Journal of Indian Acad. Math
Vol. 47, No. 2 (2025) pp. 483-493

ISSN: 0970-5120

*Deepak Rout*[1],
*Tanmoy Som*[2],
*Jayanta Sarkar*[3]

# RESULTS ON FIXED POINTS UTILIZING TWO OR MORE GENERALIZED METRICS IN $G$-METRIC SPACE

**Abstract**

The primary objective of this study is to develop the fixed point conclusions for contractions of Banach, Chatterjea, and Kannan types within a complete $G$-metric space that utilizes two generalized metrics, $G$ and $H$. Additionally, we provide all the sufficient conditions under which a fixed point exists when considering three generalized metrics, $G$, $H$, and $I$, for the corresponding contractions in the conclusion section. To reinforce the findings, we include illustrative examples and numerical calculations, which also serve to extend certain theorems found in recent literature.

**Keywords**: $G$-metric space; fixed point theory; generalized metric space; complete metric space.

**Mathematics Subject Classification (2020)**: 54H25; 47H10.

## 1 Introduction

In 1922, Banach [1] established both the conditions for existence and uniqueness of fixed points for a class of certain types of maps in metric space. This result was a remarkable contribution due to its wide theoretical and numerical applications. The conceptualization of spaces or, in particular generalization of metric spaces and studying properties on them has consistently been an engrossing area of maths. Moreover, studying fixed point theory on such spaces has always been an interesting area of work for mathematicians due to its applications not only in other areas of mathematics but also in a few other disciplines.

Results related to fixed points in $G$-metric spaces involving two or more generalized metrics are a fascinating area of research within the broader field of fixed point theory and generalized metric spaces. These types of results generally extend classical fixed-point theorems to spaces that are more generalized than traditional metric spaces, and they have applications in various fields, including analysis, optimization, and even differential equations.

Mustafa and Sims [2] in 2006 have proposed the framework of $G$-metric space. Later, several generalized fixed point conclusions in $G$-metric space were obtained by [4–11, 17, 21]. Recently, many authors have generalized several fixed point results with two or more generalized metrics in complete $G$-metric space. See, [3, 12, 13] for related results and extensions. The $G$-metric generalizes the classical metric by considering the distance between three points at once rather than two, which allows for more nuanced distance calculations and fixed-point theorems.

The introduction of two or more generalized metrics in the $G$-metric space extends the G-metric concept further, providing an even broader framework. In this context, instead of a single metric governing the space, multiple metrics are used. Each of these metrics could capture different distance properties or topological structures of the space.

Let us recall some of the preliminary results on the metric space for different types of contractions namely Banach, Chatterjea and Kannan contractions.

**Theorem 1.1.** [1] *Suppose $\mathcal{T}$ is a self map defined on a complete metric space $(X, \rho)$ for which there exists a constant $k \in [0, 1)$ with*

$$\rho(\mathcal{T}u, \mathcal{T}v) \leq k\rho(u, v) \,\forall\, u, v \in X.$$

*Then, there exists a single fixed point of $\mathcal{T}$.*

**Theorem 1.2.** [15] *Suppose $\mathcal{T}$ is a self mapping on complete metric space $(\mathcal{X}, \rho)$ for which $\exists\, k \in [0, \frac{1}{2})$ with*

$$\rho(\mathcal{T}u, \mathcal{T}v) \leq k[\rho(u, \mathcal{T}u) + \rho(v, \mathcal{T}v)] \ \ \forall u, v \in X.$$

*Then, $\mathcal{T}$ admits a unique fixed point.*

**Theorem 1.3.** [16] *Suppose $\mathcal{T}$ be a self mapping on complete metric space $(\mathcal{X}, \rho)$ for which $\exists\, k \in [0, \frac{1}{2})$ with*

$$\rho(\mathcal{T}u, \mathcal{T}v) \leq k[\rho(u, \mathcal{T}v) + \rho(v, \mathcal{T}u)] \ \ \forall u, v \in X.$$

*Then, there exists a single fixed point of $\mathcal{T}$*

In the present work, we have generalized Banach, Chatterjea, and Kannan type contractions in space equipped with two generalized metrics $\mathcal{G}$ and $H$. For complementary and related results, see, [17–19]. Here, we discuss some of the necessary lemmas and definitions in $\mathcal{G}$-metric space, which are required to prove our results. However, for complementary and related information, see [2, 20, 21].

**Definition 1.1.** *Suppose $\mathcal{X} \neq \emptyset$ and $\mathcal{G} : \mathcal{X}^3 \to [0, \infty)$ is a map. Then $\mathcal{G}$ is called a generalized metric on $\mathcal{X}$ if it satisfies the following conditions:*

*(i)* $\mathcal{G}(u, v, w) = 0$, *if* $u = v = w$

*(ii)* $0 \leq \mathcal{G}(u, v, w)$, *with* $u \neq v$

*(iii)* $\mathcal{G}(u, u, v) \leq \mathcal{G}(u, v, w)$, *with* $v \neq w$

*(iv)* $\mathcal{G}(u, v, w) = \mathcal{G}(u, w, v) = \mathcal{G}(v, u, w) = \mathcal{G}(v, w, u) = \mathcal{G}(w, u, v) = \mathcal{G}(w, v, u)$

*(v)* $\mathcal{G}(u, u, v) \leq \mathcal{G}(u, u, w) + \mathcal{G}(w, w, v)$

*(vi)* $\mathcal{G}(u, v, w) \leq \mathcal{G}(u, \xi, \xi) + \mathcal{G}(\xi, v, w)$ *for any point* $\xi \in X$

*Then, $(\mathcal{X}, \mathcal{G})$ is called a $\mathcal{G}$-metric space.*

**Definition 1.2.** *Suppose $\{u_n\}$ is a sequence in $\mathcal{G}$-metric space $(\mathcal{X}, \mathcal{G})$ and $\{u_n\}$ converges to $u \in \mathcal{X}$ if $\lim_{n,m\to\infty} \mathcal{G}(u, u_n, u_m) = 0$, i.e, $\forall\ \epsilon > 0$, $\exists\ N_0 \in \mathbb{N}$ such that $\mathcal{G}(u, u_n, u_m) < \epsilon$, $\forall\ n, m \geq N_0$.*

**Definition 1.3.** *A sequence $\{u_n\}$ in $(\mathcal{X}, \mathcal{G})$ is called a $\mathcal{G}$-Cauchy sequence on $\mathcal{X}$, if for every $\epsilon > 0$ $\exists\ N_0 \in \mathbb{N}$ such that $\mathcal{G}(u_n, u_m, u_l) < \epsilon$, $\forall\, l, m, n \geq N_0$. If every $\mathcal{G}$-Cauchy sequence is convergent then $(\mathcal{X}, \mathcal{G})$ is complete.*

**Lemma 1.4.** *Suppose $(\mathcal{X}, \mathcal{G})$ is a $\mathcal{G}$-metric space then $\mathcal{G}(u, u, v) \leq 2\mathcal{G}(u, v, v), \forall u, v \in \mathcal{X}$.*

*Proof.* We know that by properties of $\mathcal{G}$-metric space $\mathcal{G}(u, v, w) \leq \mathcal{G}(u, \xi, \xi) + \mathcal{G}(\xi, v, w)$ for any $\xi \in \mathcal{X}$. Now $\mathcal{G}(u, u, v) \leq \mathcal{G}(u, v, v) + \mathcal{G}(v, u, v)$ for any $v \in \mathcal{X}$. Therefore $\mathcal{G}(u, u, v) \leq 2\mathcal{G}(u, v, v), \forall u, v \in \mathcal{X}$. □

**Lemma 1.5.** *Consider $(\mathcal{X}, \mathcal{G})$ is a $\mathcal{G}$-metric space and $\{u_l\}$ is a sequence in $\mathcal{X}$, then each of the following statements implies others:*

*(i)* $\{u_l\}$ *converges to* $u \in \mathcal{X}$ *with respect to $\mathcal{G}$-metric.*

*(ii)* $\mathcal{G}(u_l, u_m, u) < \epsilon, \forall \epsilon > 0$, *as* $l, m \geq n_0$, *for some* $n_0 \in \mathbb{N}$.

*(iii)* $\mathcal{G}(u_l, u_l, u) < \epsilon, \forall \epsilon > 0$, *as* $l \geq n_0$, *for some* $n_0 \in \mathbb{N}$.

*(iv)* $\mathcal{G}(u_l, u, u) < \epsilon, \forall \epsilon > 0$, *as* $l \geq n_0$, *for some* $n_0 \in \mathbb{N}$.

## 2 Main Results

**Theorem 2.1.** (Banach type contraction with two generalized metrics) *Suppose $\mathcal{X} \neq 0$ and $H, \mathcal{G} : \mathcal{X}^3 \to [0, \infty)$ are two maps such that $\mathcal{X}$ is complete with respect to both $H$ and $\mathcal{G}$ generalized metrics. Suppose $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is a map such that $\mathcal{G}(u, \mathcal{T}v, \mathcal{T}v) \leq H(u, v, v)$ or $H(u, \mathcal{T}v, \mathcal{T}v) \leq \mathcal{G}(u, v, v)$ implies*

*(i)* $\mathcal{G}(\mathcal{T}u, \mathcal{T}v, \mathcal{T}v) \leq \alpha\Big(H(u, v, v)\Big)H(u, v, v)$

*(ii)* $H(\mathcal{T}u, \mathcal{T}v, \mathcal{T}v) \leq \alpha\Big(\mathcal{G}(u, v, v)\Big)\mathcal{G}(u, v, v)$,

*where $\alpha : [0, \infty) \to [0, \frac{1}{2})$ such that $\limsup_{s \to r+} \alpha(s) \leq \frac{1}{2}$. Then $\exists$ a fixed point for $\mathcal{T}$ which is unique.*

*Proof.* Let $u_1 \in \mathcal{X}$. Let us construct the sequence $u_{l+1} = \mathcal{T}u_l$, $\forall\, l \in \mathbb{N}$.

$$\text{Now,} \quad H(u_l, \mathcal{T}u_{l-1}, \mathcal{T}u_{l-1}) = H(u_l, u_l, u_l) = 0 \leq \mathcal{G}(u_l, u_{l+1}, u_{l+1})$$

$$\implies H(\mathcal{T}u_l, \mathcal{T}u_{l-1}, \mathcal{T}u_{l-1}) = H(u_{l+1}, u_l, u_l) \leq \alpha\Big(\mathcal{G}(u_l, u_{l+1}, u_{l+1})\Big)\mathcal{G}(u_l, u_{l+1}, u_{l+1}) \tag{2.1}$$

$$\text{and} \quad \mathcal{G}(\mathcal{T}u_{l+1}, \mathcal{T}u_l, \mathcal{T}u_l) \leq \alpha\Big(H(u_{l+1}, u_l, u_l)\Big)H(u_{l+1}, u_l, u_l). \tag{2.2}$$

Then from (2.1) and (2.2) we get,

$$\begin{aligned}
\mathcal{G}(u_{l+2}, u_{l+1}, u_{l+1}) &\leq \alpha\Big(H(u_{l+1}, u_l, u_l)\Big)\alpha\Big(\mathcal{G}(u_l, u_{l+1}, u_{l+1})\Big)\mathcal{G}(u_l, u_{l+1}, u_{l+1}) \\
&\leq 2\alpha\Big(H(u_{l+1}, u_l, u_l)\Big)\alpha\Big(\mathcal{G}(u_l, u_{l+1}, u_{l+1})\Big)\mathcal{G}(u_{l+1}, u_l, u_l), \qquad \text{[by Lemma 1.4 ]} \\
&\leq r\mathcal{G}(u_{l+1}, u_l, u_l), \qquad \text{where } r \in [0, 1) \text{ as } \alpha : [0, \infty) \to [0, \frac{1}{2})
\end{aligned}$$

Similarly, $H(u_{l+2}, u_{l+1}, u_{l+1}) \leq rH(u_{l+1}, u_l, u_l)$.

Then $\lim_{l \to \infty} \mathcal{G}(u_l, u_l, u_{l+1}) = 0$ and $\lim_{l \to \infty} H(u_l, u_l, u_{l+1}) = 0$.

Now we prove that $\{u_l\}$ is $\mathcal{G}$-Cauchy sequence with respect to both $\mathcal{G}$ and $H$. Suppose $m > l$.

$$\begin{aligned}
\mathcal{G}(u_l, u_l, u_m) &\leq \mathcal{G}(u_l, u_l, u_{l+1}) + \mathcal{G}(u_{l+1}, u_{l+1}, u_m) \\
&\leq \mathcal{G}(u_l, u_l, u_{l+1}) + \mathcal{G}(u_{l+1}, u_{l+1}, u_{l+2}) + \mathcal{G}(u_{l+2}, u_{l+2}, u_m) \\
&\leq \mathcal{G}(u_l, u_l, u_{l+1}) + \mathcal{G}(u_{l+1}, u_{l+1}, u_{l+2}) + \cdots + \mathcal{G}(u_{m-1}, u_{m-1}, u_m).
\end{aligned}$$

Now, taking $\lim_{m,l \to \infty}$ on both sides, we obtain

$$\lim_{m,l \to \infty} \mathcal{G}(u_l, u_l, u_m) = 0.$$

Therefore $\{u_l\}$ is a $\mathcal{G}$-Cauchy sequence with respect to $\mathcal{G}$. Similarly $\{u_l\}$ is a $\mathcal{G}$-Cauchy sequence with respect to $H$. Since $\mathcal{X}$ is complete with respect to both $\mathcal{G}$ and $H$ so $\exists\, u, v \in \mathcal{X}$ such that

$$\lim_{l \to \infty} \mathcal{G}(u_l, u_l, u) = \lim_{l \to \infty} H(u_l, u_l, v) = 0.$$

Now, we claim that $u = v$. Suppose that $u \neq v$. Then we get $\mathcal{G}(u, u, v) \neq 0$, $\mathcal{G}(v, v, u) \neq 0$ and $H(u, u, v) \neq 0$, $H(v, v, u) \neq 0$. From Lemma 1.4 and Lemma 1.5, we can conclude that,

$0 = \lim_{l \to \infty} \mathcal{G}(u, \mathcal{T}u_l, \mathcal{T}u_l) \leq H(u, v, v) = \lim_{l \to \infty} H(u, u_l, u_l)$.
For a large integer value of $l$, we have, $\mathcal{G}(u, \mathcal{T}u_l, \mathcal{T}u_l) \leq H(u, u_l, u_l)$ therefore from the hypothesis of the Theorem 2.1 we have,

$$\mathcal{G}(\mathcal{T}u, u_{l+1}, u_{l+1}) = \mathcal{G}(\mathcal{T}u, \mathcal{T}u_l, \mathcal{T}u_l) \leq \alpha\Big(H(u, u_l, u_l)\Big)H(u, u_l, u_l) \tag{2.3}$$

$$H(\mathcal{T}u, u_{l+1}, u_{l+1}) = H(\mathcal{T}u, \mathcal{T}u_l, \mathcal{T}u_l) \leq \alpha\Big(\mathcal{G}(u, u_l, u_l)\Big)\mathcal{G}(u, u_l, u_l). \tag{2.4}$$

Taking $\lim_{l\to\infty}$ on both sides of (2.4) we get $\lim_{l\to\infty} H(\mathcal{T}u, u_{l+1}, u_{l+1}) = 0$. This implies $\{u_l\}$ converges to $\mathcal{T}u$. Thus $\mathcal{T}u = v$. Similarly, $\mathcal{T}v = u$. Now, $\mathcal{G}(u, u, v) = \lim_{l\to\infty} \mathcal{G}(v, u_{l+1}, u_{l+1}) = \mathcal{G}(\mathcal{T}u, u_{l+1}, u_{l+1}) = \mathcal{G}(\mathcal{T}u, \mathcal{T}u_l, \mathcal{T}u_l) \leq \alpha\Big(H(u, u_l, u_l)\Big)H(u, u_l, u_l) = \lim_{l\to\infty} \alpha\Big(H(u, u_l, u_l)\Big)H(u, v, v)$, then $\exists\, k_1 \in [0, \frac{1}{2})$ such that $\mathcal{G}(u, u, v) \leq k_1 H(u, v, v)$.

Similarly, $\exists\ k_2 \in [0, \frac{1}{2})$ such that $H(u, u, v) \leq k_2 \mathcal{G}(v, u, u) \leq 2k_2 \mathcal{G}(v, v, u)$. [by Lemma 1.4]

Thus, $\mathcal{G}(u, u, v) \leq k_1 H(u, v, v) \leq 2k_1 H(u, u, v) \leq 2k_1 k_2 \mathcal{G}(u, u, v)$. If $\mathcal{G}(u, u, v) \neq 0$, then $k_1 k_2 \geq \frac{1}{2}$. This is contradiction. Therefore, $\mathcal{G}(u, u, v) = 0$. Hence $u = v$ and $\mathcal{T}u = u$.

For uniqueness, let $u$ and $v$ be two fixed points of $\mathcal{T}$ with $u \neq v$, then $\mathcal{G}(u, v, v) \neq 0$ and $H(u, v, v) \neq 0$. So one of the following inequality holds:

$$\mathcal{G}(u, v, v) < H(u, v, v) \quad \text{or} \quad H(u, v, v) < \mathcal{G}(u, v, v).$$

Suppose $\mathcal{G}(u, \mathcal{T}v, \mathcal{T}v) = \mathcal{G}(u, v, v) \leq H(u, v, v)$, then by hypothesis of Theorem 2.1 we have $\mathcal{G}(\mathcal{T}u, \mathcal{T}v, \mathcal{T}v) \leq \alpha\Big(H(u, v, v)\Big)H(u, v, v)$ and $H(\mathcal{T}u, \mathcal{T}v, \mathcal{T}v) \leq \alpha\Big(\mathcal{G}(u, v, v)\Big)\mathcal{G}(u, v, v)$ which implies

$$\mathcal{G}(u, v, v) \leq \alpha\Big(H(u, v, v)\Big)\alpha\Big(\mathcal{G}(u, v, v)\Big)\mathcal{G}(u, v, v).$$

Thus $\alpha\Big(H(u, v, v)\Big)\alpha\Big(\mathcal{G}(u, v, v)\Big) \geq 1$. This is a contradiction. So $\mathcal{G}(u, v, v) = 0$ and $u = v$. □

**Example 2.2.** *Let $\mathcal{X} = [0, 1]$ and we define two metrics $\mathcal{G}$ and $H$ on $\mathcal{X}$ as,*

$$\mathcal{G}(u, v, w) = \max\{|u - v|, |v - w|, |w - u|\} \text{ and } H(u, v, w) = 2\max\{|u - v|, |v - w|, |w - u|\}.$$

*Let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be the map define as,*

$$\mathcal{T}(u) = \frac{u^2}{16} + \frac{u}{16} + \frac{1}{16}.$$

*Now, $\mathcal{G}(u, \mathcal{T}v, \mathcal{T}v) = |u - \mathcal{T}v| = |u - \frac{v^2}{16} - \frac{v}{16} - \frac{1}{16}|$ and $H(u, v, v) = 2|u - v|$ and $\mathcal{G}(u, v, v) = |u - v|$.*

$$\text{Therefore we have,} \quad \mathcal{G}(u, \mathcal{T}v, \mathcal{T}v) \leq H(u, v, v) \qquad \forall\, u, v \in \mathcal{X}.$$

*(i) Now, $\mathcal{G}(\mathcal{T}u, \mathcal{T}v, \mathcal{T}v) = |\mathcal{T}u - \mathcal{T}v| = \frac{1}{16}(1 + u + v)|u - v| \leq \frac{3}{16}|u - v| = \frac{3}{32}H(u, v, v)\ \forall\ u, v \in [0, 1]$.*

*(ii) Again, $H(\mathcal{T}u, \mathcal{T}v, \mathcal{T}v) = 2|\mathcal{T}u - \mathcal{T}v| \leq \frac{3}{8}|u - v| \leq \frac{3}{8}\mathcal{G}(u, v, v)\ \forall\ u, v \in [0, 1]$.*

*From these two if we choose a function $\alpha : [0, \infty) \to [\frac{3}{8}, \frac{1}{2})$ then both of the conditions of Theorem 2.1 are justified. Hence, $\mathcal{T}$ admits a fixed point on $\mathcal{X}$, which is unique.*

**Theorem 2.3.** (Chatterjea type contraction with two generalized metrics) *Suppose $\mathcal{X}$ is a non-empty set equipped with two generalized metrics $\mathcal{G}$ and $H$ and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ such that $\mathcal{X}$ is complete with respect to both $H$ and $\mathcal{G}$ and $\mathcal{G}(u, \mathcal{T}v, \mathcal{T}v) \leq H(u, v, v)$ or $H(u, \mathcal{T}v, \mathcal{T}v) \leq \mathcal{G}(u, v, v)$ implies*

*(i)* $\mathcal{G}(\mathcal{T}u, \mathcal{T}v, \mathcal{T}v) \leq \alpha\Big(H(u, v, v)\Big)[\mathcal{G}(v, v, \mathcal{T}u) + H(u, u, \mathcal{T}v)]$

*(ii)* $H(\mathcal{T}u, \mathcal{T}v, \mathcal{T}v) \leq \alpha\Big(\mathcal{G}(u, v, v)\Big)[H(v, v, \mathcal{T}u) + \mathcal{G}(u, u, \mathcal{T}v)]$,

*where $\alpha : [0, \infty) \to [0, \frac{1}{4})$ such that $\limsup_{t\to r+} \alpha(t) \leq \frac{1}{4}$. Then $\exists$ is a fixed point for $\mathcal{T}$, which is unique.*

*Proof.* Let $u_1 \in \mathcal{X}$. Construct the sequence $u_{l+1} = \mathcal{T}(u_l)$, $\forall\, l \in \mathbb{N}$.

$$\text{Now,} \quad \mathcal{G}(u_{l+1}, \mathcal{T}u_l, \mathcal{T}u_l) = \mathcal{G}(u_{l+1}, u_{l+1}, u_{l+1}) = 0 \leq H(u_{l+1}, u_l, u_l)$$

$$\implies \quad \mathcal{G}(\mathcal{T}u_{l+1}, \mathcal{T}u_l, \mathcal{T}u_l) \leq \alpha\Big(H(u_{l+1}, u_l, u_l)\Big)[\mathcal{G}(u_l, u_l, \mathcal{T}u_{l+1}) + H(u_{l+1}, u_{l+1}, \mathcal{T}u_l)]$$

$$\text{Therefore,} \quad \mathcal{G}(\mathcal{T}u_{l+1}, \mathcal{T}u_l, \mathcal{T}u_l) \leq \alpha\Big(H(u_{l+1}, u_l, u_l)\Big)\mathcal{G}(u_l, u_l, u_{l+2})$$

$$\mathcal{G}(u_{l+2}, \mathcal{T}u_l, \mathcal{T}u_l) \leq \alpha\Big(H(u_{l+1}, u_l, u_l)\Big)[\mathcal{G}(u_{l+2}, u_{l+1}, u_{l+1}) + \mathcal{G}(u_{l+1}, u_l, u_l)]$$

$$\mathcal{G}(u_{l+2}, u_{l+1}, u_{l+1}) \leq \frac{\alpha\Big(H(u_{l+1}, u_l, u_l)\Big)}{1 - \alpha\Big(H(u_{l+1}, u_l, u_l)\Big)}\mathcal{G}(u_{l+1}, u_l, u_l) \tag{2.5}$$

$$\text{and} \qquad H(\mathcal{T}u_{l+1}, \mathcal{T}u_l, \mathcal{T}u_l) \leq \alpha\Big(\mathcal{G}(u_{l+1}, u_l, u_l)\Big)[H(u_l, u_l, \mathcal{T}u_{l+1}) + \mathcal{G}(u_{l+1}, u_{l+1}, \mathcal{T}u_l)]$$

$$\text{Therefore,} \quad H(\mathcal{T}u_{l+1}, \mathcal{T}u_l, \mathcal{T}u_l) \leq \alpha\Big(\mathcal{G}(u_{l+1}, u_l, u_l)\Big)H(u_l, u_l, u_{l+2})$$

$$H(\mathcal{T}u_{l+1}, \mathcal{T}u_l, \mathcal{T}u_l) \leq \alpha\Big(\mathcal{G}(u_{l+1}, u_l, u_l)\Big)[H(u_{l+2}, u_{l+1}, u_{l+1}) + H(u_{l+1}, u_l, u_l)]$$

$$H(u_{l+2}, u_{l+1}, u_{l+1}) \leq \frac{\alpha\Big(\mathcal{G}(u_{l+1}, u_l, u_l)\Big)}{1 - \alpha\Big(\mathcal{G}(u_{l+1}, u_l, u_l)\Big)}H(u_{l+1}, u_l, u_l) \tag{2.6}$$

Now, $\limsup\limits_{t \to r+} \alpha(t) < \frac{1}{4}$, for all $t \in [0, \infty)$. It follows that, $\lim\limits_{s \to t+} \sup \frac{\alpha(s)}{1-\alpha(s)} \leq \frac{1}{3} < \frac{1}{2}$. Then $\exists\ r_1, r_2 \in [0, \frac{1}{4})$ and $p_1, p_2 \in \mathbb{N}$ such that $\frac{\alpha\Big(H(u_{l+1}, u_l, u_l)\Big)}{1-\alpha\Big(H(u_{l+1}, u_l, u_l)\Big)} < r_1\ \forall l > p_1$, and $\frac{\alpha\Big(\mathcal{G}(u_{l+1}, u_l, u_l)\Big)}{1-\alpha\Big(\mathcal{G}(u_{l+1}, u_l, u_l)\Big)} < r_2\ \forall l > p_2$. It follows that,

$$\mathcal{G}(u_{l+2}, u_{l+1}, u_{l+1}) \leq r_1\mathcal{G}(u_{l+1}, u_l, u_l) \ \text{ and } \ H(u_{l+2}, u_{l+1}, u_{l+1}) \leq r_2 H(u_{l+1}, u_l, u_l).$$

If $\lim\limits_{l \to \infty} \mathcal{G}(u_{l+1}, u_l, u_l) \neq 0$, then $r_1 \geq 1$, this is a contradiction. Therefore, $\lim\limits_{l \to \infty} \mathcal{G}(u_{l+1}, u_l, u_l) = 0$, similarly $\lim\limits_{l \to \infty} H(u_{l+1}, u_l, u_l) = 0$. By proceeding similarly as in the proof of 2.1, we can conclude that $\{u_l\}$ is a Cauchy sequence relative to both $\mathcal{G}$ and $H$ in $\mathcal{X}$. Since the given $\mathcal{X}$ is complete with respect to both $\mathcal{G}$ and $H$ so $\exists\ u, v \in \mathcal{X}$ such that $\lim\limits_{l \to \infty} \mathcal{G}(u_l, u_l, u) = 0$. Similarly $\lim\limits_{l \to \infty} H(u_l, u_l, v) = 0$, then $\lim\limits_{l \to \infty} H(u_l, u_l, u) = H(u, v, v) > 0$ and $\lim\limits_{l \to \infty} \mathcal{G}(u_l, u_l, u) = 0 \leq \lim\limits_{l \to \infty} H(u_l, u_l, u) = H(u, v, v)$. For some large value of $l$ we have, $\mathcal{G}(u, \mathcal{T}u_l, \mathcal{T}u_l) \leq H(u, u_l, u_l)$. From the hypothesis of Theorem 2.3 we obtain

$$\mathcal{G}(\mathcal{T}u, \mathcal{T}u_l, \mathcal{T}u_l) \leq \alpha\Big(H(u, u_l, u_l)\Big)[\mathcal{G}(u_l, u_l, \mathcal{T}u) + H(u, u, \mathcal{T}u_l)], \tag{2.7}$$

$$H(\mathcal{T}u, \mathcal{T}u_l, \mathcal{T}u_l) \leq \alpha\Big(\mathcal{G}(u, u_l, u_l)\Big)[H(\mathcal{T}u, u_l, u_l) + \mathcal{G}(\mathcal{T}u_l, u, u)]. \tag{2.8}$$

Taking limit $l \to \infty$ in (2.8) we get,

$$\lim_{l \to \infty} H(\mathcal{T}u, \mathcal{T}u_l, \mathcal{T}u_l) \leq \lim_{l \to \infty} \alpha\Big(\mathcal{G}(u, u_l, u_l)\Big) \lim_{l \to \infty} [H(Tu, u_l, u_l) + \mathcal{G}(\mathcal{T}u_l, u, u)].$$

If $\lim_{l\to\infty} H(\mathcal{T}u, \mathcal{T}u_l, \mathcal{T}u_l) \neq 0$, then $\lim_{l\to\infty} \sup \alpha\Big(\mathcal{G}(u, u_l, u_l)\Big) \geq 1$. Therefore, $\mathcal{T}u = v$.
Analogously, we can prove that $\mathcal{T}v = u$.

Taking limit $l \to \infty$ in (2.7) we obtain,

$$\lim_{l\to\infty} \mathcal{G}(\mathcal{T}u, \mathcal{T}u_l, \mathcal{T}u_l) \leq \lim_{l\to\infty} \alpha\Big(H(u, u_l, u_l)\Big) \lim_{l\to\infty} [\mathcal{G}(u_l, u_l, \mathcal{T}u) + H(u, u, \mathcal{T}u_l)]$$

$$\implies \mathcal{G}(\mathcal{T}u, u, u) \leq \lim_{l\to\infty} \alpha\Big(H(u, u_l, u_l)\Big)[\mathcal{G}(u, u, v) + H(v, u, u)].$$

Since $\limsup_{t\to s+} \alpha(t) < \frac{1}{4}$, then $\exists\ k_1 \in [0, \frac{1}{4})$ such that $\mathcal{G}(v, u, u) \leq k_1[\mathcal{G}(u, u, v) + H(v, u, u)]$.

$$\text{Therefore, from Lemma 1.4, } \mathcal{G}(v, u, u) \leq \frac{k_1}{1-k_1} H(v, u, u) \leq 2\frac{k_1}{1-k_1} H(v, v, u).$$

Similarly $\exists\, k_2 \in [0, \frac{1}{4})$ such that

$$H(u, v, v) \leq \frac{k_2}{1-k_2} \mathcal{G}(u, v, v) \implies H(v, v, u) \leq 2\frac{k_2}{1-k_2} \mathcal{G}(v, u, u)$$

which implies $\mathcal{G}(v, u, u) \leq \frac{k_1}{1-k_1}\frac{k_2}{1-k_2} 4\mathcal{G}(u, v, v)$. If $\mathcal{G}(v, u, u) \neq 0$, then we have $4\frac{k_1}{1-k_1}\frac{k_2}{1-k_2} \geq 1$, which contradicts our assumption. Hence $\mathcal{G}(v, u, u) = 0$ and $u = v$. So $\mathcal{T}$ admits a fixed point.

For uniqueness, let $u$ and $v$ be two fixed points of the map $\mathcal{T}$ with $u \neq v$. Then we have $\mathcal{T}u = u, \mathcal{T}v = v$ and $\mathcal{G}(u, u, v) \neq 0, \mathcal{G}(u, v, v) \neq 0, H(u, v, v) \neq 0, H(u, u, v) \neq 0$. It follows that either $\mathcal{G}(u, v, v) = \mathcal{G}(u, \mathcal{T}v, \mathcal{T}v) \leq H(u, v, v)$ or $H(u, v, v) = H(u, \mathcal{T}v, \mathcal{T}v) \leq \mathcal{G}(u, v, v)$. Then from (2.5) and (2.6) we get,

$$\mathcal{G}(u, u, v) \leq \frac{\alpha\Big(H(u, v, v)\Big)}{1 - \alpha\Big(H(u, v, v)\Big)} H(u, u, v) \quad \text{and} \quad H(u, v, v) \leq \frac{\alpha\Big(\mathcal{G}(u, v, v)\Big)}{1 - \alpha\Big(\mathcal{G}(u, v, v)\Big)} \mathcal{G}(u, u, v).$$

Now applying the lemma 1.4 we get,

$$\mathcal{G}(u, v, v) \leq \frac{\alpha\Big(H(u, v, v)\Big)}{1 - \alpha\Big(H(u, v, v)\Big)} \cdot \frac{\alpha\Big(\mathcal{G}(u, v, v)\Big)}{1 - \alpha\Big(\mathcal{G}(u, v, v)\Big)} 4\mathcal{G}(u, v, v).$$

If $\mathcal{G}(u, v, v) \neq 0$, then $\frac{\alpha\Big(H(u,v,v)\Big)}{1-\alpha\Big(H(u,v,v)\Big)} \cdot \frac{\alpha\Big(\mathcal{G}(u,v,v)\Big)}{1-\alpha\Big(\mathcal{G}(u,v,v)\Big)} \geq \frac{1}{4}$, which is a contradiction. Hence $\mathcal{G}(u, v, v) = 0$ and $u = v$.

□

**Example 2.4.** *Let $\mathcal{X} = \{0, \frac{1}{2}, 1\}$, define two metrics $\mathcal{G}$ and $H$ as,*

$$\mathcal{G}(u, v, w) = \begin{cases} 0, & \text{if } u = v = w \\ \max\{u, v, w\} & \text{else} \end{cases}$$

*and*

$$H(u, v, w) = \max\{|u - v|, |v - w|, |w - u|\}.$$

*Let $\mathcal{T}(u) = \frac{u}{5}$ be a map on $\mathcal{X}$. Since $\mathcal{X}$ is a finite set therefore, it is complete with respect to both $\mathcal{G}$ and $H$. Let $u, v$ be two arbitrary elements of $\mathcal{X}$. Therefore,*

*(i) if $u = 0, v = \frac{1}{2}$ then*

$\mathcal{G}\Big(u, \mathcal{T}(v), \mathcal{T}(v)\Big) = \frac{1}{10} \leq H(u, v, v) = \frac{1}{2}$ *or* $H\Big(u, \mathcal{T}(v), \mathcal{T}(v)\Big) = \frac{1}{10} \leq \mathcal{G}(u, v, v) = \frac{1}{2}$ *implies*

$\mathcal{G}\Big(\mathcal{T}(u), \mathcal{T}(v), \mathcal{T}(v)\Big) = \frac{1}{10} \leq \alpha\Big(H(u, v, v)\Big)[\frac{1}{2} + \frac{1}{10}]$ *and*

$H\Big(\mathcal{T}(u), \mathcal{T}(v), \mathcal{T}(v)\Big) = \frac{1}{10} \leq \alpha\Big(\mathcal{G}(u, v, v)\Big)[\frac{1}{2} + \frac{1}{10}]$,

*(ii) if* $u = 0, v = 1$ *then*

$\mathcal{G}\Big(u, \mathcal{T}(v), \mathcal{T}(v)\Big) = \frac{1}{5} \leq H(u, v, v) = 1$ *or* $H\Big(u, \mathcal{T}(v), \mathcal{T}(v)\Big) = \frac{1}{5} \leq \mathcal{G}(u, v, v) = 1$ *implies*

$\mathcal{G}\Big(\mathcal{T}(u), \mathcal{T}(v), \mathcal{T}(v)\Big) = \frac{1}{5} \leq \alpha\Big(H(u, v, v)\Big)[\frac{1}{2} + \frac{1}{10}]$ *and*

$H\Big(\mathcal{T}(u), \mathcal{T}(v), \mathcal{T}(v)\Big) = \frac{1}{5} \leq \alpha\Big(\mathcal{G}(u, v, v)\Big)[\frac{1}{2} + \frac{1}{10}]$,

*(iii) if* $u = \frac{1}{2}, v = 0$ *then*

$\mathcal{G}\Big(u, \mathcal{T}(v), \mathcal{T}(v)\Big) = \frac{1}{2} \leq H(u, v, v) = \frac{1}{2}$ *or* $H\Big(u, \mathcal{T}(v), \mathcal{T}(v)\Big) = \frac{1}{2} \leq \mathcal{G}(u, v, v) = \frac{1}{2}$ *implies*

$\mathcal{G}\Big(\mathcal{T}(u), \mathcal{T}(v), \mathcal{T}(v)\Big) = \frac{1}{10} \leq \alpha\Big(H(u, v, v)\Big)[\frac{1}{10} + \frac{1}{2}]$ *and*

$H\Big(\mathcal{T}(u), \mathcal{T}(v), \mathcal{T}(v)\Big) = \frac{1}{10} \leq \alpha\Big(\mathcal{G}(u, v, v)\Big)[1 + \frac{1}{5}]$,

*(iv) if* $u = \frac{1}{2}, v = 1$ *then*

$\mathcal{G}\Big(u, \mathcal{T}(v), \mathcal{T}(v)\Big) = \frac{1}{2} \leq H(u, v, v) = \frac{1}{2}$ *or* $H\Big(u, \mathcal{T}(v), \mathcal{T}(v)\Big) = \frac{3}{10} \leq \mathcal{G}(u, v, v) = 1$ *implies*

$\mathcal{G}\Big(\mathcal{T}(u), \mathcal{T}(v), \mathcal{T}(v)\Big) = \frac{1}{5} \leq \alpha\Big(H(u, v, v)\Big)[1 + \frac{3}{10}]$ *and*

$H\Big(\mathcal{T}(u), \mathcal{T}(v), \mathcal{T}(v)\Big) = \frac{1}{10} \leq \alpha\Big(\mathcal{G}(u, v, v)\Big)[\frac{9}{10} + \frac{1}{2}]$,

*(v) if* $u = 1, v = 0$ *then*

$\mathcal{G}\Big(u, \mathcal{T}(v), \mathcal{T}(v)\Big) = 1 \leq H(u, v, v) = 1$ *or* $H\Big(u, \mathcal{T}(v), \mathcal{T}(v)\Big) = 1 \leq \mathcal{G}(u, v, v) = 1$ *implies*

$G\Big(\mathcal{T}(u), \mathcal{T}(v), \mathcal{T}(v)\Big) = \frac{1}{5} \leq \alpha\Big(H(u, v, v)\Big)[\frac{1}{5} + 1]$ *and*

$H\Big(\mathcal{T}(u), \mathcal{T}(v), \mathcal{T}(v)\Big) = \frac{1}{5} \leq \alpha\Big(\mathcal{G}(u, v, v)\Big)[\frac{1}{5} + 1]$,

*(vi) if* $u = 1, v = \frac{1}{2}$ *then*

$\mathcal{G}\Big(u, \mathcal{T}(v), \mathcal{T}(v)\Big) = 1 \leq H(u, v, v) = \frac{1}{2}$ *(which is absurd case) but we have the another condition*
$H\Big(u, \mathcal{T}(v), \mathcal{T}(v)\Big) = \frac{9}{10} \leq \mathcal{G}(u, v, v) = 1$ *which implies*

$\mathcal{G}\Big(\mathcal{T}(u), \mathcal{T}(v), \mathcal{T}(v)\Big) = \frac{1}{5} \leq \alpha\Big(H(u, v, v)\Big)[\frac{1}{2} + \frac{9}{10}]$ *and*

$H\Big(\mathcal{T}(u), \mathcal{T}(v), \mathcal{T}(v)\Big) = \frac{1}{10} \leq \alpha\Big(\mathcal{G}(u, v, v)\Big)[\frac{3}{10} + 1]$.

*Now if we choose function* $\alpha : [0, \infty) \to [\frac{1}{6}, \frac{1}{4})$ *then all the conditions of Theorem 2.3 are satisfied. Hence, we can conclude that* $\mathcal{T}$ *admits a unique fixed point on* $\mathcal{X}$.

**Theorem 2.5.** ( Kannan type contraction with two generalized metrics) *Suppose* $\mathcal{X}$ *is a non-empty set equipped with two generalized metrics* $\mathcal{G}$ *and* $H$ *and* $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ *mapping such that* $\mathcal{X}$ *is complete relative to both* $\mathcal{G}$ *and* $H$ *and* $\mathcal{G}(u, \mathcal{T}v, \mathcal{T}v) \leq H(u, v, v)$ *or* $H(u, \mathcal{T}v, \mathcal{T}v) \leq \mathcal{G}(u, v, v)$ *implies*

*(i)* $\mathcal{G}(\mathcal{T}u, \mathcal{T}v, \mathcal{T}v) \leq \alpha\Big(H(u, v, v)\Big)[\mathcal{G}(u, u, \mathcal{T}u) + H(v, v, Tv)]$

*(ii)* $H(\mathcal{T}u, \mathcal{T}v, \mathcal{T}v) \leq \alpha\Big(\mathcal{G}(u, v, v)\Big)[H(u, u, \mathcal{T}u) + \mathcal{G}(v, v, \mathcal{T}v)]$,

*where* $\alpha : [0, \infty) \to [0, \frac{1}{4})$ *such that* $\limsup_{t \to r+} \alpha(t) \leq \frac{1}{4}$. *Then* $\exists$ *a fixed point of* $\mathcal{T}$, *which is unique.*

*Proof.* Let $u_1 \in \mathcal{X}$. Let us construct the sequence $u_{l+1} = \mathcal{T}(u_l)$, $\forall\ l \in \mathbb{N}$. Now,

$$\mathcal{G}(u_{l+1}, \mathcal{T}u_l, \mathcal{T}u_l) = 0 \leq H(u_{l+1}, u_l, u_l)$$
$$\implies \mathcal{G}(\mathcal{T}u_{l+1}, \mathcal{T}u_l, \mathcal{T}u_l) \leq \alpha\Big(H(u_{l+1}, u_l, u_l)\Big)[\mathcal{G}(u_{l+1}, u_{l+1}, Tu_{l+1}) + H(u_l, u_l, Tu_l)],$$

$$\text{and } H(\mathcal{T}u_{l+1},\mathcal{T}u_l,\mathcal{T}u_l)\leq\alpha\Big(\mathcal{G}(u_{l+1},u_l,u_l)\Big)[H(u_{l+1},u_{l+1},\mathcal{T}u_{l+1})+\mathcal{G}(u_l,u_l,\mathcal{T}u_l)]$$

then we get,

$$\mathcal{G}(\mathcal{T}u_{l+1},\mathcal{T}u_l,\mathcal{T}u_l)\leq\alpha\Big(H(u_{l+1},u_l,u_l)\Big)[\mathcal{G}(u_{l+1},u_{l+1},\mathcal{T}u_{l+1})+H(u_l,u_l,\mathcal{T}u_l)],$$

$$\text{Therefore,}\quad \mathcal{G}(u_{l+2},u_{l+1},u_{l+1})\leq\frac{\alpha\Big(H(u_{l+1},u_l,u_l)\Big)}{1-\alpha\Big(H(u_{l+1},u_l,u_l)\Big)}H(u_l,u_l,u_{l+1}),$$

$$H(u_{l+2},u_{l+1},u_{l+1})\leq\frac{\alpha\Big(\mathcal{G}(u_{l+1},u_l,u_l)\Big)}{1-\alpha\Big(\mathcal{G}(u_{l+1},u_l,u_l)\Big)}\mathcal{G}(u_l,u_l,u_{l+1}).$$

Put $a(l)=\frac{\alpha\Big(H(u_{l+1},u_l,u_l)\Big)}{1-\alpha\Big(H(u_{l+1},u_l,u_l)\Big)}$ and $b(l)=\frac{\alpha\Big(\mathcal{G}(u_{l+1},u_l,u_l)\Big)}{1-\alpha\Big(\mathcal{G}(u_{l+1},u_l,u_l)\Big)}$.

$$\text{Then we have,}\quad \mathcal{G}(u_{l+2},u_{l+1},u_{l+1})\leq a(l)H(u_l,u_l,u_{l+1})\leq a(l)b(l-1)\mathcal{G}(u_l,u_{l-1},u_{l-1}).$$

If $\lim_{l\to\infty}\mathcal{G}(u_l,u_{l-1},u_{l-1})\neq 0$ then $\lim_{l\to\infty}a(l)b(l-1)\geq 1$, which leads to a contradiction.
Therefore, $\lim_{l\to\infty}\mathcal{G}(u_{l+1},u_l,u_l)=0$. Similarly we can prove that $\lim_{l\to\infty}H(u_{l+1},u_l,u_l)=0$.

By proceeding similarly as in the proof of 2.1, we can conclude that $\{u_l\}$ is a Cauchy sequence relative to both $\mathcal{G}$ and $H$ in $\mathcal{X}$. By completeness of $\mathcal{X}$ relative to $\mathcal{G}$ and $H$ $\exists\ u,v\in\mathcal{X}$ such that $\lim_{l\to\infty}\mathcal{G}(u_l,u_l,u)=0=\lim_{l\to\infty}H(u_l,u_l,v)$.

Therefore, $\lim_{l\to\infty}H(u_l,u_l,u)=H(u,v,v)>0$ and $\lim_{l\to\infty}\mathcal{G}(u_l,u_l,u)=0<\lim_{l\to\infty}H(u_l,u_l,u)=H(u,v,v)$.
Now, $\lim_{l\to\infty}\mathcal{G}(u_l,u,u)=0\leq H(u,v,v)=\lim_{l\to\infty}H(u_l,u_l,u)$. For some large value of $l$ we have,
$\mathcal{G}(u,\mathcal{T}u_l,\mathcal{T}u_l)\leq H(u,u_l,u_l)$. From hypothesis of Theorem 2.5 we get,

$$\mathcal{G}(\mathcal{T}u,\mathcal{T}u_l,\mathcal{T}u_l)\leq\alpha\Big(H(u,u_l,u_l)\Big)[\mathcal{G}(u,u,\mathcal{T}u)+H(\mathcal{T}u_l,\mathcal{T}u_l,\mathcal{T}u_{l+1})],\tag{2.9}$$

$$H(\mathcal{T}u,\mathcal{T}u_l,\mathcal{T}u_l)\leq\alpha\Big(\mathcal{G}(u,u_l,u_l)\Big)[H(u,u,\mathcal{T}u)+\mathcal{G}(\mathcal{T}u_l,\mathcal{T}u_l,Tu_{l+1})].\tag{2.10}$$

Taking limit as $l\to\infty$ on both sides in (2.9) we have, $\mathcal{G}(\mathcal{T}u,u,u)\leq\lim_{l\to\infty}\alpha\Big(H(u_l,u_l,u)\Big)\mathcal{G}(u,u,\mathcal{T}u)$.
If $\mathcal{G}(\mathcal{T}u,u,u)\neq 0$, then $\lim_{l\to\infty}\alpha\Big(H(u_l,u_l,u)\Big)\geq 1$. This is a contradiction. So, $\mathcal{G}(u,\mathcal{T}u,\mathcal{T}u)=0$ and $\mathcal{T}u=u$. Hence $\mathcal{T}$ has a fixed point on $\mathcal{X}$.

For the uniqueness part, let $\mathcal{T}$ have two fixed points $u$ and $v$. Then $\mathcal{T}(u)=u$ and $\mathcal{T}(v)=v$.
$\mathcal{G}(u,u,v)\neq 0$, $\mathcal{G}(u,v,v)\neq 0$, $H(u,v,v)\neq 0, H(u,u,v)\neq 0$. It follows that either $\mathcal{G}(u,v,v)<H(u,v,v)$ or $H(u,v,v)<\mathcal{G}(u,v,v)$. Suppose $\mathcal{G}(u,v,v)<H(u,v,v)$, then it implies that $\mathcal{G}(u,\mathcal{T}v,\mathcal{T}v)<H(u,v,v)$. Then by hypothesis of Theorem 2.5 we get,

$$\mathcal{G}(\mathcal{T}u,\mathcal{T}v,\mathcal{T}v)\leq\alpha\Big(H(u,v,v)\Big)[\mathcal{G}(u,u,\mathcal{T}u)+H(v,v,\mathcal{T}v)],$$

$$H(\mathcal{T}u,\mathcal{T}v,\mathcal{T}v)\leq\alpha\Big(\mathcal{G}(u,v,v)\Big)[H(u,u,\mathcal{T}u)+\mathcal{G}(v,v,\mathcal{T}v)].$$

Since, $\mathcal{G}(u,u,\mathcal{T}u)=H(v,v,\mathcal{T}v)=H(v,v,v)=0$. Therefore, $\mathcal{G}(\mathcal{T}u,\mathcal{T}v,\mathcal{T}v)\leq 0$, which implies $\mathcal{G}(\mathcal{T}u,\mathcal{T}v,\mathcal{T}v)=\mathcal{G}(u,v,v)=0$. Hence $u=v$.

□

**Example 2.6.** *Let $\mathcal{X} = [0,1]$ and we define two metrics $\mathcal{G}$ and $H$ on $\mathcal{X}$ as,*

$$\mathcal{G}(u,v,w) = \max\{|u-v|, |v-w|, |w-u|\} \text{ and } H(u,v,w) = 2\max\{|u-v|, |v-w|, |w-u|\}.$$

*Let $\mathcal{T} : X \to X$ be the map defined as,*

$$\mathcal{T}(u) = \frac{u}{16}.$$

*Now, $\mathcal{G}(u,\mathcal{T}v,\mathcal{T}v) = |u-\mathcal{T}v| = |u-\frac{v}{16}|$ and $H(u,v,v) = 2|u-v|$.*

$$\textit{Therefore we have,} \quad \mathcal{G}(u,\mathcal{T}v,\mathcal{T}v) \leq H(u,v,v) \qquad \forall\, u,v \in \mathcal{X}.$$

*(i) Now, $\mathcal{G}(\mathcal{T}u,\mathcal{T}v,\mathcal{T}v) = \frac{1}{16}|u-v| \leq \frac{1}{16}$ $\forall\ u,v \in [0,1]$ and*

$$\mathcal{G}(u,u,\mathcal{T}u) + H(v,v,\mathcal{T}v) = |u-\mathcal{T}u| + 2|v-\mathcal{T}v| = \tfrac{15u+30v}{16} \leq \tfrac{45}{16} \ \forall\ u,v \in [0,1]$$

$$\textit{also, } \mathcal{G}(\mathcal{T}u,\mathcal{T}v,\mathcal{T}v) \leq \alpha\Big(H(u,v,v)\Big)[\mathcal{G}(u,u,\mathcal{T}u) + H(v,v,\mathcal{T}v)] \ \forall\ u,v \in [0,1].$$

*(ii) Again, $H(\mathcal{T}u,\mathcal{T}v,\mathcal{T}v) = \frac{2}{16}|u-v| \leq \frac{2}{16}$ $\forall\ u,v \in [0,1]$ and*

$$H(u,u,\mathcal{T}u) + \mathcal{G}(v,v,\mathcal{T}v) = 2|u-\mathcal{T}u| + |v-\mathcal{T}v| = \tfrac{30u+15v}{16} \leq \tfrac{45}{16} \ \forall\ u,v \in [0,1]$$

$$\textit{also, } H(\mathcal{T}u,\mathcal{T}v,\mathcal{T}v) \leq \alpha\Big(\mathcal{G}(u,v,v)\Big)[H(u,u,\mathcal{T}u) + \mathcal{G}(v,v,\mathcal{T}v)] \ \forall\ u,v \in [0,1].$$

*Now if we choose function $\alpha : [0,\infty) \to [\frac{2}{45}, \frac{1}{4})$ then all the conditions of Theorem 2.5 are satisfied. Hence, we can conclude that $\mathcal{T}$ has exactly one fixed point in $\mathcal{X}$.*

# 3 Concluding remarks

The broadening of fixed-point theory in the setting of $\mathcal{G}$-metric spaces with two or more generalized metrics provides a rich framework for exploring more general types of distance and convergence properties. These results not only generalize classical fixed-point theorems but also have important implications for theoretical and applied mathematics.
In this work, in Theorem 2.1, we have extended the Banach contraction principle in complete $\mathcal{G}$-metric space with two generalized metrics $\mathcal{G}$ and $H$. In Theorem 2.3 and Theorem 2.5, we have generalized Chatterjea and Kannan type of contractions, respectively, in a complete $\mathcal{G}$-metric space with two generalized metrics $\mathcal{G}$ and $H$. Further, we have given sufficient conditions for an existing unique fixed point of the self-map in the setting of a complete $\mathcal{G}$-metric space for the case of three generalized metrics $\mathcal{G}, H$, and $I$ for the contractive classes Banach, Chatterjea, and Kannan contractions.

Let $\mathcal{G}, H, I$ be three generalized metrics on complete metric space $\mathcal{X}$. Suppose $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ such that $\mathcal{G}(u,\mathcal{T}u,\mathcal{T}v) \leq H(u,v,v)$ or $H(u,\mathcal{T}v,\mathcal{T}v) \leq I(u,v,v)$ or $I(u,\mathcal{T}v,\mathcal{T}v) \leq \mathcal{G}(u,v,v)$ implies,

(1) Banach-type contraction with three generalized metrics:

(i) $\mathcal{G}(\mathcal{T}u,\mathcal{T}v,\mathcal{T}v) \leq \alpha\Big(H(u,v,v)\Big)H(u,v,v)$

(ii) $H(\mathcal{T}u,\mathcal{T}v,\mathcal{T}v) \leq \alpha\Big(I(u,v,v)\Big)I(u,v,v)$

(iii) $I(\mathcal{T}u,\mathcal{T}v,\mathcal{T}v) \leq \alpha\Big(\mathcal{G}(u,v,v)\Big)\mathcal{G}(u,v,v)$,

where $\alpha : [0,\infty) \to [0,\frac{1}{2})$ such that $\limsup_{s\to r+} \alpha(s) \leq \frac{1}{2}$. Then, $\mathcal{T}$ has only one fixed point.

(2) Chatterjea type contraction with three generalized metrics:

(i) $\mathcal{G}(\mathcal{T}u,\mathcal{T}v,\mathcal{T}v) \leq \alpha\Big(H(u,v,v)\Big)[\mathcal{G}(v,v,\mathcal{T}u) + H(u,u,\mathcal{T}v)]$

(ii) $H(\mathcal{T}u,\mathcal{T}v,\mathcal{T}v) \leq \alpha\Big(I(u,v,v)\Big)[H(v,v,\mathcal{T}u) + I(u,u,\mathcal{T}v)]$

(iii) $I(\mathcal{T}u, \mathcal{T}v, \mathcal{T}v) \leq \alpha\Big(\mathcal{G}(u,v,v)\Big)[I(v,v,\mathcal{T}u) + \mathcal{G}(u,u,\mathcal{T}v)]$,

where $\alpha : [0,\infty) \to [0, \frac{1}{4})$ such that $\limsup_{t \to r+} \alpha(t) \leq \frac{1}{4}$. Then, $T$ has a unique fixed point.

(3) Kannan type contraction with three generalized metrics:

(i) $\mathcal{G}(\mathcal{T}u, \mathcal{T}v, \mathcal{T}v) \leq \alpha\Big(H(u,v,v)\Big)[\mathcal{G}(u,u,\mathcal{T}u) + H(v,v,\mathcal{T}v)]$

(ii) $H(\mathcal{T}u, \mathcal{T}v, \mathcal{T}v) \leq \alpha\Big(I(u,v,v)\Big)[H(u,u,\mathcal{T}u) + I(v,v,\mathcal{T}v)]$

(iii) $I(\mathcal{T}u, \mathcal{T}v, \mathcal{T}v) \leq \alpha\Big(\mathcal{G}(u,v,v)\Big)[I(u,u,\mathcal{T}u) + \mathcal{G}(v,v,\mathcal{T}v)]$,

where $\alpha : [0,\infty) \to [0, \frac{1}{4})$ such that $\limsup_{t \to r+} \alpha(t) \leq \frac{1}{4}$. Then, the fixed point of $\mathcal{T}$ is exactly one.

## Acknowledgements

The work of the third author is financially supported by the CSIR, India, with grant no: 09/1217(13093)/2022-EMR-I.

## Conflict of Interest

The authors have no conflict of interest in this paper.

## References

[1] Banach, S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fundamenta mathematicae, 3 (1922), no. 1,133-181.

[2] Mustafa, Z., Sims, B. A new approach to generalized metric spaces, Journal of Nonlinear and Convex Analysis, 7 (2006), no. 2, 289.

[3] M, Abbas., Khan, A, R., T, Nazir. Coupled common fixed point results in two generalized metric spaces, Applied Mathematics and Computation, Elsevier 217 (2011), no. 13, 6328-6336.

[4] Chen, J. Zhu, C., Zhu, L. A note on some fixed point theorems on G-metric spaces, J. Appl. Anal. Comput., 11 (2021), no. 1, 101-112.

[5] Gaba, Y, U. Fixed point theorems in G-metric spaces, Journal of Mathematical Analysis and Applications, Elsevier 455 (2017), no. 1, 528-537.

[6] Gautam, P., Kumar, S., Verma, S., Gupta, G. Existence of common fixed point in Kannan F-contractive mappings in quasi-partial b-metric space with an application, Fixed Point Theory and Algorithms for Sciences and Engineering, SpringerOpen 2022 (2022), no. 1, 1-19.

[7] Mustafa, Z., Obiedat, H., Awawdeh, F. Some fixed point theorem for mapping on complete G-metric spaces, Fixed Point Theory and Algorithms for Sciences and Engineering, Springer Nature BV, (2008). doi:10.1155/2008/189870.

[8] Mustafa, Z., Shatanawi, W., Bataineh, M., Volodin, A. Existence of fixed point results in G-metric spaces, International Journal of Mathematics and Mathematical Sciences, 2009 (2009), no. 10, 10.

[9] Zheng, D. Fixed Point Theorems for Generalized $\theta$-$\varphi$-Contractions in G-Metric Spaces, Journal of Function Spaces, Hindawi Limited, 2018 (2018), 1-8.

[10] Luong, Van, W., Thuan, N, X. Coupled fixed point theorems in partially ordered G-metric spaces, Mathematical and Computer Modelling, Elsevier, 55 (2012), no. (3-4), 1601-1609.

[11] Reddy, G, Sudhaamsh, Mohan., Chary, V, Srinivas., Chary, D, Srinivasa., Radenovic, Stojan., Mitrovic, Slobodanka. Coupled fixed point theorems of JS-G-contraction on G-Metric Spaces, Boletim da Sociedade Paranaense de Matemática, 41(2023), 1-10.

[12] Maharani, E, M., Chaira, K. Fixed point theorems in a space with two metrics, Adv. Fixed Point Theory, 5 (2014), no. 1, 1-12.

[13] Shatanawi, W., Abbas, M., Nazir, T. Common coupled coincidence and coupled fixed point results in two generalized metric spaces, Fixed Point Theory and Applications, Springer 2011 (2011), 1-13.

[14] Edelstein, M. An extension of Banach's contraction principle, Proceedings of the American Mathematical Society, JSTOR, 12 (1961), no. 1, 7-10.

[15] Kannan, R. Some results on fixed points, Bull. Cal. Math. Soc. 60 (1968), 71-76.

[16] Chatterjea, S. Fixed-point theorems, Dokladi na Bolgarskata Akademiya na Naukite, 25(1972), no. 6, 727.

[17] Al-Khaleel, M., Al-Sharif, S., AlAhmad, R. On Cyclic Contractive Mappings of Kannan and Chatterjea Type in Generalized Metric Spaces, Mathematics, MDPI 11 (2023), no. 4, 890.

[18] Mohsenialhosseini, S. Approximate fixed points of operators on G-metric spaces, University Politehnica of Bucharest Scientific Bulletin Series A Applied Mathematics and Physics, 79 (2017), 85-96.

[19] Phaneendra, T., Saravanan, S. On some misconceptions and Chatterjee-type G-contraction, Int. J. Pure Appl. Math., 109 (2016), no. 4, 789-797.

[20] Mustafa, Z., Obiedat, H. A fixed point theorem of Reich in G-metric spaces, Cubo (Temuco) SciELO Chile 12(2010), no. 1, 83-93.

[21] Asadi, M., Karapinar, Erdal., Salimi, P. A new approach to G-metric and related fixed point theorems, Journal of Inequalities and Applications, SpringerOpen (2013), no. 1, 1-14.

[1] *DEPARTMENT OF MATHEMATICS*
*JAI PRAKASH UNIVERSITY,*
*CHHAPRA-841301, BIHAR, INDIA*
*E-mail: deepak.gymmaster@gmail.com*

[2,3] *DEPARTMENT OF MATHEMATICAL SCIENCES,*
*INDIAN INSTITUTE OF TECHNOLOGY (BHU),*
*VARANASI-221005, UTTAR PRADESH, INDIA*
*E-mail: tsom.apm@iitbhu.ac.in*
*E-mail: jayantasarkar.rs.mat20@itbhu.ac.in*

*(Received, February 21, 2025)*