Journal of Indian Acad. Math
Vol. 47, No. 2 (2025) pp. 418-426
ISSN: 0970-5120

*A. Deepshika[1], J. Kannan[*2], and P. Vimala Devi[3]*

ON THE MISCELLANEOUS EXPONENTIAL DIOPHANTINE EQUATIONS

Abstract : An exponential Diophantine equation is a particular form of Diophantine equations. Here, we solved the five different representations of exponential Diophantine equations such as $7^x + 47^y = z^2$, $p^x + (p+40)^y = z^2$, $7^x + 168^y = z^2$, $7^x + 282^y = z^2$ and $19.12^x + 19^y = z^2$ where $p, p+40$ are primes with $p > 7$ and $x, y, z \in \mathbb{Z}^+$. This paper contains the actual solution sets for these equations, together with the proof and python programming is provided.

2020 MSC Classification:11A07, 11D61, 11D72

Keywords: Diophantine equation, Exponential Diophantine equation, Mihailescu's Theorem, congruences.

1 Introduction

An exponential Diophantine equation is a sort of Diophantine equation in which the variables are in the exponent. There are several ways to solve the exponential Diophantine problem, such as the modulo arithmetic approach, the factoring method, method of congruences as in [12, 13]. There has been some interesting study effort on these equations so far [1–6].

Especially, the Diophantine equation $7^x + 7^{2y} = z^2$ was examined by Manju Somanath et.al [9]. William Sobredo Gayo,. Jr. et.al [13] solved the Diophantine equation of the form $M_p^x + (M_q+1)^y = z^2$ with M_p and M_q+1 are mersenne primes. In [7], Mahalakshmi.et.al. worked on the exponential Diophantine equation over triangular numbers.

This paper has divided into six sections. Section(2) discusses the solvability of the Diophantine equation $7^x + 47^y = z^2$. It is proved that it has no solution on the positive integers x, y, z. In the following section (3), the equation $p^x + (p+40)^y = z^2$ where $p > 7$ and $p + 40$ are primes has no positive integer solutions. In section (4), the two forms of Diophantine equations, namely $7^x + 168^y = z^2$ and $7^x + 282^y = z^2$ are solved for all $x, y, z \in \mathbb{Z}^+$. The interesting fact is that the Diophantine equation $7^x + 282^y = z^2$ has exactly one solution and $7^x + 168^y = z^2$ has exactly two solutions. In the final section(5), some theorems are discussed about the solvability of the Diophantine equation $19.12^x + 19^y = z^2$. One only solution exists for this Diophantine equation is the obtained result.

2 The Exponential Diophantine equation $7^x + 47^y = z^2$

This section gives various theorems for the solvability of the Diophantine equation $7^x + 47^y = z^2$ over positive integers.

Lemma 2.1. *(**Mihailescu's Theorem**) [13] $(a, x, b, y) = (3, 2, 2, 3)$ is the unique solution for the exponential Diophantine equation $a^x - b^y = 1$, where $a, b, x, y \in \mathbb{Z}$ such that $min\{a, b, x, y\} \geq 2$.*

Theorem 2.1. *The Diophantine equation $7^x + 47^y = z^2$ has no solution in positive integers x, y and z with x and y are of same parity.*

Proof. Here 7^x and 47^y are odd then $7^x + 47^y$ is even which implies z^2 is even.

Case (i) Suppose x and y are even. Let $x = 2k$, and $y = 2s$, where $k \geq 1, s \geq 1$ Then, $7^x + 47^y \equiv 2 \pmod 4$ implies $z^2 = 2 \pmod 4$ which is a contradiction.

Case (ii) Suppose x and y are odd Let $x = 2k + 1$ and $y = 2s + 1$, $k, s \geq 0$. $7^x + 47^y \equiv -2 \pmod 4$. We get $z^2 = -2 \pmod 4$ Again we get a contradiction. Hence $7^x + 47^y = z^2$ has no solution in positive integers x and y of same parity.

□

Theorem 2.2. *The Diophantine equation $7^x+47^y = z^2$ has no solution $\forall x, y, z \in Z^+$ with x and y of different parity.*

Proof. Consider the equation $7^x + 47^y = z^2$ Now x is even and y is odd. Take $x = 2k, k \geq 1$ and $y = 2s + 1, s \geq 0$. The equation $7^x + 47^y = z^2$ becomes $7^{2k} + 47^{2s+1} = z^2$ and it changes into $(z - 7^k)(z + 7^k) = 47^{2s+1}$. Then there exist a non-negative positive integer α, β such that $47^\alpha = z - 7^k$ and $47^\beta = z + 7^k$ where $\alpha < \beta$ and $\alpha+\beta = 2s + 1$. Now $2.7^k = 47^\beta - 47^\alpha = 47^\alpha(47^{\beta+\alpha} - 1)$ which implies $\alpha = 0$. Then we get $2.7^k = 47^{2s+1} - 1$. For $s = 0$, this is impossible, and for $s > 1$, we have $7^k = 23(47^{2s}+47^{2s-1}+\cdots+47+1)$ which is an impossible. Suppose x is odd and y is even and choose $x = 2k + 1, k \geq 0$ and $y = 2s, s \geq 1$. The given equation becomes $7^{2k+1} = (z + 47^s)(z - 47^s)$. Then there exists $\alpha, \beta \in Z^+ \cup \{0\}$ such that $7^\alpha = z - 47^s$ and $7^\beta = z + 47^s$ with $\alpha < \beta$ and $\alpha + \beta = 2k + 1$. Now $2.47^s = 7^\beta - 7^\alpha$ becomes $2.47^s = 7^{2k+1} - 1$. The above equality fails for $k = 0$ and for $k > 1$, the equality $2.47^s = 7^{2k+1}$ changes into $47^s = 3(7^{2k} + 7^{2k-1} + \cdots + 7 + 1)$ which is not possible. The given Diophantine's equation has no solution for $x, y, z \in Z^+$ □

3 The Exponential Diophantine equation $p^x + (p + 40)^y = z^2$

This section discusses the solvability of the special form of the Diophantine equation $p^x + (p + 40)^y = z^2$ where $p(> 7)$ and $p + 40$ are primes.

Theorem 3.1. *The Diophantine equation $p^x + (p + 40)^y = z^2$ where $p > 7$ and $p + 40$ are primes, has no solution in positive integers x, y and z with x and y are of opposite parity.*

Proof. Suppose x is even ($x = 2k, k \geq 1$) and y is odd ($y = 2s+1, s \geq 0$). Then $p^x + (p+40)^y = z^2$ is equivalent to $(p+40)^{2s+1} = (z+p^k)(z-p^k)$. Thus there exists non-negative integers α and β such that $(p+40)^\alpha = (z-p^k)$ and $(p+40^\beta) = (z+p^k)$ where $\alpha < \beta$ and $\alpha + \beta = 2s+1$. By solving, the above two equations becomes $2p^k = (p+40)^\alpha[(p+40)^{\beta-\alpha} - 1]$. This implies $\alpha = 0$ and therefore $2p^k = (p+40)^{2s+1} - 1$. When $s = 0$, $p+39 = 2p^k \implies p(2p^{k-1} - 1) = 39$, we obtain $p = 39$ and $k = 1$, which is a contradiction as p is prime. For $s \geq 1$, $2p^k = (p+40)^{2s} - 1 = (p+39)\,[(p+40)^{2s} + \cdots + (p+40) + 1]$. Here $p+39 = 2p^j$, where j is an integer such that $0 \leq j < k$. For $j = 0$, $p+39 = 2$ which is an abstract one. For $1 \leq j < k$, $p+39 = 2p^j$ and so $p(2p^{j-1} - 1) = 39$, which is not possible. Suppose x is odd and y is even,*i.e.*$x = 2k+1, k \geq 0$ and $y = 2s, s \geq 1$, then $p^x + (p+40)^y = z^2$ is equivalent to $p^{2k+1} = (z + (p+40)^s)(z - (p+40)^s)$. As same in the above case we obtain $2(p+40)^s = p^\alpha(p^{\beta-\alpha} - 1)$, which implies $\alpha = 0$. Thus $2(p+40)^s = p^{2k+1} - 1$. Now for $k = 0$, $2(p+40)^s = p - 1 \implies [2(p+40)^s] + 41 = p + 40$ which is an impossible ($s > 1$) Now $k > 1$, $2(p+40)^s = (p-1)(p^{2k} + p^{2k-1} + \cdots + p + 1)$, since $(p-1)$ is an even positive divisor of $2(p+40)$. Take $p - 1 = 2(p+40)^j$, $j \in Z$ such that $0 \leq j < s$ when $j = 0$, Then value of $p = 3 < 7$ not possible. Therefore $1 \leq j < k$ and so we obtain $2(p+40)^j + 41 = (p+40)$ and $j > 1$ This is again an absurd one.

□

Theorem 3.2. *The Diophantine equation $p^x + (p+40)^y = z^2$ where $p > 7$ and $p+40$ are primes, has no solution in positive integers x, y and z.*

Proof. Consider the equation $p^x + (p+40)^y = z^2$. Here $p^x + (p+40)^y$ are primes and even and $p > 7$. We have two choices of p , *i.e.*$p \equiv 1 \pmod 4$ or $p \equiv 3 \pmod 4$

Case 1: If $p \equiv 1 \pmod 4$, by the above lemma, it has no solution.

Case 2: If $p \equiv -1 \pmod 4$, then we have different cases.

- Sub-case 1: Suppose x and y are even, then $x = 2k, k \geq 1$ and $y = 2s, s \geq 1$. The equation 2 implies $p^x + (p+40)^y = 2 \pmod 4$. We get a contradiction.
- Sub-case 2: Suppose x and y are odd , then $x = 2k+1, k > 0$ and $y = 2s+1, s > 0$. The given equation leads to $p^x + (p+40)^y = 2 \pmod 4$. Again this is contradiction. Hence it has no solution.
- Sub-case 3: If x is odd and y is even, By theorem(3.1),the sub-case fails.
- Sub-case 4: If x is even and y is odd, again by theorem(3.1) this is not possible.

□

4 The Exponential Diophantine equations $7^x + 168^y = z^2$ and $7^x + 282^y = z^2$

Here we analyse the two variants of the Diophantine equation; one is $7^x + 168^y = z^2$ and the other is $7^x + 282^y = z^2$. A few of the lemmas and theorems were shown,

and Python code for the corresponding equations was also given. .

Lemma 4.1. *The Diophantine equation* $7^x + 1 = z^2$ *has no non negative integer solutions.*

Proof. Suppose that there are non negative integers x and z such that $7^x + 1 = z^2$. If $x = 0$, the $z^2 = 2$ which is an impossible one. If $x \geq 1$, then by Mihailescu's theorem, we get $x = 1$, which is not possible. □

Lemma 4.2. $(y, z) = (1, 13)$ *is the unique solution for the Diophantine equation* $1 + 168^y - z^2 = 0$ *where* y, z *are non negative integers.*

Proof. Suppose there are non negative integers y, z such that $1+168^y = z^2$ If $y = 0$, then $z^2 = 2$ which is impossible. If $y = 1$, then $z = 13$ and $y > 1$ implies $z \geq 13$. Now, we consider the equation, $z^2 - 168^y = 1$ By Mihailescu's theorem, $z = 2$ which is impossible. $(1, 13)$ is a unique solution (y, z) for the equation $1 + 168^y = z^2$ □

Lemma 4.3. *The Diophantine equation* $1 + 282^y - z^2 = 0$ *has no non negative integer solutions.*

Proof. For $y = 0$, it has no solution. For $y \geq 1$, then $z \geq 18$. By using theorem (2.1), we obtain $z = 3$. This is not possible as $z \geq 18$. □

4.1 For the Diophantine equation $7^x + 168^y = z^2$

In this subsection, we provided the Python programming and the solutions to the Diophantine equation $7^x + 168^y = z^2$.

Theorem 4.1. *The Diophantine equation* $7^x + 168^y = z^2$ *has exactly two solutions in non negative integers* $(x, y, z) \in \{(0, 1, 13), (4, 2, 175)\}$.

Proof. Let x, y and z be non negative integers such that $7^x + 168^y = z^2$.

Case 1: $x = 0$ and By Lemma (4.2),we get $(x, y, z) = (0, 1, 13)$

Case 2: When y is even and $x > 1$, then there exists a positive integer l such that $y = 2l$. Now the equation becomes $7^x + 168^{2l} = z^2$ and we get $z - 168^l = 7^u$, $(z + 168^l) = 7^{x-u}$ with $x > 2u$. Now, $7^{x-u} - 7^u = 2 \times 168^l$ implies $7^u(7^{x-2u} - 1) = 2 \times 168^l$. For $l = 1$, $7^u(7^{x-2u} - 1) = 2 \times 24 \times 7$. Therefore $u = 1$ and $7^{x-2} - 1 = 48 \implies x = 4$. Hence $(x, y, z) = (4, 2, 175)$.

Case 3: When y is odd and $x > 1$. For a positive integer l, such that $y = 2l + 1$. Now the equation becomes $7^x - (168)^2 l = (z - 13(168)^l)(z + 13(168)^l)$. Let $(z - 13(168)^l) = 1$ and $z + 13(168)^l = 7^x - (168)^{2l}$. On solving this we obtain $(168)^l[26 + (168)^l] = 7^x - 1$ which implies $l = 0$ and $7^x = 26$. This is not possible.

□

The occurrence of the Diophantine equation's solution, $7^x + 168^y = z^2$, as illustrated below:

```
#For the equation 7^x+168^y=z^2
import math
def diophantine():
 print('x\ty\tz')
 for x in range(0,n+1):
    for y in range (0,n+1):
        for z in range (0,n+1):
         if 7**x+168**y==z**2 :
             print(x,'\t',y,'\t',z,'\t')
n =int(input("Enter the maximum range:"))
#n is the maximum range
diophantine()
```

Coding 1: Calculating the solution for $7^x + 168^y = z^2$

```
Enter the maximum range:500
x       y       z
0        1       13
4        2       175
>>>
```

Figure 1: Output: Coding 1

4.2 For the Diophantine equation $7^x + 282^y = z^2$

The theorem and Python code for the Diophantine equation $7^x + 282^y = z^2$ are provided in this subsection.

Theorem 4.2. *The Diophantine equation $7^x + 282^y = z^2$ has exactly two non negative integer solution $(x, y, z) = (1, 1, 17), (3, 1, 25)$*

Proof. Let x, y and z be non negative integers such that $7^x + 282^y = z^2$. The Diophantine equation $7^x + 282^y = z^2$ has no solution on putting $x = 0$ and $y = 0$ (Using lemma (4.3) and (4.1)). When y is odd ($i.e., y = 2l + 1, l \in \mathbb{N} \cup \{0\}$), For $l = 0$, then $y = 1$. If $x = y = 1$, then we have $z = 17$. For $l > 0$ and $y > 1$, then the Diophantine equation becomes $7^x - 7(282)^{2l} = z^2 - 289(282)^{2l}$. Let $z - 17(282)^l = 1$ and $z + 17(282)^l = 7^x - 7(282)^{2l}$. This implies $(z + 17(282)^l) - (z - 17(282)^l) = 7^x - 7(282)^{2l} - 1$. On solving this we get $(282)^l[34 + 7(282)^l] = 7^x - 1$ and this implies $l = 0$ which is an impossible one. In other way we can solve $7^x + 282^y = z^2$ and it becomes $7^x - 294(282)^{2l} = z^2 - 24^2(282)^{2l}$. Now choose $z - 24(282)^l = 1$ and $z + 24(282)^l = 7^x - 294(282)^{2l}$. Now the equation reduces to $282^l[48 + (296)(282)^l] = 7^x - 1$ and this implies $l = 0, x = 3$ and $z = 25$. Now, the solution is $(x, y, z) = (3, 1, 25)$. Suppose y is even ($i.e., 2l, l \in \mathbb{Z}^+ \cup \{0\}$), then equation $7^x + 282^y = z^2$ reduces to $7^x = z^2 - (282)^{2l}$. Choose $z - 282^l = 7^u$ and $z + 282^l = 7^{x-u}$, then it becomes $7^u(7^{x-2u} - 1) = 2(282)^l$. For $l = 1$, we get $u = 0 \implies 7^x = 565$. This is not possible. □

Python programming validates the preceding theorem's solution, which is provided as:

```
#For the equation 7^x+282^y=z^2
import math
def diophantine():
 print('x\ty\tz')
 for x in range(0,n+1):
    for y in range (0,n+1):
        for z in range (0,n+1):
         if 7**x+282**y==z**2 :
             print(x,'\t',y,'\t',z,'\t')
n =int(input("Enter the maximum range:"))
#n is the maximum range
diophantine()
```

Coding 2: Calculating the solution for $7^x + 282^y = z^2$

```
---------------------- RESTART
Enter the maximum range:100
x       y       z
1        1       17
3        1       25
>>>
```

Figure 2: Output: Coding 2

5 The Exponential Diophantine equation $19.12^x + 19^y = z^2$

This section displayed certain lemmas and theorems that were required to obtain the positive integer solutions of the Diophantine equation $19.12^x + 19^y = z^2$.

Lemma 5.1. $(x, z) = (1, 20)$ *is the only solution for the Diophantine equation* $19.21^x + 1 - z^2 = 0$ *where* x, z *are non negative integers.*

Proof. Consider the equation

$$19.21^x = z^2 - 1 \tag{1}$$

From this, we have either 19 divides $z-1$ or $z+1$. If $19|(z-1)$, then there exists a positive integer $r < x$ such that $z - 1 = 19.21^r$ and $z + 1 = 19.21^r + 2$. The equation (1) becomes

$$21^{x-r} = 19.21^r + 2. \tag{2}$$

When $r = 0$, we get $x = 1$. Hence $(x, z) = (1, 20)$. When $r > 0$, we have $x - r > 0$. Now $21^{x-r} \equiv 0 \pmod{21}$ implies $21^{x-r} \equiv 0 \pmod{3}$. Also $19.21^r \equiv 0 \pmod{3}$ and by using the equation (2) we get a contradiction. If $19|z + 1$ and 19 does not divides $z - 1$, then there exists positive integers r and s such that $z + 1 = 19.21^r$ and $z - 1 = 21^s$, where $r + s = x$. On solving this we obtain

$$2 = 19.21^r - 21^s \tag{3}$$

On Substituting $s = 0$, it becomes $3 = 19.21^r$ which is an impossible one. Similarly for $r = 0$, we get an impossible one. If $s, r > 0$ then $19.21^r - 21^s \equiv 0 \pmod 3$. From the equation (3) we get contradiction. □

Lemma 5.2. *Let b be a positive integer such that $b \equiv 5 \pmod{40}$. Then the Diophantine equation $1 + b^y = z^2$ has no non integer solution.*

Proof. Suppose that there are non negative integers y and z such that $1 + b^y = z^2$. This contradicts when $y = 0$. For $y = 1$, we get $b = z^2 - 1$. Since $b \equiv 5 \pmod{40} \implies b \equiv \pmod 4$. Then $z^2 \equiv 2 \pmod 4$. The case fails for $y > 1$, using theorem(2.1). □

Lemma 5.3. *Let b be a positive integer such that $b \equiv 8 \pmod{40}$. Then $(y, z) = (1, \sqrt{b+1})$ represent the non negative integer solutions of the Diophantine equation $1 + b^y = z^2$, where $\sqrt{b+1}$ is an integer.*

Proof. Let y and z be non negative integers such that $1+b^y = z^2$. There is no solution for the equation when $y = 0$ and $y > 1$(By theorem(2.1). Suppose $y = 1$, then we get $z = 7$ and hence the solution for the Diophantine equation is $(1, \sqrt{b+1})$. □

5.1 For the Diophantine equation $19.21^x + 19^y = z^2$

The Diophantine equation $19.21^x + 19^y = z^2$ is solved here using the theory and Python programming.

Theorem 5.1. *$(x, y, z) = (1, 0, 20)$ is the unique solution to the Diophantine equation $19.21^x + 19^y = z^2$ where x, y, z are non-negative integers.*

Proof. Let x, y, z be non-negative integers such that $19.21^x + 19^y = z^2$. We consider the following cases.

Case 1: For $x = y = 0$, we have $z^2 = 20$ which is impossible.

Case 2: For $x = 0$ and $y > 0$, we have $19(1 + 19^{y-1}) = z^2$. When $y = 1$, it has no solution. If $y > 1$, then $19|z^2$. There is some positive integer k such that $z = 19k$ and it becomes $19(k^2 - 19^{y-2}) = 1$. This is not possible.

Case 3: For $x > 0$ and $y = 0$, we obtain $19.21^x + 1 = z^2$. By lemma (5.1), the only solution is $(x, y, z) = (1, 0, 20)$.

Case 4: When $x > 0$ and $y > 0$, the equation $19.21^x + 19^y = z^2$ implies $19|z$. Therefore $z = 19k$ and it becomes

$$21^x + 19^{y-1} = 19k^2 \tag{4}$$

. When $y = 1$, we get $k^2 \equiv 2 \pmod 4$ (which impossible). When $y > 1$, from equation(4), $21^x \equiv 0 \pmod{19}$ which is not possible.

□

Here, the python programming is provided for the existence of the solution of the exponential Diophantine equation $19.21^x+19^y = z^2$ as shown:

```
#For the equation 19*21**x+19**y=z**2
import math
def diophantine():
 print('x\ty\tz')
 for x in range(0,n+1):
    for y in range (0,n+1):
        for z in range (0,n+1):
         if 19*21**x+19**y==z**2 :
             print(x,'\t',y,'\t',z,'\t')
n =int(input("Enter the maximum range:"))
#n is the maximum range
diophantine()
```

Coding 3: Calculating the solution for $19.21^x + 19^y == z^2$

```
Enter the maximum range:200
x       y       z
1        0       20
>>>
```

Figure 3: Output: Coding 3

6 Conclusion

Finally, we solved and established the existence of the positive integer solutions to the miscellaneous Diophantine equations with the Python programming. One can try to solve for any alternative forms of the Diophantine equation.

References

[1] Abdelkader Hamtat, An exponential diophantine equation on triangular numbers, Mathematica Applicanda. 51, 1, 2023, 99–107.

[2] Kannan. J, Manju Somanath, and Raja. K, On a class of solutions for the hyperbolic diophantine equation, International Journal of Apllied Mathematics. 32, 3, 2019, 443-449.

[3] Kaleeswari.K, Kannan. J, and Narasimman, G, Exponential diophantine equations involving isolated primes, Advances and Applications in Mathematical Sciences. 22, 1, 2022, 169-177.

[4] Manju Somanath, Kannan. J, and Raja. K, Exponential diophantine equation in two or three variables, Global Journal of Pure and Applied Mathematics. 13, 5, 2017, 128-132.

[5] Manju Somanath, Raja. K, Kannan. J, and Mahalakshmi. M, On a class of solution for a quadratic diophantine equation, Advances and Applications in Mathematical Sciences. 19, 11, 2022, 1097-1103.

[6] Manju Somanath, Raja. K, Kannan. J, and Nivetha. S, Exponential diophantine equation in three unknowns, Advances and Applications in Mathematical Sciences, 19, 11, 2020, 1113-1118.

[7] Mahalakshmi. M, Kannan. J, Deepshika. A, and Kaleeswari, K, (2023). Existence and non - existence of exponential diophantine triangles over triangular numbers, Indian Journal of Science and Technology. 16, 41, 2023, 3599–3604.

[8] Kannan. J, and Manju Somanath, Fundamental perceptions in contemporary number theory, Nova Science Publisher, New York, 2023.

[9] Manju Somanath, Kannan. J, and Raja, K, Exponential diophantine equation in three variables $7^x + 7^{2y} = z^2$, International Journal of Engineering Research-Online, 5, 4, 2017, 91-93.

[10] Rosen. K.H., Elementary number theory and its applications, Addison-Wesley Publishing Company, Boston, 1987.

[11] Telang, S.G., Number theory, Tata McGraw - Hill Publishing Company Limited, New York, 1996.

[12] Titu Andreescu, Dorin Andrica, and Ion Cucurezeanu,An introduction to diophantine equations: a problem - based approach, Birkhauser Boston, 2010.

[13] William Sobredo Gayo, Jr., and Jerico Bravo Bacani, On the diophantine equation $M_p^x + (M_q + 1)^y = z^2$, European Journal of Pure and Applied Mathematics. 14, 2, 2021, 396-403.

[1,2] *Department of Mathematics,*
Ayya Nadar Janaki Ammal College (Autonomous, affiliated to
Madurai Kamaraj University, Madurai), Sivakasi - 626124, India.
Email: deepshi20mar@gmail.com

Email: jayram.kannan@gmail.com

[3] *School of Advanced Sciences,*
Kalasalingam Academy of Research and Education,
Krishnankoil - 626 126, Tamil Nadu, India.
Email: vimalkathir 15@gmail.com

(Received, February 25, 2025)