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STUDY OF FRAME STEWART NUMBERS OF THE MULTI-PEG TOWER OF HANOI PROBLEM

ABSTRACT. In this paper we have studied the set of best known solutions (called Frame Stewart Numbers) obtained so far, for the multipeg Tower of Hanoi problem for r rings, $r \geq 1$ and p pegs, $p \geq 3$. These solutions obtained using the Frame Stewart algorithm are denoted by $T(r,p)$. Some interesting observations of the set $T = \{T(r,p) | r \geq 1, p \geq 3\}$ have been made. In this study we have proved that the set of all odd natural numbers and the set T are equal. For a fixed p, we define $T_p = \{T(r,p) \mid r \geq 1\}$. We have proved that $T_p \nsubseteq T_{p+1}$. A combinatorial identity for a specific value of r depending on p has been proved. Three conjectures about the set T_p have been made.

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Key words: Multi-peg Tower of Hanoi Problem; Frame Stewart Algorithm; Frame Stewart Numbers.

1. INTRODUCTION

The Tower of Hanoi problem was proposed over a hundred years ago by Lucas [1], in which, a player is given 3 pegs and a certain number r of rings of distinct sizes. The player is required to transfer the rings from one peg to another. Initially all rings are stacked (composing a tower) on the first peg (the source) ordered monotonically by size, with the smallest at the top and the largest at the bottom. The goal is to transfer them to the third peg (the destination), while observing the following rules:

(1) at each step only one ring can be moved;
(2) the moved ring must be the topmost one;
(3) at any moment, a ring cannot reside on a smaller one.

The well-known recursive algorithm that accomplishes this task requires $2^r - 1$ steps, and is the unique optimal algorithm for the Tower of Hanoi problem [6].
One natural generalization is the multi-peg case with ($p \geq 4$) pegs, $P_1, P_2, \ldots, P_p$. This generalization was first proposed by Lucas in 1889. Later, it was revived by Stewart [2]. It has been shown that seven different approaches, including those by Frame and Stewart, to the p peg tower of Hanoi problem are all equivalent i.e. achieve the same number of moves, though their optimality has not yet been proved [9]. Thus the optimality of the solution stands as an interesting and challenging open problem [4]. Let $T(r,p)$ denote the minimum number of (legal) moves required to solve the Tower of Hanoi problem with $r \geq 1$ rings and $p \geq 3$ pegs using the Frame Stewart algorithm. The minimal solution obtained using Frame Stewart Algorithm for different values of r and p are called as Frame Stewart numbers in literature [5][8].

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Section 2 describes the Frame Stewart algorithm. In section 3, Table 1, we have obtained the values of $T(r,p)$ for $3,4,\ldots,10$ pegs for $1 \leq r \leq 30$. Observing table 1, we have denoted the size of a block for $p \geq 3$ pegs by $B_p(k)$ for $k \geq 0$, with the condition that the consecutive entries of $T(r,p)$ in the k^{th} block for p-pegs differ by 2^k. Table 2 gives the sizes of different blocks for different values of r and p. In Table 3, we have calculated the Frame Stewart numbers when the block described above is of full size.

In section 4, some interesting observations of the set $T = \{T(r,p)|r \geq 1, p \geq 3\}$ have been made. In this study we have proved that the set of all odd natural numbers and the set T are equal. For a fixed p, we define $T_p = \{T(r,p)|r \geq 1\}$. We have proved that $T_p \nsubseteq T_{p+1}$. A combinatorial identity for specific value of r depending on p has been proved. Three conjectures about the set T_p have been made.

2. The Frame Stewart Algorithm

The Frame Stewart algorithm is described as follows:
For $p \geq 4$, the transfer of rings from P_1 to P_p may be carried out using the following steps:

(1) Move optimally the topmost t (smallest) rings from P_1 to some intermediate peg, using all the pegs, in $T(t,p)$ number of moves.
(2) Transfer the remaining $(r-t)$ largest rings from P_1 to P_p, using the $(p-1)$ pegs available (the intermediate peg cannot be used by the condition of the problem) in an optimal way in $T(r-t,p-1)$ number of moves.
(3) Shift optimally the t rings from the intermediate peg to P_p, using all the p pegs, again in $T(t,p)$ number of moves.

Then, t is to be determined so as to minimize the total number of minimum moves in the above three steps.

For the 3-peg Tower of Hanoi problem, Wood [6] has shown that the policy leading to the recurrence relation is indeed optimum. The recursive solution describing the above steps is given by the following equation:

$$T(r,p) = \min_{0 \leq t \leq (r-1)} \{2T(t,p) + T(r-t,p-1)\}; r \geq 1, p \geq 4,$$
$$T(0,p) = 0, T(1,p) = 1, \ \forall p \geq 3.$$

For $r \geq 1, p \geq 4$,

$$T(r,p) = \min_{0 \leq t \leq (r-1)} \{2T(t,p) + T(r-t,p-1)\};$$
$$T(0,p) = 0, T(1,p) = 1, \ \forall\, p \geq 3.$$

The values of $T(r,p)$ for $r \geq 1$ rings and $p \geq 3$ pegs obtained using the above recursive algorithm are called Frame Stewart numbers.

3. Tables using Frame Stewart Algorithm

The values tabulated below have been obtained using the **Frame Stewart Algorithm.** Some of the values for $T(r,p)$ have been obtained in [3]. In this section, using the Frame Stewart Algorithm we find the number of moves required to move r rings for $3,4,\ldots,10$ pegs where $1 \leq r \leq 30$.

TABLE 1 : T(r, p)

p \ r	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
3	1	3	7	15	31	63	127	255	511	1023	2047	4095	8191	..	..	..
4	1	3	5	9	13	17	25	33	41	49	65	81	97	113	129	161
5	1	3	5	7	11	15	19	23	27	31	39	47	55	63	71	79
6	1	3	5	7	9	13	17	21	25	29	33	37	41	45	49	57
7	1	3	5	7	9	11	15	19	23	27	31	35	39	43	47	51
8	1	3	5	7	9	11	13	17	21	25	29	33	37	41	45	49
9	1	3	5	7	9	11	13	15	19	23	27	31	35	39	43	47
10	1	3	5	7	9	11	13	15	17	21	25	29	33	37	41	45

p \ r	17	18	19	20	21	22	23	24	25	26	27	28	29	30
3	..	..	..	..	.. ..	..	..	..	..	..	..	..	..	..
4	193	225	257	289	321	385	449	513	577	641	705	769	897	1025
5	87	95	103	111	127	143	159	175	191	207	223	239	255	271
6	65	73	81	89	97	105	113	121	129	137	145	153	161	169
7	55	59	63	67	71	79	87	95	103	111	119	127	135	143
8	53	57	61	65	69	73	77	81	85	89	93	97	105	113
9	51	55	59	63	67	71	75	79	83	87	91	95	99	103
10	49	53	57	61	65	69	73	77	81	85	89	93	97	101

Let us denote the size of a block for $p \geq 3$ pegs by $B_p(k)$ for $k \geq 0$, with the condition that the consecutive entries of $T(r,p)$ in the k^{th} block for p-pegs differ by 2^k. Note that $B_p(0) = 1,\ \forall p \geq 3$. For a fixed p these are precisely the cells with the same colour as shown in TABLE 1.
Using Frame Stewart algorithm, we get

$$B_p(k) = \binom{p+k-3}{k}.$$

Note that $B_p(k)$ is the coefficient of x^k in the expansion of $\dfrac{1}{(1-x)^{p-2}}$.
This gives the following generating function for $B_p(k)$.

$$\frac{1}{(1-x)^{p-2}} = B_p(0) + B_p(1)x + B_p(2)x^2 + \ldots + B_p(l-1)x^{l-1} + B_p(l)x^l + \ldots.$$

We enumerate the values of the blocks $B_p(k)$ in the table given below for $3 \leq p \leq 10$ and $0 \leq k \leq 12$.

TABLE 2: $B_p(k)$

p \ k	0	1	2	3	4	5	6	7	8	9	10	11	12
3	1	1	1	1	1	1	1	1	1	1	1	1	1
4	1	2	3	4	5	6	7	8	9	10	11	12	13
5	1	3	6	10	15	21	28	36	45	55	66	78	91
6	1	4	10	20	35	56	84	120	165	220	286	364	455
7	1	5	15	35	70	126	210	330	495	715	1001	1365	1820
8	1	6	21	56	126	252	462	792	1287	2002	3003	4368	6188
9	1	7	28	84	210	462	924	1716	3003	5005	8008	12376	18564
10	1	8	36	120	330	762	1716	3432	6435	11440	19448	31824	50388

Using a standard combinatorial identity we get,

$$B_{p+1}(l) = B_p(0) + B_p(1) + B_p(2) + \ldots + B_p(l).$$

This can also be observed in the above table.
TABLE 2 gives the sizes of blocks for different values of r and p. The table below shows the Frame Stewart numbers for various block-ends or when the block is of full size. The Frame Stewart number for the case when $r = 6435$ and $p = 9$ is $T(B_{9+1}(8), 9) = T(6435, 9) = 1066495$.

TABLE 3: $T(B_{p+1}(k), p)$

p \ k	0	1	2	3	4	5	6	7	8
3	1	3	7	15	31	63	127	255	511
4	1	5	17	49	129	321	769	1793	4097
5	1	7	31	111	351	1023	2815	7423	18943
6	1	9	49	209	769	2561	7137	23297	65537
7	1	11	71	351	1471	5503	18943	61183	187903
8	1	13	97	545	2561	10625	40193	141569	471041
9	1	15	127	799	4159	18943	78079	297727	**1066495**
10	1	17	161	1121	6401	31745	141569	580965	2228225

The results and notation of this section can be found in [10]

4. Observations About the Frame Stewart numbers

Observing TABLE 1, TABLE 2 and TABLE 3, we arrive at few observations related to the Frame Stewart numbers of the multi-peg Tower of Hanoi problem.

Proposition 4.1. *a)* $T(r+1,p) - T(r,p)$ *is some non-negative integral power of 2.*
b) For r *satisfying*
$\binom{p+l-3}{l-1} \leq r < \binom{p+l-2}{l}$
$$T(r,p) = \sum_{k=0}^{l-1} B_p(k)2^k + \left\{ r - \sum_{k=0}^{l-1} B_p(k) \right\} 2^l.$$

For a proof of the above theorem refer to [7].

Lemma 4.2. $T(r,p)$ *is always odd for* $r \geq 1$ *and* $p \geq 3$.

Proof. For r satisfying $\binom{p+l-3}{l-1} \leq r < \binom{p+l-2}{l}$,
$$T(r,p) = \sum_{k=0}^{l-1} B_p(k)2^k + \left\{ r - \sum_{k=0}^{l-1} B_p(k) \right\} 2^l.$$
Since $B_p(0) = 1$, the first term on the right is 1 and the remaining terms are all multiples of 2. Thus the right hand side is odd. □

Theorem 4.3. *The set* $T = \{T(r,p) | r \geq 1, p \geq 3\}$ *and the set of all odd positive integers are equal.*

Proof. Using Lemma 4.2, it is clear that T is the subset of odd positive integers. Conversely, let n be an odd positive integer. Clearly $n = 1 \in T$. Let $n > 1$ where $n = 2q + 1$ for some $q \geq 1$. Take $p = q + 2$. Thus $p \geq 3$. Consider $T(B_{p+1}(1), p) = T(B_p(0)+B_p(1)2, p) = 1+\binom{p+1-3}{1}2 = 1+2(p-2) = 2p-3 = 2(q+2)-3 = 2q+1 = n$. Therefore for all odd $n \in \mathbb{N}$, $n \in T$.
Thus the set T and the set of all odd positive integers are equal. □

For a fixed p, let $T_p = \{T(r,p) | r \geq 1\}$, the set of Frame Stewart numbers of the multi-peg Tower of Hanoi problem for p-pegs.

Theorem 4.4. $T_p \nsubseteq T_{p+1}$ *for any* $p \geq 3$.

Proof. For any $p \geq 3$, take $n = 1 + 2B_p(1) + 4$. Clearly $n \in T_p$. Note that $n - 2 = 1 + 2B_{p+1}(1) \in T_{p+1}$ and is the last number in the block of size 1. Therefore $n - 2 + 2 = n \notin T_{p+1}$. □

From the TABLES 1, 3 and using Mathematica we have computed values of Frame Stewart numbers for the end of the block for $3 \leq p \leq 150$ and observed that:

(1) $T(B_{p+3}(p-2), p+2) = T(B_{p+1}(p-1), p)$.
(2) $T(B_{p+3}(2p-4), p+2) = T(B_{p+1}(2p-2), p)$.

We prove the first observation using the identity $B_{p+3}(p-2) = B_{p+2}(0) + B_{p+2}(1) + B_{p+2}(2) + \ldots + B_{p+2}(p-2) = \sum_{i=0}^{p-2} B_{p+2}(i)$ followed by a combinatorial argument.

Theorem 4.5. $T(B_{p+3}(p-2), p+2) = T(B_{p+1}(p-1), p), p \geq 3$.

Proof. We begin with the LHS,

$$\begin{aligned} &T(B_{p+3}(p-2), p+2) \\ &= 2^0 B_{p+2}(0) + 2^1 B_{p+2}(1) + 2^2 B_{p+2}(2) + \ldots + 2^{p-2} B_{p+2}(p-2) \\ &= 2^0 \binom{p-1}{0} + 2^1 \binom{p}{1} + 2^2 \binom{p+1}{2} + \ldots + 2^{p-2} \binom{2p-3}{p-2}. \end{aligned}$$

Now RHS $= T(B_{p+1}(p-1), p)$

$$\begin{aligned} &= 2^0 B_p(0) + 2^1 B_p(1) + 2^2 B_p(2) + \ldots + 2^{p-1} B_p(p-1) \\ &= 2^0 \binom{p-3}{0} + 2^1 \binom{p-2}{1} + 2^2 \binom{p-1}{2} + \ldots + 2^{p-1} \binom{2p-4}{p-1}. \end{aligned}$$

We thus need to prove

$$\binom{p}{1} + 2^1 \binom{p+1}{2} + \ldots + 2^{p-3} \binom{2p-3}{p-2} = \binom{p-2}{1} + 2^1 \binom{p-1}{2} + \ldots + 2^{p-2} \binom{2p-4}{p-1}.$$

By using Pascal's identity and simplifying, the problem reduces to proving

$$1 + \left[\binom{p-1}{0} + 2\binom{p-1}{1}\right] + 2^1 \left[\binom{p}{1} + 2\binom{p}{2}\right] + \ldots + 2^{p-4} \left[\binom{2p-5}{p-4} + 2\binom{2p-5}{p-3}\right] = 2^{p-3} \binom{2p-4}{p-1}.$$

Right hand side is the number of sequences of length $2p-4$ with exactly $p-1$ $a's$ and either b or c in the remaining $p-3$ places.

We shall now characterize all these sequences with the occurrence of the rightmost a or c and followed by all $b's$.

For the first sequence, place a in the first $p-1$ places and b in all the remaining $p-3$ places. This can be done in 1 way.

The second type of sequence can be formed by placing a in the first $p-1$ places, c in the p^{th} place and b in the remaining $p-4$ places. This can be done in $\binom{p-1}{0}$ ways.

Next we fix a in the p^{th} position followed by all b's and in one position out of the first $p-1$ places we place either b or c. This can be done in $2\binom{p-1}{1}$ ways.

Next we count the number of sequences with rightmost c in the $(p+1)^{\text{th}}$ position followed by all $b's$. Among the first p positions, we choose to fill exactly one place with either b or c and the remaining with $a's$. This can be done in $2\binom{p}{1}$ ways. We place b in the remaining $p-5$ places. We then count the number of sequences with the rightmost a in the $(p+1)^{\text{th}}$ position. Out of the first p places, we have to place

$a's$ in $p-2$ places and two places need to be filled with either b or c. This can be done in $2^2 \binom{p}{2}$ ways. We continue this process.
Now the last place is filled with either a or c. If it is filled with c then out the first $2p-5$ places we need to fill $(p-1)$ places with $a's$ and remaining $p-4$ places with b or c. This can be done in $2^{p-4} \binom{2p-5}{p-4}$ ways. If the last place is filled with a then among the first $2p-5$ places we have to place $(p-2)$ $a's$ and $p-3$ places have to be filled with either b or c. This can be done in $2^{p-3} \binom{2p-5}{p-3}$ ways.
Thus all the strings are completely characterized by the occurrence of the rightmost a or c which are followed by b at all remaining places, if any. This proves the identity. □

Observation 2:
$T(B_{p+3}(2p-4), p+2) = T(B_{p+1}(2p-2), p)$.

Observing the values of $T(r,p)$ in TABLES 1, 2 and 3; we propose the following three conjectures.

Conjecture 4.6. *$T_p \subsetneq T_{p+2}$ for any $p \geq 3$.*

Conjecture 4.7. *$T(B_p(2), p-1) + 4 \notin \cup_{i=3}^{p} T_i$ for any $p \geq 4$.*

Conjecture 4.8. *There does not exist an $n \in \mathbb{N}$ such that $\cup_{p=3}^{n} T_p$ is equal to the set of all odd natural numbers.*

The proof of the first conjecture will lead to many combinatorial identities related to the Frame Stewart numbers of the multi-peg Tower of Hanoi problem.
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