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SUMS INVOLVING A FAMILY OF GIBONACCI POLYNOMIAL SQUARES: GRAPH-THEORETIC CONFIRMATIONS

Abstract

We confirm a generalized sum of a family of gibonacci polynomial squares using graph-theoretic techniques, and its graph-theoretic and Pell consequences.

Key words: *Gibonacci Polynomial, Fibonacci Polynomial Pell Polynomials, Lukas Polynomial. Benet Formula. Graph- theoretic Confirmation.*

MSC2020: *Primary 11B37, 11B39, 11C08*

1 INTRODUCTION

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x), b(x), z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. They can also be defined by the *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 2].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively [2].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n; $b_n = p_n$ or q_n; $\Delta = \sqrt{x^2+4}$; and $2\alpha = x + \Delta$ [6].

It follows by the Binet-like formulas that $\lim_{m\to\infty} \dfrac{1}{g_{m+r}} = 0$ and $\lim_{m\to\infty} \dfrac{g_{m+r}}{g_m} = \alpha^r$.

1.1 Fundamental Gibonacci Identities

Gibonacci polynomials satisfy the following properties [2, 3, 4, 5, 6]:

$$g_{n+k}g_{n-k} - g_n^2 = \begin{cases} (-1)^{n+k+1}f_k^2, & \text{if } g_n = f_n \\ (-1)^{n+k}\Delta^2 f_k^2, & \text{otherwise;} \end{cases} \tag{1}$$

$$g_{n+k+r}g_{n-k} - g_{n+k}g_{n-k+r} = \begin{cases} (-1)^{n+k+1}f_r f_{2k}, & \text{if } g_n = f_n \\ (-1)^{n+k}\Delta^2 f_r f_{2k}, & \text{otherwise,} \end{cases} \tag{2}$$

where k and r are positive integers. These properties can be confirmed using the Binet-like formulas.

It follows from these two identities that

$$g_{(2pn+t)k}g_{(2pn+t-2p)k} - g_{(2pn+t-p)k}^2 = \begin{cases} (-1)^{tk+1}f_{pk}^2, & \text{if } g_n = f_n \\ (-1)^{tk}\Delta^2 f_{pk}^2, & \text{otherwise;} \end{cases} \tag{3}$$

$$g_{(2pn+t)k+r}g_{(2pn+t-2p)k} - g_{(2pn+t)k}g_{(2pn+t-2p)k+r} = \begin{cases} (-1)^{tk+1}f_r f_{2pk}, & \text{if } g_n = f_n \\ (-1)^{tk}\Delta^2 f_r f_{2pk}, & \text{otherwise,} \end{cases} \tag{4}$$

where k, p, r, and t are positive integers and $t \leq 2p$.

2 A TELESCOPING GIBONACCI SUM

Using recursion, we established the following telescoping gibonacci sum in [6]. In the interest of brevity, we omit its proof here.

Lemma 1. *Let k, p, r, t, and λ be positive integers, where $t \leq 2p$. Then*

$$\sum_{n=1}^{\infty}\left[\frac{g^{\lambda}_{(2pn+t-2p)k+r}}{g^{\lambda}_{(2pn+t-2p)k}} - \frac{g^{\lambda}_{(2pn+t)k+r}}{g^{\lambda}_{(2pn+t)k}}\right] = \frac{g^{\lambda}_{tk+r}}{g^{\lambda}_{tk}} - \alpha^{\lambda r}. \tag{5}$$

3 A FAMILY OF GIBONACCI SUMS

Using identities (3) and (4), Lemma 1 with $\lambda = 1$ played a major role in the development of the following theorem. To this end, in the interest of brevity, we let [6]:

$$\mu = \begin{cases} 1, & if\ g_n = f_n \\ \Delta^2, & otherwise; \end{cases} \quad \text{and} \quad \nu^* = \begin{cases} 1, & if\ g_n = f_n \\ -1, & otherwise. \end{cases}$$

These tools served as building blocks of the discourse, as the theorem shows [6].

Theorem 1. *Let k, p, r, and t be positive integers, where $t \leq 2p$. Then*

$$\sum_{n=1}^{\infty}\frac{(-1)^{tk}\mu\nu^* f_r f_{2pk}}{g^2_{(2pn+t-p)k} - (-1)^{tk}\mu\nu^* f^2_{pk}} = \frac{g_{tk+r}}{g_{tk}} - \alpha^r. \tag{6}$$

The objective of our discourse is to confirm this result using graph-theoretic techniques. To this end, first we present the needed tools.

4 GRAPH-THEORETIC TOOLS

Consider the *Fibonacci digraph* in Figure 1 with vertices v_1 and v_2, where a *weight* is assigned to each edge [2, 5]. It follows from its *weighted adjacency matrix* $Q = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$ that

$$Q^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix},$$

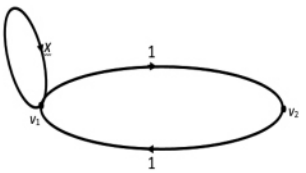


Figure 1: Weighted Fibonacci Digraph

where $n \geq 1$ [2, 3, 4, 5]. We extend the exponent n to 0, which is consistent with the *Cassini-like formula* $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$ [2, 5].

A *walk* from vertex v_i to vertex v_j is a sequence v_i-e_i-v_{i+1}-$\cdots$-v_{j-1}-e_{j-1}-v_j of vertices v_k and edges e_k, where edge e_k is incident with vertices v_k and v_{k+1}. The walk is *closed* if $v_i = v_j$; and *open*, otherwise. The *length* of a walk is the number of edges in the walk. The *weight* of a walk is the product of the weights of the edges along the walk.

The ijth entry of Q^n gives the sum of the weights of all walks of length n from v_i to v_j in the weighted digraph, where $1 \leq i, j \leq n$ [2, 3, 4]. Consequently, the sum of the weights of closed walks of length n originating at v_1 in the digraph is f_{n+1} and that of those originating at v_2 is f_{n-1}. So, the sum of the weights of all closed walks of length n in the digraph is $f_{n+1} + f_{n-1} = l_n$ [2, 5].

Let A and B denote sets of walks of varying lengths originating at a vertex v. Then the sum of the weights of the elements (a, b) in the *product set* $A \times B$ is *defined* as the product of the sums of weights from each component [3, 4]. This definition can be extended to any finite number of component sets. In particular, let A, B, C, and D denote the sets of walks of varying lengths originating at a vertex v, respectively. Then the sum of the weights of the elements (a, b, c, d) in the product set $A \times B \times C \times D$ is the product of the sums of weights from each component [3, 4].

We now make an interesting observation. Let $A = \{u\}$ and $B = \{v\}$, where u denotes the closed walk v_1-v_1 and v denotes the closed walk v_1-v_2-v_1. The weight of the element (u, u) in $A \times A$ is x^2, and that in $B \times B$ is 1. Consequently, the sum w of the weights of the elements in $C^* = (A \times A) \cup (B \times B) \cup (B \times B) \cup (B \times B) \cup (B \times B)$ is given by $w = x^2 + 4 = \Delta^2$.

These tools play a major role in the discourse. With them at our finger tips, we are now ready for pursuing the graph-theoretic confirmation.

5 GRAPH-THEORETIC CONFIRMATION

Let T_n^* denote the set of closed walks of length n in the digraph originating at v_1, and U_n^* the set of all closed walks of the same length n in the digraph. Correspondingly, let T_n denote the sum of the weights of all elements in T_n^*, and U_n that of those in U_n^*. Clearly, $T_n = f_{n+1}$ and $U_n = f_{n+1} + f_{n-1} = l_n$ [2, 5]. With this brief background, we now begin the proof of the gibonacci sum (6) in cases, where $k, p, r, t \geq 1$ and $t \leq 2p$.

Proof. Case 1. Suppose $g_n = f_n$. The sum of the weights of the elements in the product set $T_{r-1}^* \times T_{2pk-1}^*$ is $T_{r-1}T_{2pk-1} = f_r f_{2pk}$; the sum of those in $T_{(2pn+t-p)k-1}^* \times T_{(2pn+t-p)k-1}^*$ is $T_{(2pn+t-p)k-1}^2 = f_{(2pn+t-p)k}^2$; and that of those in $T_{pk-1}^* \times T_{pk-1}^*$ is $T_{pk-1}^2 = f_{pk}^2$.

Combining the two parts, we now let

$$\begin{aligned} S_n &= \frac{T_{r-1}T_{2pk-1}}{T_{(2pn+t-p)k}^2 - (-1)^k T_{pk-1}^2} \\ &= \frac{f_r f_{2pk}}{f_{(2pn+t-p)k}^2 - (-1)^k f_{pk}^2}. \end{aligned}$$

Using identities (3) and (4), and the lemma, this yields

$$\begin{aligned} \frac{(-1)^{tk} f_r f_{2pk}}{f_{(2pn+t-p)k}^2 - (-1)^{tk} f_{pk}^2} &= \frac{f_{(2pn+t)k} f_{(2pn+t-2p)k+r} - f_{(2pn+t)k+r} f_{(2pn+t-2p)k}}{f_{(2pn+t)k} f_{(2pn+t-2p)k}} \\ \sum_{n=1}^{\infty} \frac{(-1)^{tk} f_r f_{2pk}}{f_{(2pn+t-p)k}^2 - (-1)^{tk} f_{pk}^2} &= \sum_{n=1}^{\infty} \left[\frac{f_{(2pn+t-2p)k+r}}{f_{(2pn+t-2p)k}} - \frac{f_{(2pn+t)k+r}}{f_{(2pn+t)k}} \right] \\ &= \frac{f_{tk+r}}{f_{tk}} - \alpha^r. \end{aligned} \tag{7}$$

Now, we turn to the flip side.

Case 2. Let $g_n = l_n$. Recall that the sum w of the weights of the elements in $C^* = (A \times A) \cup (B \times B) \cup (B \times B) \cup (B \times B) \cup (B \times B)$ is given by $w = x^2 + 4 = \Delta^2$, and that of the elements in the product set $C^* \times T_{r-1}^* \times T_{2k-1}^*$ is given by $wT_{r-1}T_{2pk-1} = \Delta^2 f_r f_{2pk}$. The sum of the weights of the elements in the product set $U_{(2pn+t-p)k}^* \times U_{(2pn+t-p)k}^*$ is $U_{(2pn+t-p)k}^2 = l_{(2pn+t-p)k}^2$; and that of those in $T_{2pk-1}^* \times T_{pk-1}^*$ is $T_{2pk-1}^2 = f_{2pk}^2$.

As above, we now let

$$\begin{aligned} S_n &= \frac{wT_{r-1}T_{2pk-1}}{U_{(2pn+t-p)k}^2 + (-1)^k w T_{pk-1}^2} \\ &= \frac{\Delta^2 f_r f_{2pk}}{l_{(2pn+t-p)k}^2 + (-1)^k \Delta^2 f_{pk}^2}. \end{aligned}$$

It then follows by identities (3) and (4), and the lemma that

$$\begin{aligned} \frac{(-1)^{tk+1} \Delta^2 f_r f_{2pk}}{l_{(2pn+t-p)k}^2 + (-1)^{tk} \Delta^2 f_{pk}^2} &= \frac{l_{(2pn+t)k} l_{(2pn+t-2p)k+r} - l_{(2pn+t)k+r} l_{(2pn+t-2p)k}}{l_{(2pn+t)k} l_{(2pn+t-2p)k}} \\ \sum_{n=1}^{\infty} \frac{(-1)^{tk+1} \Delta^2 f_r f_{2pk}}{l_{(2pn+t-p)k}^2 + (-1)^{tk} \Delta^2 f_{pk}^2} &= \sum_{n=1}^{\infty} \left[\frac{l_{(2pn+t-2p)k+r}}{l_{(2pn+t-2p)k}} - \frac{l_{(2pn+t)k+r}}{l_{(2pn+t)k}} \right] \\ &= \frac{l_{tk+r}}{l_{tk}} - \alpha^r. \end{aligned} \tag{8}$$

This equation, coupled with equation (7), yields Theorem 1, as desired. □

For the curious-minded, we now add that equation (6) can be rewritten in terms of graph-theoretic tools. To this end, we define $T_0 = 1, U_0 = 2$; $H_n = T_n$ or U_n;

$$\mu = \begin{cases} 1 & if\ H_n = T_n \\ w & otherwise; \end{cases} \qquad \nu^* = \begin{cases} 1, & if\ H_n = T_n \\ -1, & otherwise; \end{cases} \quad \text{and} \quad \nu = \begin{cases} 1, & if\ H_n = T_n \\ 0, & otherwise. \end{cases}$$

With these new tools, and integers k, p, r, and t as before, we now present the graph-theoretic version of equation (6):

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk}\mu\nu^* T_{r-1}T_{2pk-1}}{H^2_{(2pn+t-p)k-\nu} - (-1)^{tk}\mu\nu^* T^2_{pk-1}} = \frac{H_{tk+r-\nu}}{H_{tk-\nu}} - \alpha^r. \tag{9}$$

Next, we turn to the Pell implications of the graph-theoretic techniques.

6 PELL CONSEQUENCE

With the gibonacci-Pell relationship $b_n(x) = g_n(2x)$, we can construct the graph-theoretic proof of the Pell version of Theorem 1 independently by changing the weight of the loop at v_1 from x to $2x$. We encourage gibonacci enthusiasts to explore this route.

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