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EDGE STABLE SETS IN GRAPH

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Abstract. An edge set F of $E(G)$ is an edge stable set if each vertex $v \in V(G)$ has at least one incident edge e such that $e \notin F$. An edge H-dominating set is defined and discussed in [5]. The maximal edge stable set is a minimal edge H-dominating set. The maximal edge stable set with minimum cardinality is an edge stability number $\alpha_e(G)$. The upper bound of $\alpha_e(G)$ is obtained using an edge covering number $\alpha_1(G)$. The effect of a vertex(edge) removal operation on $\alpha_e(G)$ is observed. The inequality chain corresponding these extreme edge parameters is discussed.

Keywords: edge stable set, edge cover, edge H-dominating set.

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1. Introduction

Let $G = (V, E)$ be a finite, simple, undirected graph. The vertex set of G is denoted by $V(G)$ (or simply V) and edge set by $E(G)$ (or E). Each edge $e \in E$ is an unordered pair of distinct vertices of V. If an edge $e = uv$ then we say that a vertex u is adjacent to vertex v or the vertices u and v are neighbors and that e is incident to u and v.

A *subgraph* H of G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a vertex $v \in V(G)$, the graph $G - v$ is called a vertex-deleted subgraph of G. If $vu \in E(G)$ then $G - vu$ is called an edge-deleted subgraph of G.

The degree of a vertex v (denoted by $deg(v)$) is equal to the number of vertices that are adjacent to v. If there is a vertex $v \in V(G)$ such that $deg(v) = 0$ then v is called an *isolated vertex*. If $deg(v) = 1$ then v is called a *pendent vertex*. An edge $e = uv$ is an *isolated edge* of G if $deg(u) = 1$ and $deg(v) = 1$ in G also $e = uv$ is a *pendent edge* of G if the degree of exactly one of u and v is 1 in G. The *minimum degree* of the graph G is denoted by $\delta(G)$. $|V|$ denotes the cardinality of V.

A concept called an edge stable set of graph [4] is considered in this paper. The various results of an edge variant of the graph called edge H-domination have been discussed and proved in [5]. In this paper, the relations between these two edge variants namely edge H-domination and edge stability are observed and proved. The change in the edge stability number of the graph is observed under the operation vertex(edge) removal from the graph. The maximal edge stable set with minimum cardinality is defined. The edge parameters considered in this paper are expressed in terms of an inequality chain.

Example 1.1. Consider the following graph with vertices 1, 2, 3, 4, 5, 6 and edges 12, 23, 34, 45, 53, 36 and 16.

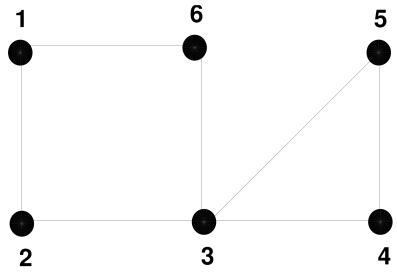


FIGURE 1. Graph G with 6 vertices

Some examples of edge stable sets of the graph G are

$$\{16, 23, 34\}, \{12, 36, 45\}, \{16, 23, 34, 35\}.$$

Obviously, every subset of an edge stable set is an edge stable set.

An edge stable set exists only in those graphs G for which $\delta(G) \geq 1$. In this paper, we consider only those graphs which has $\delta(G) \geq 2$.

Definition 1.2. (maximal edge stable set) An edge stable set F is a *maximal* edge stable set if $F \cup \{e\}$ is not an edge stable set for any edge $e \in E - F$.

Example 1.3. The edge stable set $F = \{16, 23, 34, 35\}$ is a maximal edge stable set of the graph given in figure 1.

Definition 1.4. An edge stable set with maximum cardinality is a *maximum* edge stable set. The cardinality of a maximum edge stable set is *the edge stability number* of G, denoted by $\alpha_e(G)$.

2. Upper bound of an edge stability number

Definition 2.1. [3] A subset F of $E(G)$ is said to be an *edge cover* of G if every vertex of G is an end vertex of some edge of F. An edge cover F is said to be a *minimal* edge cover if for every edge $e \in F$, $F - \{e\}$ is not an edge cover. An edge cover with minimum cardinality is a *minimum* edge cover of G. The cardinality of a minimum edge cover is called the *edge covering number* of G and it is denoted by $\alpha_1(G)$.

Remark 2.2. (**1**) If F is an edge stable set of the graph G with $\delta(G) \geq 2$ then $E - F$ is an edge cover of G. The complement of a maximum edge stable set of G is a minimum edge cover of G and vice versa. Therefore,

$$\alpha_1(G) + \alpha_e(G) = |E(G)|.$$

(**2**) If G is a graph with $\delta(G) \geq 2$ then any edge cover of G must contain atleast $\frac{n}{2}$ edges if n is even and $\frac{n+1}{2}$ edges if n is odd, where $n = |V(G)|$. Therefore, any edge stable set of G contains atmost $m - \frac{n}{2}$ edges, where $m = |E(G)|$. Therefore, $\alpha_e(G) \leq m - \frac{n}{2}$.

3. Relation between edge H-domination and edge stability

Definition 3.1. [5] (edge H-dominating set) A set $F \subseteq E(G)$ is said to be an *edge H-dominating set* of G if the following conditions are satisfied by any edge $e = uv$ in $E(G)$.

(1) If e is an isolated edge then $e \in F$.
(2) If e is a pendant edge with v as a pendant vertex and u is not a pendant vertex. If $uv \notin F$ then all the edges incident at u (except e) are in F.
(3) If e is a pendant edge with u as a pendant vertex and v is not a pendant vertex. If $uv \notin F$ then all the edges incident at v (except e) are in F.
(4) If neither u nor v is a pendant vertex and $uv \notin F$ then all the edges incident at u (except e) are in F or all the edges incident at v (except e) are in F.

Example 3.2. Consider the following graph with vertices 1, 2, 3, 4, 5, 6 and edges 12, 23, 34, 45, 51 and 56.

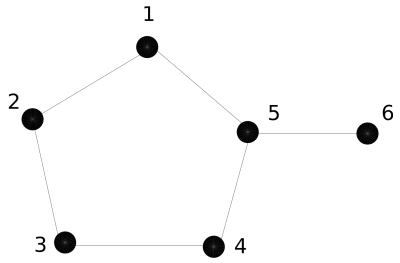


Figure 2. Graph G with 6 vertices

The edge sets $\{23, 45, 15\}$, $\{23, 45, 56\}$ are edge H-dominating sets of the graph given in figure 2.

Theorem 3.3. Let G be a graph with $\delta(G) \geq 2$. An edge stable set is a maximal edge stable set if and only if it is an edge H-dominating set.

Proof. Suppose that F is an edge stable set and suppose that F is maximal. Let $e = uv$ be an edge such that $e \notin F$. Since F is maximal $F \cup \{e\}$ is not an edge stable set. Therefore there is a vertex x such that all the edges incident at x are in $F \cup \{e\}$.

Suppose that $x \neq u$ and $x \neq v$. Since F is an edge stable set, there is an edge f incident at x such that $f \notin F$. Therefore $f \notin F \cup \{e\}$ because $f \neq uv$. This contradicts the above statement that all the edges incident at x are in $F \cup \{e\}$. Thus $x = u$ or $x = v$. Suppose that $x = u$. Since all the edges incident at x are in $F \cup \{e\}$ and e is an edge incident at u such that $e \notin F$. Therefore all the edges incident at u (except e) are in F. Similarly, if $x = v$ then all the edges incident at v (except e) are in F. This proves that F is an edge H-dominating set.

Conversely, suppose that F is an edge stable set which is also an edge H-dominating set. To prove that F is maximal. Let $e = uv$ be an edge such that $e \notin F$. Suppose that all the edges incident at u (except e) are in F then it implies that all the edges incident at u are in $F \cup \{e\}$. Similarly, if all the edges incident at v (except e) are in F then all the edges incident at v are in $F \cup \{e\}$. This proves that F is maximal. □

Definition 3.4. [5] Let G be a graph and F be an edge H-dominating set of G. The set F is a *minimal edge H-dominating set* of G if $F - \{e\}$ is not an edge H-dominating set for every edge $e \in F$.

Theorem 3.5. Let G be a graph with $\delta(G) \geq 2$. An edge set F is a maximal edge stable set of G then F is a minimal edge H-dominating set of G.

Proof. Let F be a maximal edge stable set of G and $e = uv \in F$. Since F is a maximal edge stable set, by theorem 3.3, F is an edge H-dominating set. Now consider $F - \{uv\}$ $= F_1$. Since $deg(u) \geq 2$ and $deg(v) \geq 2$ also F is an edge stable set of G, there are edges f and g incident at u and v respectively (other than uv) such that $f \notin F$ and $g \notin F$. Therefore, there is an edge incident at u which is not in F_1 and also there is an edge incident at v which is not in F_1. Thus, F_1 is not an edge H-dominating set of G. That is, $F - \{uv\} = F_1$ is not an edge H-dominating set of G. Thus, F is a minimal edge H-dominating set of G. □

Definition 3.6. [5] An edge H-dominating set with minimum cardinality is a *minimum* edge H-dominating set. The cardinality of a minimum edge H-dominating set is the *edge H-domination number* of G, denoted by $\gamma_H'(G)$.

Remark 3.7. Let G be a graph and an edge set F is a maximum edge stable set of G. The edge stability number $\alpha_e(G) = |F|$. Since F is also a maximal edge stable set of G, by theorem 3.5, F is a minimal edge H-dominating set of G. Therefore, $\gamma_H'(G) \leq \alpha_e(G)$.

Example 3.8. Consider the cycle graph C_6 with vertices 1, 2, 3, 4, 5, 6.

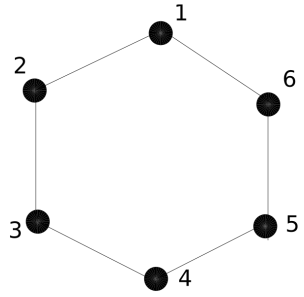


FIGURE 3. Cycle graph C_6

The minimum edge H-dominating sets of C_6 are $\{12, 45\}$, $\{23, 56\}$, $\{34, 61\}$ also each set is maximal edge stable set. But the maximum edge stable sets of C_6 are $\{12, 34, 56\}$, $\{23, 45, 61\}$. Therefore $\gamma_H'(C_6) = 2$ and $\alpha_e(C_6) = 3$. In this example, $\gamma_H'(C_6) < \alpha_e(C_6)$.

4. Vertex removal from the graph

We consider the operation vertex removal from the graph. Let G be a graph with $\delta(G)$ ≥ 3 and a vertex $v \in V(G)$.

Example 4.1. Consider 3-regular graph G with vertices 1, 2, 3, 4, 5, 6 and edges 12, 23, 34, 45, 56, 61, 14, 25 and 36.

The edge set $F_1 = \{25, 36, 34\}$ is a maximal edge stable set of $G - 1$, however F_1 is not a maximal edge stable set of G because by theorem 3.3, F_1 is not an edge H-dominating set of G. Also it is observed from the graph of figure 4 that the edge set $F = \{14, 12, 25, 36, 34, 56\}$ is a larger edge stable set of G containing F_1.

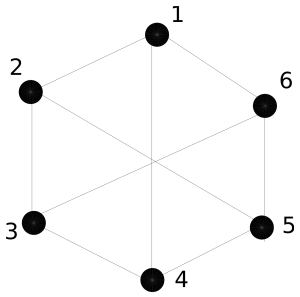


Figure 4. 3-regular graph G with 6 vertices

Proposition 4.2. If $F \subseteq E(G)$ is an edge stable set of $G - v$ then F is also an edge stable set of G.

Proof. Let x be any vertex of G. If $x \neq v$ then x is a vertex of $G - v$ and therefore there is an edge e incident at x such that $e \notin F$. If $x = v$ then any edge incident at v is not in F. Therefore F is an edge stable set of G. □

Proposition 4.3. Let G be a graph and a vertex $v \in V(G)$ then $\alpha_e(G - v) \leq \alpha_e(G)$.

Proof. Let F be a maximum edge stable set of $G - v$ then by proposition 4.2, F is also an edge stable set of G. Therefore $\alpha_e(G - v) = |F| \leq \alpha_e(G)$. □

5. Edge removal from the graph

We consider the graph G with $\delta(G) \geq 3$.

Proposition 5.1. Let G be a graph with $\delta(G) \geq 3$ and M is a maximal edge stable set of G. For $e = uv \in M$, $M - \{e\}$ is a maximal edge stable set of $G - e$.

Proof. It is obvious that $M - \{e\}$ is an edge stable set of $G - e$. To prove that $M - e$ is maximal, let xy be any edge of $G - e$. If $x, y \notin \{u, v\}$ then since M is a maximal edge stable set of G, all the edges incident at x are in M or all the edges incident at y are in M. These edges are the edges of $G - e$. Therefore, all the edges incident at x are in $M - e$ or all the edges incident at y are in $M - e$.

Suppose that $x = u$ and $y \neq v$. Now uy is an edge of G which is not in M. Since M is maximal, all the edges incident at u are in M or all the edges incident at y are in M. Since the edge $e = uv$ is not in $M - \{e\}$. It follows that all the edges of $G - e$ incident at u are in $M - \{e\}$ or all the edges incident at y are in $M - \{e\}$. Therefore, $M - \{e\}$ is a maximal edge stable set of $G - e$. □

Theorem 5.2. Let $e = uv$ be any edge of G then $\alpha_e(G - e) < \alpha_e(G)$.

Proof. Let M be any maximum edge stable set of $G - e$. There is an edge h incident at u and there is an edge h' incident at v such that $h, h' \notin M$. Let $M_1 = M \cup \{e\}$ then obviously, M_1 is an edge stable set of G. Therefore, $\alpha_e(G) \geq |M_1| > |M| = \alpha_e(G - e)$. □

Theorem 5.3. Let $e = uv$ be any edge of G then $\alpha_e(G - e) = \alpha_e(G) - 1$ if and only if there is a maximum edge stable set M of G such that $e \in M$.

Proof. Suppose that $\alpha_e(G - e) = \alpha_e(G) - 1$. Let M_1 be a maximum edge stable set of $G - e$ and let $M = M_1 \cup \{e\}$. Since $\alpha_e(G - e) = \alpha_e(G) - 1$, M is a maximum edge stable set of G and $e \in M$.

Conversely, suppose that there is a maximum edge stable set M of G such that $e \in M$. Let $M_1 = M - \{e\}$ then M_1 is a maximum edge stable set of $G - e$. Therefore, $\alpha_e(G - e) = |M_1| = |M| - 1 = \alpha_e(G) - 1$. □

6. Maximal edge stable set with minimum cardinality

Definition 6.1. A maximal edge stable set with minimum cardinality is called an $I^{'}$-set. The cardinality of $I^{'}$-set is denoted by $I^{'}(G)$.

Theorem 6.2. Let G be a graph with $\delta(G) \geq 3$ and $e = uv$ be any edge of G then $I^{'}(G - e) < I^{'}(G)$ if and only if there is an $I^{'}-$ set M of G such that $e \in M$.

Proof. Suppose that there is an $I^{'}-$ set M of G such that $e \in M$. Let $M_1 = M - \{e\}$ then by proposition 5.1, M_1 is a maximal edge stable set of $G - e$. Therefore, $I^{'}(G - e) \leq |M_1| < |M| = I^{'}(G)$.

Conversely, suppose that $I^{'}(G - e) < I^{'}(G)$. Let M_1 be an $I^{'}-$ set of $G - e$. Let $N = M_1 \cup \{e\}$ then N is a maximal edge stable set of G. Since $I^{'}(G - e) < I^{'}(G)$, N is an $I^{'}-$ set of G. Obviously, $e \in N$. □

Corollary 6.3. Let G be a graph with $\delta(G) \geq 3$. If $I^{'}(G - e) < I^{'}(G)$ then $I^{'}(G - e) = I^{'}(G) - 1$.

Remark 6.4. Let G be a graph with $\delta(G) \geq 2$ then By remark 2.2(2) and remark 3.7, we have the following inequality chain.

$$\gamma_H^{'}(G) \leq I^{'}(G) \leq \alpha_e(G) \leq m - \tfrac{n}{2}$$
$$\gamma_H^{'}(G) \leq I^{'}(G) \leq \alpha_e(G) \leq \Gamma_H^{'}(G)$$

7. Concluding remarks and further scope

The edge stability in graph is a new edge variant and hereditary property which represents a particular edge set of the graph. In this paper, it is proved that a vertex (or an edge) removal operation may decrease (does not increase) the edge stability number of the graph. How many units the number decreases is the further scope of investigation. The relation between edge H-domination number and edge stability number is written as an inequality chain. This chain can be improved further with other edge parameters like edge domination number, edge independence number, edge covering number etc. of the graph.

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