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CERTAIN CURVATURE CONDITIONS ON $(LCS)_N$-MANIFOLD ACKNOWLEDGING CONFORMAL η-EINSTEIN SOLITON

Srabani Debnath

Abstract. The present paper focuses on the study of conformal η-Einstein soliton on $(LCS)_n$-manifold satisfying certain curvature conditions.The torse-forming vector field has also been considered for conformal η-Einstein soliton on $(LCS)_n$-manifold. Finally, an example has been set up to verify the results obtained.

Keywords: $(LCS)_n$- manifold, Einstein soliton, Conformal η-Einstein soliton, W_2-curvature tensor . **2010 AMS Subject Classification:** 53C15, 53C25, 53C44.

1. Introduction

In 2012, Bhattacharyya and Debnath [2] introduced a type of generalized Ricci flow whose special case is the Einstein flow. Later, in 2016, the concept of Einstein soliton was established in differential geometry by G.Catino and L.Mazzieri [4] which represents a self-similar solution to the Einstein flow

$$\frac{\partial g}{\partial t} = -2(S - \frac{r}{2}g)$$

where g represents the Riemannian metric, S stands for the Ricci tensor and r stands for the scalar curvature.

In 2018 [3], the concept of η-Einstein soliton on a Riemannian manifold (M, g) was developed which is represented by the following equation:

$$\pounds_\xi g + 2S + (2\lambda - r)g + 2\mu\eta \otimes \eta = 0. \tag{1}$$

Here $\pounds_\xi$ denotes the Lie derivative along the direction of the vector field ξ and λ, μ are considered to be real constants. When μ becomes 0, the soliton reduces to the Einstein soliton. If the manifold admits a constant scalar curvature, the η-Einstein soliton converts to η-Ricci soliton [5] and furthermore, if we consider μ to be zero, it becomes a Ricci soliton ([7], [8], [22]).

The conformal η-Ricci soliton was initiated by M.D.Siddiqi [20] in 2018 and it is defined by,

$$\pounds_\xi g + 2S + (2\lambda - (p + \frac{2}{n}))g + 2\mu\eta \otimes \eta = 0. \tag{2}$$

Here p is a scalar non-dynamical field and n represents the dimension of the manifold.

Later on, the concept of conformal Einstein soliton was introduced by Roy et al. [12]

which was introduced on an n-dimensional manifold by,

$$\pounds_V g + 2S + (2\lambda - r + (p + \frac{2}{n}))g = 0. \tag{3}$$

Next, focusing on the case where the potential vector field ξ is of gradient type, i.e., $\xi = grad(f)$, for a non-constant smooth function f on the manifold M, Roy et al. [13] established the concept of conformal η-Einstein soliton on an n-dimensional manifold given by,

$$\pounds_\xi g + 2S + (2\lambda - r + (p + \frac{2}{n}))g + 2\mu\eta \otimes \eta = 0. \tag{4}$$

The soliton is happened to be shrinking, steady or expanding depending on $\lambda < 0, \lambda = 0$ or $\lambda > 0$.

Ricci soliton, Einstein soliton and their generalisations and related investigations have been done by many authors ([1], [**?**], [3], [**?**], [6], [19]) in different contexts, such as on Kahler manifolds, on contact and Lorentzian manifolds, on k-contact manifolds etc.

Being inspired by their study, here conformal η-Einstein soliton on $(LCS)_n$-manifold have been considered. After giving a brief introduction in the first section and recalling some basic ideas on $(LCS)_n$-manifolds in section 2, section 3 focuses on some curvature conditions on an $(LCS)_n$-manifold acknowledging conformal η-Einstein soliton. The section 4 emphasises on the $(LCS)_n$-manifold which is Einstein semi-symmetric. In section 5, the study was chronicled on $(LCS)_n$-manifold admitting conformal η-Einstein soliton whose vector field is torse-forming. Finally, in the last section, an example has been set up to verify the results.

2. Preliminaries

The concept of Lorentzian concircular structure manifold (briefly, $(LCS)_n$-manifold) was initiated by Shaikh [14] as a generalization of LP-Sasakian manifold which was established by Matsumoto [9] and later on by Mihai and Rosca [10].

Let us consider a Lorentzian manifold (M, g) of dimension n which admits a unit timelike concircular vector field ξ and it satisfies, $g(\xi, \xi) = -1$. ξ, for being a unit concircular vector field, there should be a nonzero 1-form η which satisfies $g(X, \xi) = \eta(X)$. Also $\nabla\xi = \alpha(I + \eta \otimes \xi)$ where α is a nowhere zero smooth function on the manifold which obey the following relations:

$$(\nabla_X \alpha) = (X\alpha) = d\alpha(X) = \rho\eta(X), \tag{5}$$

$\rho = -(\xi\alpha)$, a scalar function, ∇ denotes the Levi-Civita connection with respect to g and X is a vector field. The $(1, 1)$ tensor field ϕ satisfies,

$$\alpha\phi X = \nabla_X \xi. \tag{6}$$

For a $(LCS)_n$ manifold $(n > 2)$, the following relations are satisfied([15], [16], [17], [18], [23]):

$$\eta(\xi) = -1,\ \phi\xi = 0,\ \eta(\phi X) = 0,\ g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y) \tag{7}$$

$$\begin{aligned}
\phi^2 X &= X + \eta(X)\xi && (8)\\
S(X,\xi) &= (n-1)(\alpha^2-\rho)\eta(X) && (9)\\
R(X,Y)\xi &= (\alpha^2-\rho)[\eta(Y)X - \eta(X)Y] && (10)\\
R(\xi,Y)Z &= (\alpha^2-\rho)[g(Y,Z)\xi - \eta(Z)Y] && (11)\\
(\nabla_X\phi)Y &= \alpha[g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X] && (12)\\
(X\rho) &= d\rho(X) = \beta\eta(X), && (13)\\
R(X,Y)Z &= \phi R(X,Y)Z + (\alpha^2-\rho)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi && (14)
\end{aligned}$$

for any vector fields X, Y, Z on M and β being a scalar function represented by,

$$\beta = -(\xi\rho),$$

where R stands for the curvature tensor and S denotes the Ricci tensor of the manifold. Again,

$$\begin{aligned}
R(X,Y)Z &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, && (15)\\
W_2(X,Y)Z &= R(X,Y)Z + \frac{1}{n-1}[g(X,Z)QY - g(Y,Z)QX] && (16)
\end{aligned}$$

are the expressions for the Riemannian-Chistoffel curvature tensor R and the W_2-curvature tensor in a $(LCS)_n$ manifold (M^n, g), and Q, the Ricci operator is defined by $S(X,Y) = g(QX,Y)$, for $X, Y, Z \in \chi(M)$, the Lie algebra of vector fields of M.
If the manifold further admits a conformal η-Einstein soliton, then the expression for $S(X,Y)$ becomes [6],

$$S(X,Y) = [\frac{r}{2} - \lambda - \alpha - (\frac{p}{2} + \frac{1}{n})]g(X,Y) - (\alpha+\mu)\eta(X)\eta(Y). \tag{17}$$

All these definitions and results will be required in next sections.

3. Conformal η-Einstein Soliton on $(LCS)_n$-Manifolds Satisfying Projective Curvature Tensor, $R(\xi,X).S = 0$ and $W_2(\xi,X).S = 0$

Theorem 3.1. *Let (M,g) be an $(LCS)_n$-manifold which admits a conformal η-Einstein soliton (g,ξ,λ,μ). If the manifold fulfils the curvature condition $R(\xi,X).S = 0$, then the manifold turns into an Einstein manifold.*

Proof. Let us consider an $(LCS)_n$-manifold which admits a conformal η-Einstein soliton (g,ξ,λ,μ) which meets the curvature condition $R(\xi,X).S = 0$, then it can be written as,

$$S(R(\xi,X)Y,Z) + S(Y,R(\xi,X)Z) = 0 \tag{18}$$

for all $X, Y, Z \in \chi(M)$. Using (17) into the above expression we have,

$$\begin{aligned}
[\frac{r}{2} - \lambda - \alpha - (\frac{p}{2} + \frac{1}{n})](g(R(\xi,X)Y,Z) + g(Y,R(\xi,X)Z)) && (19)\\
-(\alpha+\mu)[\eta(R(\xi,X)Y)\eta(z) + \eta(Y)\eta(R(\xi,X)Z)] &= 0.
\end{aligned}$$

With the help of (11) we arrive at,

$$(\alpha+\mu)(\alpha^2-\rho)[g(X,Y)\eta(Z) + g(X,Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z)] = 0. \tag{20}$$

Substituting Z by ξ in (20) and considering (7) we get,

$$(\alpha+\mu)(\alpha^2-\rho)g(\phi X,\phi Y)=0 \tag{21}$$

for all $X,Y\in\chi(M)$, which gives,

$$\alpha+\mu=0,$$

since $g(\phi X,\phi Y)\neq 0$ and considering $\alpha^2\neq\rho$.
Hence (17) reduces to,

$$S(X,Y)=[\frac{r}{2}-\lambda-\alpha-(\frac{p}{2}+\frac{1}{n})]g(X,Y), \tag{22}$$

for all $X,Y\in\chi(M)$ and hence the theorem follows.

□

The next result in this section focuses on W_2-curvature tensor which was first considered in 1970 by Pokhariyal and Mishra [11].

Theorem 3.2. *Let (M,g) be an $(LCS)_n$-manifold which admits a conformal η-Einstein soliton (g,ξ,λ,μ). If the curvature condition $W_2(\xi,X).S=0$ is satisfied by the manifold, then either the manifold becomes an Einstein manifold or the manifold is of constant curvature $r=2\lambda+2\alpha+2(\frac{p}{2}+\frac{1}{n})$ or both.*

Proof. Let us consider an $(LCS)_n$-manifold which admits a conformal η-Einstein soliton (g,ξ,λ,μ) and satisfies the curvature condition $W_2(\xi,X).S=0$. Then it can be written as,

$$S(W_2(\xi,X)Y,Z)+S(Y,W_2(\xi,X)Z)=0, \tag{23}$$

for all $X,Y,Z\in\chi(M)$. Using (17) into the above expression we obtain,

$$\begin{aligned}[\frac{r}{2}-\lambda-\alpha-(\frac{p}{2}+\frac{1}{n})](g(W_2(\xi,X)Y,Z)+g(Y,W_2(\xi,X)Z)) \\ -(\alpha+\mu)[\eta(W_2(\xi,X)Y)\eta(z)+\eta(Y)\eta(W_2(\xi,X)Z)] &= 0.\end{aligned} \tag{24}$$

(17) also gives us,

$$QX=[\frac{r}{2}-\lambda-\alpha-(\frac{p}{2}+\frac{1}{n})]X-(\alpha+\mu)\eta(X)\xi. \tag{25}$$

Considering X to be ξ in (16) and using (11), (24), (25), we have,

$$W_2(\xi,Y)Z=Ag(Y,Z)\xi-B\eta(Z)Y+(A-B)\eta(Y)\eta(Z)\xi \tag{26}$$

where

$$\begin{aligned}A &= \alpha^2-\rho-\frac{1}{n-1}(\frac{r}{2}-\lambda+\mu-(\frac{p}{2}+\frac{1}{n})),\\ B &= \alpha^2-\rho-\frac{1}{n-1}(\frac{r}{2}-\lambda-\alpha-(\frac{p}{2}+\frac{1}{n})).\end{aligned}$$

Taking inner product with the vector field ξ and using (7) the above equation yields

$$\eta(W_2(\xi,Y)Z)=-Ag(\phi Y,\phi Z). \tag{27}$$

Using (26) and (27) in (24) and replacing Z by ξ and recalling (7) to obtain,

$$[(B-A)(\frac{r}{2}-\lambda-\alpha-(\frac{p}{2}+\frac{1}{n}))-A(\alpha+\mu)]g(\phi X,\phi Y)=0 \tag{28}$$

$\forall X, Y \in \chi(M)$. Since $g(\phi X, \phi Y) \neq 0$, so we can reach to the conclusion that

$$(B - A)(\frac{r}{2} - \lambda - \alpha - (\frac{p}{2} + \frac{1}{n})) - A(\alpha + \mu) = 0. \tag{29}$$

Recalling the expressions of A and B and simplifying, we arrive at,

$$(\alpha + \mu)[\frac{1}{n-1}(r - 2\lambda - \alpha + \mu - 2(\frac{p}{2} + \frac{1}{n})) - \alpha^2 + \rho], \tag{30}$$

which yields,
either

$$\alpha + \mu = 0,$$

or,

$$r - 2\lambda - \alpha + \mu - 2(\frac{p}{2} + \frac{1}{n}) = (n-1)(\alpha^2 - \rho).$$

Now for $\alpha + \mu = 0$, (17) indicates that the manifold reduces to an Einstein manifold. Otherwise, substituting Y by ξ in (17) and comparing with (9) we have

$$r = 2\lambda - 2\mu + 2(\frac{p}{2} + \frac{1}{n}) + 2(n-1)(\alpha^2 - \rho). \tag{31}$$

Comparing and combining these two expressions of r, we have,

$$r = 2\lambda + 2\alpha + 2(\frac{p}{2} + \frac{1}{n}) \tag{32}$$

and thus the theorem is established.

□

4. Conformal η-Einstein Soliton On An Einstein Semi-Symmetric $(LCS)_n$-Manifold

Definition 4.1. *An $(LCS)_n$-manifold (M, g) is known to be Einstein semi-symmetric* [21] *if $R.E = 0$ where the Einstein tensor E is given by,*

$$E(X, Y) = S(X, Y) - \frac{r}{n} g(X, Y) \tag{33}$$

where $X, Y \in \chi(M)$ and r is the scalar curvature of the manifold.

Lemma 4.2. *An Einstein semi-symmetric $(LCS)_n$-Manifold is an Einstein manifold.*

Proof. Let us consider an $(LCS)_n$-Manifold which is Einstein semi-symmetric. So it obeys the curvature condition $R.E = 0$. So, we can write,

$$E(R(X, Y)Z, W) + E(Z, R(X, Y)W) = 0 \tag{34}$$

for all vector fields X, Y, Z, W on M. Then taking the help of (33), (34) becomes,

$$S(R(X, Y)Z, W) + S(Z, R(X, Y)W) = \frac{r}{n}[g(R(X, Y)Z, W) + g(Z, R(X, Y)W)]. \tag{35}$$

Substituting X and Z by ξ in (35) and using (9) and (11), we obtain,

$$S(Y, W) = (n-1)(\alpha^2 - \rho)g(Y, W) \tag{36}$$

for all vector fields Y, W on M, which establishes that the manifold is Einstein. □

Theorem 4.3. *Let (M, g) be an $(LCS)_n$-manifold acknowledging a conformal η-Einstein soliton (g, ξ, λ, μ). If the manifold is Einstein semi-symmetric, then the soliton is happened to be steady, shrinking or expanding according as $(n-1)(n-2)(\alpha^2-\rho)-2\alpha-p-\frac{2}{n} = 0, < 0$ or > 0 respectively.*

Proof. Let us consider an $(LCS)_n$-manifold which admits a conformal η-Einstein soliton (g, ξ, λ, μ). If the manifold is Einstein semi-symmetric, then Lemma 1 assures that the manifold is Einstein. So, (17) gives,

$$\alpha + \mu = 0. \tag{37}$$

Replacing Y by ξ in (17) on using (37) and comparing with (9) to obtain,

$$\frac{r}{2} - \lambda - \alpha - (\frac{p}{2} + \frac{1}{n}) = (n-1)(\alpha^2 - \rho). \tag{38}$$

Again, taking an orthonormal basis $\{e_1, e_2, ..., e_n\}$ of (M, g) and then replacing X and Y by e_i in (17) and summing over i to arrive at,

$$r = \frac{2}{n-2}[\lambda n + n\alpha + \frac{pn}{2} + 1] \tag{39}$$

with the help of (37). So, comparing the two expressions of r, we finally came to the conclusion that

$$\lambda = \frac{1}{2}[(n-1)(n-2)(\alpha^2 - \rho) - 2\alpha - p - \frac{2}{n}] \tag{40}$$

and this establishes the result. □

5. Conformal η-Einstein Soliton On An $(LCS)_n$-Manifold with Torse-Forming Reeb Vector Field

In this section, the nature of conformal η-Einstein soliton on $(LCS)_n$-manifold with torse-forming vector field will be studied.

Definition 5.1. *A vector field ξ on an $(LCS)_n$-manifold is said to be a torse-forming vector field* [24] *if*

$$\nabla_X \xi = fX + \gamma(X)\xi \tag{41}$$

$\forall X \in \chi(M)$ for a smooth function $f \in C^\infty(M)$ and γ is a 1-form.

A torse-forming vector field ξ is said to be recurrent when $f = 0$.

Definition 5.2. *A vector field V is considered as a concurrent vector field when it satisfies*

$$\nabla_X V = 0$$

for any $X \in \chi(M)$.

Theorem 5.3. *Let (M, g) be an $(LCS)_n$-manifold acknowledging a conformal η-Einstein soliton (g, ξ, λ, μ) with torse-forming vector field ξ, then the manifold reduces to η-Einstein manifold with scalar curvature*

$$r = 2\lambda - 2\mu + 2(\frac{p}{2} + \frac{1}{n}) + (n-1)(\alpha^2 - \rho).$$

Proof. Let (M, g) be an $(LCS)_n$-manifold which admits a conformal η-Einstein soliton and consider that the Reeb vector field ξ of the manifold to be torse-forming. Then, taking the inner product of (41) with ξ, we have,

$$g(\nabla_X \xi, \xi) = f\eta(X) - \gamma(X) \ \forall X \in \chi(M). \tag{42}$$

Again, taking the inner product of (6) with ξ, with the help of (7), yields,

$$g(\nabla_X \xi, \xi) = 0. \tag{43}$$

Comparing the above two equations, we conclude that,

$$\gamma = f\eta,$$

which results, for an $(LCS)_n$-manifold with torse-forming vector field,

$$\nabla_X \xi = f[X + \eta(X)\xi]. \tag{44}$$

Now considering that the manifold admits a conformal η-Einstein soliton and keeping in mind that

$$(\pounds_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi), \tag{45}$$

we obtain from (4),

$$S(X, Y) = [\frac{r}{2} - \lambda - f - (\frac{p}{2} + \frac{1}{n})]g(X, Y) - (f + \mu)\eta(X)\eta(Y). \tag{46}$$

which establishes the manifold to be an η-Einstein manifold. From(44) substituting $Y = \xi$ we get,

$$S(X, \xi) = [\frac{r}{2} - \lambda + \mu - (\frac{p}{2} + \frac{1}{n})]\eta(X). \tag{47}$$

Comparing the above expression with (9), we have,

$$\frac{r}{2} - \lambda + \mu - (\frac{p}{2} + \frac{1}{n}) = (n-1)(\alpha^2 - \rho), \tag{48}$$

which results,

$$r = 2\lambda - 2\mu + 2(\frac{p}{2} + \frac{1}{n}) + (n-1)(\alpha^2 - \rho). \tag{49}$$

□

Theorem 5.4. *Let (g, ξ, λ, μ) be a conformal η-Einstein soliton on an $(LCS)_n$-manifold where ξ is a torse-forming vector field, then the following results hold:*
(i) $f = -(b + \mu)$
(ii) if ξ is recurrent, then it is also concurrent and Killing vector field.

Proof. Let us consider an $(LCS)_n$-manifold (M, g) which admits a conformal η-Einstein soliton (g, ξ, λ, μ), where ξ is a torse-forming vector field. Considering (17) with $a = \frac{r}{2} - \lambda - \alpha - (\frac{p}{2} + \frac{1}{n})$ and $b = -(\alpha + \mu)$ and recalling (45) the expression (4) yields,

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) + [2a + 2\lambda - r + (p + \frac{2}{n})]g(X, Y) + [2(b + \mu)]\eta(X)\eta(Y) = 0 \tag{50}$$

Setting $X = Y = \xi$ in (50) and using (6) and (7), we obtain,

$$2a + 2\lambda - r + (p + \frac{2}{n}) = 2(b + \mu), \tag{51}$$

which makes (50),

$$g(\nabla_X\xi, Y) + g(X, \nabla_Y\xi) + 2(b+\mu)g(\phi X, \phi Y) = 0. \tag{52}$$

Applying (44) in (52) we have,

$$2(f+b+\mu)g(\phi X, \phi Y) = 0, \tag{53}$$

which results, $f = -(b+\mu)$, since, $g(\phi X, \phi Y) \neq 0$.
If we consider ξ to be recurrent, then from (44), we have,

$$\nabla_X\xi = 0$$

$\forall X \in \chi(M)$ and which proves the vector field ξ to be concurrent. (45) also indicates that ξ is a Killing vector field in that case and hence the theorem is established. □

6. Example

Consider the 4-dimensional manifold $M^4 = (x, y, z, t) \in \mathbb{R}^4$ with the metric $g = -dt^2 + e^{2t}(dx^2 + dy^2 + dz^2)$ where (x, y, z, t) are standard co-ordinates in $\mathbb{R}^4$.
Let us consider the vector fields $\{e_1, e_2, e_3, e_4\}$ on M given by, $e_1 = \frac{\partial}{\partial t}, e_2 = e^{-t}\frac{\partial}{\partial x}, e_3 = e^{-t}\frac{\partial}{\partial y}, e_4 = e^{-t}\frac{\partial}{\partial z}$ which are linearly independent.
Let us consider the Riemannian metric g as,

$$g(e_1, e_1) = -1,\ g(e_i, e_j) = \delta_{ij},\ i, j \neq 1.$$

Let the 1-form η be defined by,

$$\eta(Z) = g(Z, e_1)$$

for any vector field Z on M and the (1,1)-tensor field ϕ is defined by,

$$\phi(e_1) = 0, \phi(e_i) = e_i\ (i = 2, 3, 4).$$

Then with the help of the linear property of ϕ and g, we have,

$$\eta(e_1) = -1,\ \phi^2(X) = X + \eta(X)e_1 \text{ and } g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$$

for any $X, Y \in \chi(M)$.
Considering $\xi = e_1$, using Koszul's formula,

$$\nabla_{e_i}e_1 = e_i,\ \nabla_{e_i}e_i = e_1,\ (i = 2, 3, 4)$$

and all others are zero.
Consequently, we have $(M, g, \xi, \eta, \phi, \alpha)$ is an $(LCS)_n$-manifold, where $\alpha = 1$.
Using the above relations, we can compute the components of the curvature tensor R and the Ricci tensor S as follows:

$$R(e_1, e_i)e_1 = -e_i,\ R(e_1, e_i)e_j = -\delta_{ij}e_1,$$
$$R(e_i, e_j)e_k = \delta_{ik}e_j - \delta_{jk}e_i,\ R(e_i, e_j)e_1 = 0,\ i, j, k = 2, 3, 4$$
$$S(e_1, e_1) = -3,\ S(e_i, e_j) = 3\delta_{ij},\ i, j = 2, 3, 4.$$

So, we can deduce from here that the scalar curvature $r = 12$.
Next, let us consider the vector field $V = \xi$. Then we can easily calculate,

$$(\pounds_\xi g)(e_1, e_1) = 0,\ (\pounds_\xi g)(e_i, e_i) = 2,\ i = 2, 3, 4.$$

then considering that the manifold admits a conformal η-Einstein soliton and replacing X and Y by $e_i,\ i = 1, 2, 3, 4$ we arrive at,

$$2\lambda + (p + \frac{2}{n}) = 4,$$

which yields,

$$\mu = -1.$$

Now we see that in this manifold $R(\xi, X).S = 0$ and the above result gives, $\alpha + \mu = 0$, which results the manifold to be Einstein and verifies theorem 1.

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Department of Mathematics
Budge Budge College
Kolkata-700137, India
Email: srabani.1986@gmail.com