Journal of Indian Acad. Math
Vol. 47, No. 2 (2025) pp. 397-000
ISSN: 0970-5120

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GROWTH OF THE SOLUTIONS OF Nth ORDER LINEAR SHIFT DIFFERENTIAL EQUATIONS WITH UNCERTAIN COEFFICIENT

Abstract. After applying shift, we look at the complex oscillation solutions of the nth order linear differential equation and demonstrate by a counter example that the conclusion of the earlier result does not hold. We also establish the result for the shift differential equation with a supporting a example.

Key words and phrases: Entire function, meromorphic function, growth, linear differential equation, shift differential equation. **Mathematical Subject Classification 2020**: 34M10, 30D35.

1. Introduction

The standard definitions and notations of Nevanlinna value distribution theory are adopted in this paper. For further information, see [9], [11], [14], [15]. As for example $\rho(f)$ denote the order of a meromorphic function f. The growth of the solution of complex linear differential equation and difference equation have been studied by many numerous researchers since 1960s. Based on the Nevanlinna value distribution theory, some result on the growth of all transcendental entire solutions has been found in [2]. Nevanlinna value distribution theory and the maximum modulus principle have been used to examine the zero problems of second order algebraic differential equations with meromorphic coefficients in [13]. The fundamental theorem of Nevanlinna has been used to explore the problems of certain kinds of complex differential equations [1].

In the year 1986 langley [12] consider the differential equation

$$f^{(2)} + Ae^{-z}f^{(1)} + Q(z)f = 0 \tag{1.1}$$

and obtained the following results.

Theorem A. [12] Let $Q(z)$ be a non-constant polynomial. Then all non-trivial solutions of equation (1.1) have infinite order, where A is any non-zero constant.

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Gunderson [7] established the following for the situation when $Q(z)$ is a transcendental entire function.

Theorem B. [7]Consider a transcendental entire function $Q(z)$ with order $\rho(Q) \neq 1$, then every non-zero solution of the equation

$$f^{(2)} + e^{-z} f^{(1)} + Q(z) f = 0$$

has infinite order.

Using the previously mentioned concept, Chen [3] takes a look at the differential equation of the form

$$f^{(2)} + A_1(z) e^{az} f^{(1)} + A_0(z) e^{bz} f = 0 \tag{1.2}$$

and got the following results.

Theorem C. [3] Let $A_j(z) (\neq 0) (j = 0, 1)$ be entire functions with $\rho(A_j) < 1$, a, b are complex constants such that $ab \neq 0$ and $a = cb$ $(c > 1)$, then every non-zero solution of equation (1.2) has infinite order.

Theorem D. [3] Let $A_j(z) (\neq 0)$, $D_j(z)$ be entire functions with $\rho(A_j) < 1, \rho(D_j) < 1$, a, b are complex constants such that $ab \neq 0$ and $\arg a \neq \arg b$ or $a = cb$ $(0 < c < 1)$, then every non-zero solution of equation

$$f^{(2)} + (A_1(z) e^{az} + D_1) f^{(1)} + \left(A_0(z) e^{bz} + D_2\right) f = 0$$

has infinite order.

In the recent year 2023, He and Gao [10] consider $n - th$ order differential equation instead of second order differential equation of the form

$$f^{(n)} + A_{n-1} e^{a_{n-1} z^m} f^{(n-1)} + \ldots + A_1 e^{a_1 z^m} f^{(1)} + A_0 e^{a_0 z^m} f = 0 \tag{1.3}$$

and improved the result of Chen [3] as follows :

Theorem E. [10]Let $A_j(z) (j = 0, 1, \ldots, (n-1))$ are meromorphic functions and a_j are complex constants such that $\rho(A_j) < m$ $(\forall\ j = 0, 1, \ldots, (n-1))$. Then all the non-zero solutions of equation (1.3) have infinite order.

The shift version of Nevanlinna theory was investigated by Hulburd-Korhonen [8], Chiang-Feng [5] and numerous other researchers and produce some remarkable results.

Now, a natural question arises: "Is the conclusion of Theorem E still valid if we apply shift on the equation (1.3)?" Applying shift, the equation (1.3) takes the form

$$f^{(n)}(z+\xi_n)+A_{n-1}e^{P_{n-1}(z)}f^{(n-1)}\left(z+\xi_{n-1}\right)+...+A_0e^{P_0(z)}f(z+\xi_0)=0 \tag{1.4}$$

where $\xi_i(\neq 0)\,(i=0,1,...,n)$ are complex constants and $P_i(z)\,(i=0,1,...,(n-1))$ are the polynomial of degree $m\geq 1$ with $\rho(A_j)<m\ (\forall\ j=0,1,...,(n-1))$.

Unfortunately, the answer to the above question is no. The following counter-example supports our claim.

Example 1. *The function $f(z)=e^{z^3}$ satisfies the equation*

$$f^{(3)}(z+1)+A_2e^{P_2(z)}f^{(2)}(z-1)+A_1e^{P_1(z)}f^{(1)}(z-i)+A_0e^{P_0(z)}f(z+i)=0$$

where

$$\begin{aligned}
A_2(z) &= -3z^2+2z,\ P_2(z)=6z^2\\
A_1(z) &= -54z^3-108iz^2+54z,\ P_1(z)=3iz^2+3z^2+6z-i+1\\
A_0(z) &= 504z^4-270z^3-567z^2-162z+61,\ P_0(z)=-3iz^2+3z^2+6z+i+1
\end{aligned}$$

But $\rho(f)=3$, which is finite.

We now present the main result of our paper.

Theorem 1. *Let $A_j(z)\,(j=0,1,...,(n-1))$ are meromorphic functions, $\xi_j(\neq 0)\,(j=0,1,...,n)$ are complex constants and $P_j(z)\,(j=0,1,...,(n-1))$ are the polynomial of degree $m\geq 1$ such that $\rho(A_j)<m\ (\forall\ j=0,1,...,(n-1))$. Then every non-zero solutions f of equation (1.4) satisfy $\rho(f)\geq m+1$.*

Remark 1. *The conclusion of the above theorem may not be true if we relax the condition $\rho(A_j)<m\ (\forall\ j=0,1,...,(n-1))$. To support this let us consider the following example.*

Example 2. *The equation*

$$f^{(2)}(z-2i)+A_1e^{P_1(z)}f^{(1)}(z+i)+A_0e^{P_0(z)}f(z+1)=0$$

is satisfied by the function $f(z) = e^{2z^2}$, *where*

$$\begin{aligned} A_1(z) &= (-4z)\, e^{-z^2-5iz-2},\ P_1(z) = z^2 - 7iz - 4 \\ A_0(z) &= (80iz + 60)\, e^{3z^2-(3i+1)z-3},\ P_0(z) = -3z^2 - (5i+3)\, z - 7. \end{aligned}$$

Here we see that $\rho(A_0) = \rho(A_1) = 2$ *and* $\deg(P_j) = 2\,(\, j = 0, 1)$. *Which does not satisfy the condition of the above theorem. So the conclusion of the above theorem is not true as* $\rho(f) = 2 \ngeq 2 + 1$.

2. Lemmas

Several lemmas that will be required later are presented in this section.

Lemma 2. [3]*Let* $P(z) = (\alpha + i\beta)\, z^n + (\alpha + i\beta)\, z^{n-1} + ... + (\alpha + i\beta)\, z + (\alpha + i\beta)$, $\alpha, \beta \in \mathbb{R}$ *such that* $|\alpha| + |\beta| \neq 0$, *is a polynomial of degree* n *and* $A(z)$ *is a meromorphic function of order less than* n. *Then for the function* $g(z) = A(z)\, e^{P(z)}$, $z = re^{i\theta}$, $\delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta$, *there exist sets* $H_1 \subset [0, 2\pi)$ *that has linear measure zero and* $H_2 = \{\theta \in [0, 2\pi); \delta(P, \theta) = 0\}$ *and* $R > 1$ *such that for all* $|z| = r > R$ *and for all* $\theta \in [0, 2\pi) \setminus \{H_1 \cup H_2\}$, *we have*

$$(i)\ \exp\{(1-\varepsilon)\,\delta(P,\theta)\, r^n\} < g\left(re^{i\theta}\right) < \exp\{(1+\varepsilon)\,\delta(P,\theta)\, r^n\},\ \textit{when}\ \delta(P,\theta) > 0$$

and

$$(ii)\ \exp\{(1+\varepsilon)\,\delta(P,\theta)\, r^n\} < g\left(re^{i\theta}\right) < \exp\{(1-\varepsilon)\,\delta(P,\theta)\, r^n\},\ \textit{when}\ \delta(P,\theta) < 0.$$

Lemma 3. [6]*Let* $f(z)$ *be a transcendental meromorphic function of finite order* ρ *and* $\varepsilon > 0$ *be a given constant. Then, there exist a subset* $E \subset (1, \infty)$ *that has finite logarithmic measure, such that for all* z *satisfying* $|z| = r \notin [0, 1] \cup E$, *and for all* $k, j,\ 0 \leq j < k$, *we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq r^{(k-j)(\rho-1+\varepsilon)}.$$

Lemma 4. [4]*Let* $f(z)$ *be a meromorphic function of finite order* ρ *and* $\varepsilon > 0$ *be any given constant. Then, there exist a subset* $E \subset (1, \infty)$ *that has finite linear measure or finite logarithmic measure, such that*

$$|f(z)| \leq e^{r^{\rho+\varepsilon}}$$

holds for all z *satisfying* $|z| = r \notin [0, 1] \cup E$ *as* $r \to \infty$.

Lemma 5. [5]*Let η_1, η_2 be two arbitrary complex constant and let $f(z)$ be a meromorphic function of finite order ρ and $\varepsilon > 0$ be a given constant. Then, there exist a subset $E \subset (1, \infty)$ that has finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0,1] \cup E$, the following double inequality holds*

$$e^{-r^{\rho-1+\varepsilon}} \leq \left| \frac{f(z+\eta_1)}{f(z+\eta_2)} \right| \leq e^{r^{\rho-1+\varepsilon}}.$$

3. Proof of Theorem

We will proof the theorem by contradiction.

Let us assume that all the solution of the equation (1.4) are transcendental meromorphic function of order $0 < \rho < k+1$. Let $Q_0(z) = (\gamma_0 + i\delta_0) z^m + (\gamma_0 + i\delta_0) z^{m-1} + ... + (\gamma_0 + i\delta_0)$ be the polynomial among $P_j(z) = \left(\alpha_j + i\beta_j\right) z^m + \left(\alpha_j + i\beta_j\right) z^{m-1} + ... + \left(\alpha_j + i\beta_j\right)$, $j = 1, 2, ..., n-1$, which satisfies

$$\delta(Q_0, \theta) = \max_{1 \leq j \leq n-1} \delta(P_j, \theta), \ z = re^{i\theta}.$$

Then two cases may arise.

Case-1 : Let $\alpha_0 + i\beta_0$ and $\gamma_0 + i\delta_0$ satisfy $\arg(\alpha_0 + i\beta_0) \neq \arg(\gamma_0 + i\delta_0)$. Then there exist $\varepsilon > 0$ and a ray $\arg z = \theta$ such that for all $\theta \in [0, 2\pi) \setminus \{E_1 \cup H_1 \cup H_2\}$,

$$\delta(P_0, \theta) > 0 \text{ and } \delta(P_j, \theta) < 0, \text{ for all } j = 1, 2, ..., n-1.$$

Where $E_1 \cup H_1 \cup H_2$ has linear measure zero with $E_1 \subset (1, \infty)$,
$H_1 = \{\theta \in [0, 2\pi); \delta(Q_0, \theta) = 0\}$ and $H_2 = \{\theta \in [0, 2\pi); \delta(P_0, \theta) = 0\}$. Therefore by Lemma 2, we have

$$\left| A_k(z) e^{P_k(z)} \right| \leq \exp\{(1-\varepsilon) \delta(P_k, \theta) r^m\}, \ k = 1, 2, ..., n-1 \tag{3.1}$$

and

$$\left| A_0(z) e^{P_0(z)} \right| \geq \exp\{(1-\varepsilon) \delta(P_0, \theta) r^m\} \tag{3.2}$$

Now from equation (1.4),

$$\begin{aligned}\left|A_0(z)e^{P_0(z)}\right| &\leq \left|\frac{f^n(z+\xi_n)}{f(z+\xi_0)}\right| + \left|\sum_{k=1}^{n-1}\frac{f^k(z+\xi_k)}{f(z+\xi_0)}A_k(z)e^{P_k(z)}\right| \\ &= \left|\frac{f^n(z+\xi_n)}{f(z+\xi_n)}\frac{f(z+\xi_n)}{f(z+\xi_0)}\right| + \left|\sum_{k=1}^{n-1}\frac{f^k(z+\xi_k)}{f(z+\xi_k)}\frac{f(z+\xi_k)}{f(z+\xi_0)}A_k(z)e^{P_k(z)}\right|.\end{aligned}$$

Now using Lemma 3, Lemma 5 and (3.1) we have,

$$\begin{aligned}\left|A_0(z)e^{P_0(z)}\right| &\leq r^{n(\rho-1+\varepsilon)}e^{r^{(\rho-1+\varepsilon)}} + r^{(n-1)(\rho-1+\varepsilon)}e^{r^{(\rho-1+\varepsilon)}}\exp\{(1-\varepsilon)\delta(P_{n-1},\theta)r^m\} + \\ &\quad ... + r^{(\rho-1+\varepsilon)}e^{r^{(\rho-1+\varepsilon)}}\exp\{(1-\varepsilon)\delta(P_1,\theta)r^m\} \\ &\leq nr^{n(\rho-1+\varepsilon)}e^{r^{(\rho-1+\varepsilon)}}\exp\{(1-\varepsilon)\delta(Q_0,\theta)r^m\}. \qquad (3.3)\end{aligned}$$

Hence from (3.2) and (3.3) we have,

$$\begin{aligned}\exp\{(1-\varepsilon)\delta(P_0,\theta)r^m\} &\leq nr^{n(\rho-1+\varepsilon)}e^{r^{(\rho-1+\varepsilon)}}\exp\{(1-\varepsilon)\delta(Q_0,\theta)r^m\} \\ \text{or, } (1-\varepsilon)\delta(P_0,\theta)r^m &\leq \log n + n(\rho-1+\varepsilon)\log r + r^{(\rho-1+\varepsilon)} + (1-\varepsilon)\delta(Q_0,\theta)r^m \\ \text{or, } (1-\varepsilon)\delta(P_0,\theta) &\leq \frac{\log n}{r^m} + \frac{n(\rho-1+\varepsilon)\log r}{r^m} + \frac{r^{(\rho-1+\varepsilon)}}{r^m} + (1-\varepsilon)\delta(Q_0,\theta)\end{aligned}$$

Which contradict the fact that $\delta(P_0,\theta) > 0$, if the arbitrary chosen $\varepsilon \in (0,1)$.

Case-2 : Let $\alpha_0 + i\beta_0$ and $\gamma_0 + i\delta_0$ satisfy $(\gamma_0 + i\delta_0) = c(\alpha_0 + i\beta_0)$, where $0 < c < 1$. Then obviously $\delta(Q_0,\theta) = c\delta(P_0,\theta)$ and there exist a ray $\arg z = \theta$ for which

$$\delta(Q_0,\theta) = c\delta(P_0,\theta) > 0$$

holds. Therefore by Lemma 2, we have

$$\left|A_k(z)e^{P_k(z)}\right| \leq \exp\{(1+\varepsilon)\delta(P_k,\theta)r^m\},\ k = 0,1,2,...,n-1 \qquad (3.4)$$

Hence from equation (1.4), using (3.2) and (3.4), we get

$$\exp\{(1-\varepsilon)\delta(P_0,\theta)r^m\} \leq \left|A_0e^{P_0(z)}\right| \leq nr^{n(\rho-1+\varepsilon)}e^{r^{(\rho-1+\varepsilon)}}\exp\{(1+\varepsilon)\delta(Q_0,\theta)r^m\}$$

which is a contradiction if we choose arbitrary $\varepsilon = \frac{1-c}{1+c},\ c \in (0,1)$.

We now claim that any solution of equation (1.4) with finite order can not have polynomial structure. If possible let $f(z)$ be a solution of equation (1.4), which is a polynomial of degree d. In this case two the following possibilities may arise:

Case-3 : For a given integer t, if $d \geq t$, then there exist a ray $\arg z = \theta$ for which

$$\delta(P_j, \theta) > 0, \text{ for all } j = 0, 1, ..., n-1.$$

Therefore by Lemma 2, we have

$$\left| A_t e^{P_t(z)} \right| \geq \exp\{(1-\varepsilon)\,\delta(P_t, \theta)\, r^m\}. \tag{3.5}$$

Now equation (1.4) can be written as,

$$\begin{aligned} A_t e^{P_t(z)} f^{(t)}(z+\xi_t) &= [f^{(n)}(z+\xi_n) + A_{n-1} e^{P_{n-1}(z)} f^{(n-1)}(z+\xi_{n-1}) \\ &\quad + ... + A_{t-1} e^{P_{t-1}(z)} f^{(t-1)}(z+\xi_{t-1}) + A_{t+1} e^{P_{t+1}(z)} f^{(t+1)}(z+\xi_{t+1}) \\ &\quad + ... + A_0 e^{P_0(z)} f(z+\xi_0)] \end{aligned} \tag{3.6}$$

Hence combining (3.4), (3.5) and (3.6), we get

$$\begin{aligned} \exp\{(1-\varepsilon)\,\delta(P_t, \theta)\, r^m\} r^{d-t} &\leq r^{d-n} M_1 + \exp\{(1+\varepsilon)\,\delta(P_{n-1}, \theta)\, r^m\} r^{d-n+1} M_2 + ... \\ &\quad + \exp\{(1+\varepsilon)\,\delta(P_0, \theta)\, r^m\} r^d M_{n-1} \\ &\leq n r^d \exp\{(1+\varepsilon)\,\delta(P, \theta)\, r^m\} \max_{1 \leq j \leq n-1} M_j, \end{aligned} \tag{3.7}$$

where $\delta(P_t, \theta) > \delta(P, \theta) = \max_{0 \leq j \leq n-1, j \neq t} \delta(P_j, \theta)$ is a specified constant with regard to the polynomial coefficient.

Now choosing $0 < \varepsilon < \frac{\delta(P_t,\theta) - \delta(P,\theta)}{\delta(P_t,\theta) + \delta(P,\theta)}$, we have a contradiction with (3.7) as $r \to \infty$.

Case-4 : For a given integer t, if $d < t$, then there exist a ray $\arg z = \theta$ for which

$$\delta(P_0, \theta) > 0 \text{ and } \delta(P_j, \theta) < 0, \text{ for all } j = 1, 2, ..., n-1$$

holds.

Note that in this case equation (1.4) can be written as,

$$A_d e^{P_d(z)} f^{(d)}(z+\xi_d) + ... + A_1 e^{P_1(z)} f^{(1)}(z+\xi_1) + A_0 e^{P_0(z)} f(z+\xi_0) = 0.$$

Now using the same method discuss above, we can get

$$r^d \exp\{(1-\varepsilon)\,\delta(P_0, \theta)\, r^m\} M_{n-1} \leq d r^d \exp\{(1+\varepsilon)\,\delta(Q_0, \theta)\, r^m\} \max_{1 \leq j \leq t} M_j.$$

which give us a contradiction if we choose $0 < \varepsilon < \frac{\delta(P_0,\theta) - \delta(Q_0,\theta)}{\delta(P_0,\theta) + \delta(Q_0,\theta)}$.

Therefore we can conclude that $f(z)$ cannot be a polynomial. Hence all non-zero solutions f of equation (1.4) satisfy $\rho(f) \geq m+1$.

4. Statements and Declarations

Conflict of interest

Authors declares that we have no conflict of interest.

Funding

For the first author this research is supported by the Council of Scientific and Industrial Research (CSIR-HRDG), India, Grant No. 09/025(0255)/2018-EMR-I.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Data Availability

Not applicable.

5. Acknowledgement

The first author is thankful to the Council of Scientific and Industrial Research (CSIR-HRDG), India, Grant No. $09/025(0255)/2018-EMR-I$ for the financial support and the fourth author sincerely acknowledges the financial support rendered by DST-FIST $2024-2025$ running at the Department of Mathematics, University of Kalyani, P.O.: Kalyani, Dist: Nadia, Pin: 741235, West Bengal, India.

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