i=1}^{n} \mathcal{P}_2(G) \setminus h_2(|a_i|)\right)$. Then there exists $(P_1, P_2, \ldots, P_{n-1}) \in$

$\prod_{i=1}^{n-1} \mathcal{P}_2(G) \setminus h_2(|a_i|)$ such that

$\phi(P_1, P_2, \ldots, P_{n-1}) = \bigcap_{i=1}^{n-1} P_i = Q$. Since $(P_1, P_2, \ldots, P_{n-1}) \in \prod_{i=1}^{n-1} \mathcal{P}_2(G) \setminus h_2(|a_i|)$, we have $|a_i| \notin P_i$ for each i. Hence $|a_i| \notin \bigcap_{i=1}^{n-1} P_i = Q$, implies that $Q \in \mathcal{P}_n(G) \setminus \bigcup_{i=1}^{n-1} h_n(|a_i|)$. Let $R \in \mathcal{P}_n(G) \setminus \left(\bigcup_{i=1}^{n-1} h_n(|a_i|) \right)$. Then $|a_i| \notin R$ for each i. By the Theorem 1.15, we have $R = \bigcap_{i=1}^{n-1} P_i$, where P_i's are minimal prime ℓ subgroups. Let $B_i = \{|a_j| \ : \ |a_j| \notin P_i\}$. Since $|a_i| \wedge |a_j| = 0$ for $i \neq j$, we have $|B_i| = 1$ for each i. Thus there is one to one correspondence between $|a_1|, |a_2|, \ldots, |a_{n-1}|$ and prime ℓ-subgroups $P_1, P_2, \ldots, P_{n-1}$ such that each prime subgroup misses precisely one element. Hence $R \in \phi \left(\prod_{i=1}^{n} (\mathcal{P}_2(G) \setminus h_2(|a_i|)) \right)$ and so, ϕ is an open map. By Lemma 2.5, ϕ is onto.

Let $(P_1, P_2, \ldots, P_{n-1}) \in \pi_n(G)$ be any element. By Lemma 3.11, there exists a basic open set $U = \prod_{i=1}^{n-1} \mathcal{P}_2(G) \backslash h_2(|a_i|)$ of $(P_1, P_2, \ldots, P_{n-1})$, where $|a_i| \wedge |a_j| = 0$ for $i \neq j$. We show $\phi|_U$ is one-to-one. Let $(Q_1, Q_2, \ldots, Q_{n-1}), (R_1, R_2, \ldots, R_{n-1}) \in U$ be such that $\phi(Q_1, Q_2, \ldots, Q_{n-1}) = \phi(R_1, R_2, \ldots, R_{n-1})$, that is, $\bigcap_{i=1}^{n-1} Q_i = \bigcap_{i=1}^{n-1} R_i$. First, we claim that $\{Q_1, Q_2, \ldots, Q_{n-1}\} = \{R_1, R_2, \ldots, R_{n-1}\}$. Suppose not; then there exists R_k such that $R_k \notin \{Q_1, Q_2, \ldots, Q_{n-1}\}$. Then $R_k \neq Q_i$ for each i. Let $|b_i| \in Q_i \setminus R_k$ for each i and put $b = |b_1| \wedge |b_2| \wedge \ldots \wedge |b_{n-1}|$. Then $|b| \in \bigcap_{i=1}^{n-1} Q_i = \bigcap_{i=1}^{n-1} R_i$, implies that $|b| \in R_k$, a contradiction. Suppose that, $(Q_1, Q_2, \ldots, Q_{n-1}) \neq (R_1, R_2, \ldots, R_{n-1})$. Then there exists some i such that $Q_i \neq R_i$. It is also clear that there exists some k such that $Q_i = R_k$ for some $i \neq k$. Since $(Q_1, Q_2, \ldots, Q_{n-1}), (R_1, R_2, \ldots, R_{n-1}) \in U = \prod_{i=1}^{n-1} \mathcal{P}_2(G) \backslash h_2(|a_i|)$, we have $|a_i| \notin Q_i$ and $|a_k| \notin R_k = Q_i$ for $i \neq k$. But $|a_i| \wedge |a_k| = 0 \in Q_i$, a contradiction. Hence $\phi|_U$ is one-to-one. □

Definition 3.14. *(Munkres [8]) Let X and Y be topological spaces. A surjective map $f : X \to Y$ is said to be a perfect map, if it is continuous, closed and, for each $y \in Y$, the set $f^{-1}(y)$ is compact.*

Although the proof of the following theorem closely follows the proof presented in Mundlik and Joshi [7], we include it here for the sake of completeness.

Theorem 3.15. *Let G be an ℓ-group with $|\mathcal{P}_2(G)| = |Min(G)| > n - 2$. Then the following statements are equivalent for $n > 2$.*

1) $\mathcal{P}_n(G)$ *is compact.*
2) $\pi_n(G)$ *is compact.*
3) $\mathcal{P}_2(G)$ *is finite.*
4) $\mathcal{P}_n(G)$ *is finite.*

Proof. 1) $\Rightarrow$ 2) Let G be an ℓ-group with $|\mathcal{P}_2(G)| = |Min(G)| > n-2$. By the Theorem 3.13, the map $\phi : \pi_n(G) \to \mathcal{P}_n(G)$ defined by $\phi(P_1, P_2, \ldots, P_{n-1}) = \bigcap_{i=1}^{n-1} P_i$ is $(n-1)!$ to one, onto and continuous. We prove that ϕ is a closed map. Let C be a closed set in $\pi_n(G)$. Choose $y \in \mathcal{P}_n(G) \setminus \phi(C)$. Since ϕ is $(n-1)!$ to one, there exist distinct points $x_1, x_2, \ldots, x_{(n-1)!} \in \pi_n(G) \setminus C$ with $\phi(x_i) = y$. Since C is closed in $\pi_n(G)$, for each x_i there is a neighbourhood U_i such that $U_i \cap C = \emptyset$. Observe that, $\prod_{i=1}^{n-1} \mathcal{P}_2(G)$ is a Hausdorff space, and $\pi_n(G)$ is a subspace of $\prod_{i=1}^{n-1} \mathcal{P}_2(G)$, implies that $\pi_n(G)$ is a Hausdorff space. Hence for $x_i \neq x_j$ there exists disjoint neighbourhoods V_i and V_j of x_i and x_j (respectively). Now, local homeomorphism property of ϕ, implies that $\phi(V_i)$ is open in $\mathcal{P}_n(G)$ and contains y for each $i \in \{1, 2, \ldots, (n-1)!\}$. Thus $\bigcap_{i=1}^{(n-1)!} \phi(V_i)$ is a neighbourhood of y.

We now demonstrate that $\left(\bigcap_{i=1}^{(n-1)!} \phi(V_i)\right) \bigcap \phi(C) = \emptyset$. Assume, for the sake of contradiction, that $z \in \left(\bigcap_{i=1}^{(n-1)!} \phi(V_i)\right) \bigcap \phi(C)$. Therefore for each $i, 1 \leq i \leq (n-1)!$, there exists $v_i \in V_i$ and $c \in C$ such that $\phi(v_i) = z = \phi(c)$, a contradicting the fact that ϕ is an $(n-1)!$ to one map. Therefore $\left(\bigcap_{i=1}^{(n-1)!} \phi(V_i) \bigcap \phi(C)\right) = \emptyset$, implies that $\bigcap_{i=1}^{(n-1)!} \phi(V_i) \subseteq \mathcal{P}_n(G) \setminus \phi(C)$. This establishes that ϕ is a closed map. Since $\phi^{-1}(y) = \left\{x_1, x_2, \ldots, x_{(n-1)!}\right\}$ is a finite set for each $y \in \mathcal{P}_n(G)$, it is compact in $\pi_n(G)$. Therefore ϕ is a perfect map. Hence $\phi^{-1}(\mathcal{P}_n(G)) = \pi_n(G)$ is compact by Munkres[[8], Ex 12, Page 172].

2) $\Rightarrow$ 3) Assume that $\pi_n(G)$ is compact. As $\pi_n(G) \subseteq \prod_{i=1}^{n-1} \mathcal{P}_2(G)$ and $\prod_{i=1}^{n-1} \mathcal{P}_2(G)$ is Housdorff implies that $\pi_n(G)$ is closed. Hence, $\Delta_n(G)$ is open in $\underbrace{\mathcal{P}_2(G) \times \cdots \times \mathcal{P}_2(G)}_{(n-1)-times}$. Now, we prove that $\mathcal{P}_2(G)$ is discrete. Consider an arbitrary point $P \in \mathcal{P}_2(G)$. Choose $(P, P, Q_1, Q_2, \ldots, Q_{n-3}) \in \Delta_n(G)$, where all $Q_1, Q_2, \ldots, Q_{n-3}$ are distinct and $Q_i \neq P$. Since $\Delta_n(G)$ is open, there exists a neighbourhood $U \times U \times U_1 \times U_2 \times \cdots \times U_{n-3}$ of $(P, P, Q_1, Q_2, \ldots, Q_{n-3})$ such that $U \times U \times U_1 \times U_2 \times \cdots \times U_{n-3} \subseteq \Delta_n(G)$, where U is a neighbourhood of P. As each $Q_i \neq P$ using the Hausdorff property of $\mathcal{P}_2(G)$, we can find neighbourhoods U_i of Q_i and W_i of P such that $U_i \cap W_i = \emptyset$. Then

$U' = \bigcap_{i=1}^{n-3} W_i$ is a neighbourhood of P which is disjoint from each $U_i, i = 1, 2, \ldots, n-3$. We claim that $U' = \{P\}$. Suppose that U' contains another point, say $R(\neq P)$ then $(P, R, Q_1, Q_2, \ldots, Q_{n-3}) \in (U' \times U' \times U_1 \times U_2 \times \cdots \times U_{n-3}) \subseteq \Delta_n(G)$, a contradiction. Therefore $\{P\}$ is open in $\mathcal{P}_2(G)$ making $\mathcal{P}_2(G)$ is discrete and hence $\pi_n(G)$ is discrete. As it is compact, $\pi_n(G)$ is finite. Thus $\mathcal{P}_2(G)$ is finite.

3) $\Rightarrow$ 4) If $\mathcal{P}_2(G)$ is finite then there are only finitely many choices of subsets of $\mathcal{P}_2(G)$ of size $n-1$, hence only finitely many possible intersections. Consequently $\mathcal{P}_n(G)$ is finite.

4) $\Rightarrow$ 1) If $\mathcal{P}_n(G)$ is finite. It is well known that every finite set of topological space is compact. Consequently $\mathcal{P}_n(G)$ is compact. □

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