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MODULO TWO SQUARE MEAN LABELING OF HURDLE GRAPHS, COMB GRAPHS AND $F(P_n)$ GRAPHS AND ITS ALGORITHMS

Abstract: A graph is said to be modulo two square mean labeling, if there is a function ∅ from the vertex set of G to $\{1,2,\ldots n\}$to the edge set of G to {1}where$\emptyset'(uv) = \left\lceil \frac{f(u)^2+f(v)^2}{2} \right\rceil mod\ 2$. A graph that admits modulo two square mean labeling is called modulo two square mean graphs. In this paper we provethat hurdle graphs, comb graphs and $F(P_n)$graphs are modulo two square mean labeling and its algorithms.

Key words: MeanLabeling, Root Mean Square Labeling, Hurdle graph, comb graph.

Mathematics Subject Classification 2020 : 05C78and 05C85.

1 Introduction

Let$G\ (V, E)$ be a graph with p vertices and q edges. A graph labeling is an assignment of integers to the vertices or edges, or both,subject to certain conditions. Most graph labeling methods trace their origin to one introduced by Rosa [7] in 1967, or one given by Graham and Sloane [2] in 1980. Rosa [7] called a function f a β-valuation of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1 \ldots q\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct.Variouslabeling methods have been introduced by Acharya, Arumugam and Rosa [1].

The concept of mean labeling has been introduced by S.Somasundaram and R.Ponraj[5] in 2004. A graph G with p vertices and q edges is called a mean graph if there is an injective function f from the vertices of G to $\{0, 1, 2, \ldots, q\}$ such that when each edge uv is labeled with $(f(u) + f(v))/2$ if $f(u) + f(v)$ is even, and $(f(u) + f(v) + 1)/2$ if $f(u) + f(v)$ is odd, then the resulting edge labels are distinct.

V.Ramachandran and C.Sekar [6] introduced the concept of one modulo N graceful where N is any positive integer. In the case of N = 2, the labeling is odd graceful and In the case of N = 2the labeling is graceful. Motivated by 0- Edge Magic Labeling [4] and 1- Edge Magic Labeling [3] we introduce modulo two square mean labeling of hurdle graphs, comb graphs and $F(P_n)$ graphs and its algorithms.

Definition1.1: A walk in which $u_1, u_2 \ldots u_n$ are distinct is called a path. A path on n vertices is denoted by P_n

Definition1.2: The graph obtained by joining a single pendent edge to each vertex of a path is called as Comb.

Definition1.3: A graph $G = (V, E)$ with p vertices and q edges is said to be a Root Square Mean graph if it is possible to label the vertices $x \in V$ with distinct elements $f(x)$ from $1,2, \ldots, q+1$ in such away that when each edge $e = uv$ is labeled with $f(e = uv) = \left\lceil \sqrt{\frac{f(u)^2 + f(v)^2}{2}} \right\rceil$ or $\left\lfloor \sqrt{\frac{f(u)^2 + f(v)^2}{2}} \right\rfloor$, then the resulting edge labels are distinct. In this case f is called a Root Square Mean labeling of G.

Definition 1.4 A graph obtained from a path P_n by attaching a pendant edge to every internal vertexof the path is called Hurdle graph. It is denoted by Hd_n and has n − 2 hurdles.

Definition 1.5 A Y– tree is a graph obtained from a path by appending an edge to a vertex of a path adjacent to an end point and is denoted by Y_n, where n is the number of vertices in the tree.

Definitions 1.6 A F-tree $F(P_n)$ is a graph obtained from a path on $n \geq 3$ vertices by appending two pendent edges one to an initial vertex and other vertex is adjacent to adjacent to initial vertex.

2 Main Results

Theorem 2.1 The Hurdle Graph Hd_n is a modulo two square mean labeling.

Proof:

Let $G = Hd_n$ be a graph with n vertices and n-1 edges.

Now We Define The Vertices Of G is as follow as $u_1, u_2, u_3, \ldots$ be the vertices of path P_n, and let $v_1, v_2, v_3, \ldots$ be the vertices of pendant edges attached to the path P_n.

Let $V(G) = \{u_k : 1 \leq k \leq n\} \cup \{v_k : 1 \leq k \leq n-2\}$ and

$E(G) = \{u_k u_{k+1} : 1 \leq k \leq n-1\} \cup \{v_k\ u_{k+1} : 1 \leq k \leq n-2\}$.

Then G has $2n - 2$ vertices and $2n - 3$ edges.

Define $f : V(G) \to \{1,2,3, \ldots, 2n-2\}$ as follows

$f(u_k) = 2k - 1, \quad 1 \leq k \leq n-1.$

$f(u_n) = 2n - 2$

$f(v_k) = 2k\,, \qquad 1 \le k \le n-2.$

Now we define edge function as follows.

$f(u_k u_{k+1}) = \left\lfloor \frac{f(u_k)^2 + f(u_{k+1})^2}{2} \right\rfloor \; mod\; 2\,,\; 1 \le k \le n-1.$

$f(v_k u_{k+1}) = \left\lfloor \frac{f(v_k)^2 + f(u_{k+1})^2}{2} \right\rfloor \; mod\; 2\,, 1 \le k \le n-2$

Thus, the graph G has a modulo two square mean labeling.

Example 2.2 modulo two square mean labeling of Hurdle Graph Hd_5 is shown in Fig 2.1

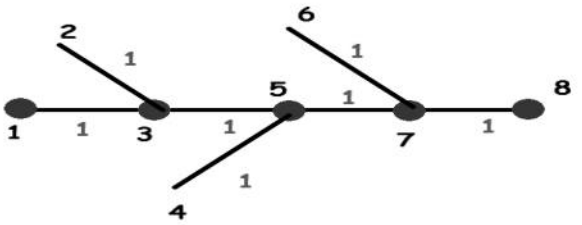


Figure 2.1

2.3. Algorithm Description

Problem Statement:

Development of a C++ algorithm for hurdle graphs using modulo-2 square mean labeling

Input:

- The Number of Vertices

Output:

- The Labeling of Vertices of path are u(1), u(2),…… and Labeling ofVertices of pendant edges are v(1), v(2),……
- The Labeling of edgese(1), e(2)…….. and its Modulo of edges of e(1)=1, e(2)=1……..

Development Process

The C++ algorithm for Hurdle graphs employs modulo-2 square mean labeling

Code Implementation:

```
#include <iostream>
#include <cmath>
int main()
```

```
{
int n,i;
int v[100],u[100];
int e1[100],e2[100],e3[100],e4[100];
std::cout<< "Enter the Number of Vertices = ";
std::cin>>n;
for (i=1;i<=n-1;i++)
{
u[i]=2*i-1;
std::cout<< " \n The Number of Vertices of u ("<<i<<") ="<<u[i];
}
u[n]=2*n-2;
std::cout<< "\n The Number of Vertices of u ("<<n<<") ="<<u[n];
std::cout<< "\n";
for (i=1;i<=n-2;i++)
{
v[i]=2*i;
std::cout<<" \n The Number of Vertices of v ("<<i<<") ="<<v[i];
}
std::cout<< "\n";
for (i=1; i<=n-2; i++)
{
e1[i]=(u[i]*u[i])+(u[(i+1)]*u[(i+1)]);
e2[i]= ceil(  float (e1[i])/2);
std::cout<<"\n Edges of e ("<<i<<") ="<<e2[i] <<"\t";
std::cout<<"\t Edges of e ("<<i<<") ="<<e2[i] %2<<"\t";
}
e1[n-1]=(u[n]*u[n])+(u[(n-1)]*u[(n-1)]);
e2[n-1]= ceil(  float (e1[n-1])/2);
std::cout<<"\n Edges of e ("<<i<<") ="<<e2[n-1] <<"\t";
std::cout<<"\t Edges of e ("<<i<<") ="<<e2[n-1] %2<<"\t";
for (i=1; i<=n-2; i++)
{
e3[i]=(v[i]*v[i])+(u[(i+1)]*u[(i+1)]);
e4[i]= ceil(  float (e3[i])/2);
std::cout<<"\n Edges of e("<<i+n-1<<")="<<e4[i] <<"\t";
std::cout<<"\t Edges of e("<<i+n-1<<")="<<e4[i] %2<<"\t";
}
return 0;
}
```

Output: The Output of Hurdle Graph H$d_5$has a modulo two square mean labeling.

Output Clear

```
Enter the Number of Vertices = 5

 The Number of Vertices of u (1) =1
 The Number of Vertices of u (2) =3
 The Number of Vertices of u (3) =5
 The Number of Vertices of u (4) =7
 The Number of Vertices of u (5) =8

 The Number of Vertices of v (1) =2
 The Number of Vertices of v (2) =4
 The Number of Vertices of v (3) =6

 Edges of e (1) =5       Edges of e (1) =1
 Edges of e (2) =17      Edges of e (2) =1
 Edges of e (3) =37      Edges of e (3) =1
 Edges of e (4) =57      Edges of e (4) =1
 Edges of e(5)=7         Edges of e(5)=1
 Edges of e(6)=21        Edges of e(6)=1
 Edges of e(7)=43        Edges of e(7)=1

=== Code Execution Successful ===
```

Explanation:

The algorithm begins by initializing two sets of vertices: one representing the path and the other corresponding to the pendant edges, which together form the labeling structure of the hurdle graph. The output includes the labeled edges of the hurdle graph and its modulo result are equal to 1. This labeling approach ensures greater efficiency and consistency compared to traditional methods.

Theorem 2.4 The comb Graph ($P_n \odot K_1$) is a modulo two square mean labeling.

Proof:

Let $G = P_n \odot K_1$ be a graph with n vertices and n edges.

Now We Define the Vertices of $V(G)$ is as follow as

$u_1, u_2, u_3, \ldots$ as the vertices of path P_n, and let $v_1, v_2, v_3, \ldots$ be the vertices of K_1 attached to the path P_n.

Let $V(G) = \{u_k : 1 \le k \le n\} \cup \{v_k : 1 \le k \le n\}$ and

$E(G) = \{u_k u_{k+1} : 1 \le k \le n-1\} \cup \{v_k u_{k+1} : 1 \le k \le n-1\}$.

Then G has 2n vertices and $2n - 1$ edges.

Define $f : V(G) \to \{1,2,3, \ldots, 2n\}$ as follows

$f(u_k) = 2k - 1, \quad 1 \le k \le n.$

$f(v_k) = 2k, \qquad 1 \le k \le n.$

Now we define edge function as follows.

$f(u_k u_{k+1}) = \left\lfloor \frac{f(u_k)^2 + f(u_{k+1})^2}{2} \right\rfloor \ mod\ 2\ , 1 \le k \le n-1.$

$f(v_k u_k) = \left\lfloor \frac{f(v_k)^2 + f(u_k)^2}{2} \right\rfloor \ mod\ 2, 1 \le k \le n$

Hence, thecomb Graph $(P_n \odot K_1)$ admits modulotwosquare mean labeling.

Example 2.5 modulo two square mean labeling of comb Graph $(P_5 \odot K_1)$ is shown in Fig 2.2

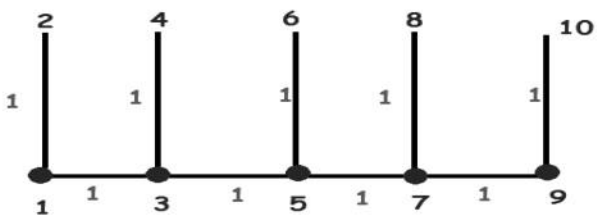


Figure 2.2

2.6. Algorithm Description

Problem Statement:

Development of C++ Algorithm forcomb Graph $(P_n \odot K_1)$ using modulo-2 square mean labeling

Input:

- The Number of Vertices

Output:

- The Labeling of Vertices of path are u(1), u(2),…… and Labeling ofVertices of pendant edges are v(1), v(2),……

- The Labeling of edgese(1), e(2)..... and its Modulo of edges of e(1)=1, e(2)=1.....

Development Process

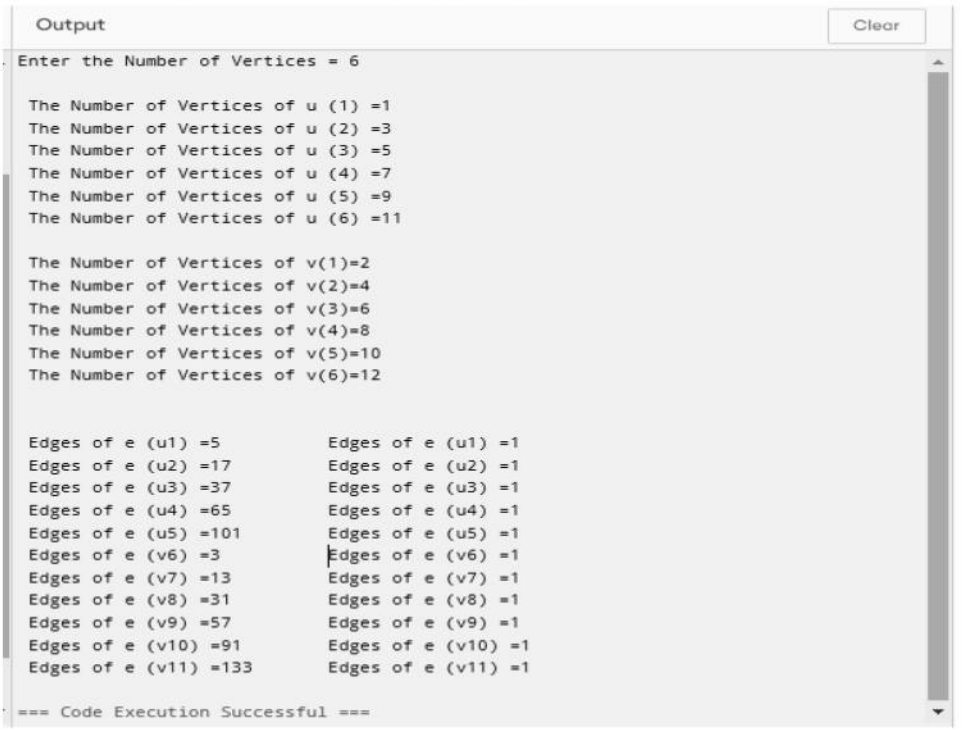
The C++ Algorithm forcomb Graph ($P_n \odot K_1$) employs modulo-2 square mean labeling.

Code Implementation:

```
#include <iostream>
#include <cmath>
int main()
{
int n,i;
int v[100],u[100];
int e1[100],e2[100],e3[100],e4[100];
std::cout<< "Enter the Number of Vertices = ";
std::cin>>n;
for (i=1;i<=n;i++)
{
u[i]=2*i-1;
std::cout<< " \n The Number of Vertices of u ("<<i<<") ="<<u[i];
}
std::cout<< "\n";
for (i=1;i<=n;i++)
{
v[i]=2*i;
std::cout<<" \n The Number of Vertices of v("<<i<<")="<<v[i];
}
std::cout<< "\n";
std::cout<<"\n";
for (i=1; i<=n-1; i++)
{
e1[i]=(u[i]*u[i])+(u[(i+1)]*u[(i+1)]);
e2[i]= ceil(  float (e1[i])/2);
std::cout<<"\n Edges of e (u"<<i<<") ="<<e2[i] <<"\t";
std::cout<<"\t Edges of e (u"<<i<<") ="<<e2[i] %2<<"\t";
}
for (i=1; i<=n; i++)
{
e3[i]=(v[i]*v[i])+(u[(i)]*u[(i)]);
e4[i]= ceil(  float (e3[i])/2);
std::cout<<"\n Edges of e (v"<<i+n-1<<") ="<<e4[i] <<"\t";
std::cout<<"\t Edges of e (v"<<i+n-1<<") ="<<e4[i] %2<<"\t";
}
```

```
return 0;
}
```

Output: The Output of the comb Graph ($P_n \odot K_1$)has a modulo two square mean labeling.



```
Output                                                    Clear

Enter the Number of Vertices = 6

 The Number of Vertices of u (1) =1
 The Number of Vertices of u (2) =3
 The Number of Vertices of u (3) =5
 The Number of Vertices of u (4) =7
 The Number of Vertices of u (5) =9
 The Number of Vertices of u (6) =11

 The Number of Vertices of v(1)=2
 The Number of Vertices of v(2)=4
 The Number of Vertices of v(3)=6
 The Number of Vertices of v(4)=8
 The Number of Vertices of v(5)=10
 The Number of Vertices of v(6)=12


 Edges of e (u1) =5          Edges of e (u1) =1
 Edges of e (u2) =17         Edges of e (u2) =1
 Edges of e (u3) =37         Edges of e (u3) =1
 Edges of e (u4) =65         Edges of e (u4) =1
 Edges of e (u5) =101        Edges of e (u5) =1
 Edges of e (v6) =3          Edges of e (v6) =1
 Edges of e (v7) =13         Edges of e (v7) =1
 Edges of e (v8) =31         Edges of e (v8) =1
 Edges of e (v9) =57         Edges of e (v9) =1
 Edges of e (v10) =91        Edges of e (v10) =1
 Edges of e (v11) =133       Edges of e (v11) =1

=== Code Execution Successful ===
```

Explanation:

The algorithm begins by initializing two sets of vertices: one representing the path and the other corresponding to the pendant edges, which together form the labeling structure of the **comb Graph ($P_n \odot K_1$)**. The output includes the labeled edges of the **comb Graph ($P_n \odot K_1$)** and its modulo result are equal to 1. This labeling approach ensures greater efficiency and consistency compared to traditional methods.

Theorem 2.7 The F-Tree $F(P_n)$ hasa modulotwo square mean labeling.

Proof: Let $G = F(P_n)$ be a graph.

$V(G) = \{v_k : 1 \leq k \leq n\} \cup \{u, w\}$ and

$E(G) = \{(v_k v_{k+1}): 1 \leq k \leq n-1\} \cup \{(v_1 u)\} \cup \{(v_2 w)\}$.

So, $|V(G)| = n + 2$ & $|E(G)| = n + 1$.

Define $f: V(G) \to \{1,2,3,\ldots,n+2\}$as follows

Now $f(v_1) = 1$

$f(v_2) = 4$

$f(v_{2+k}) = 4 + k \quad 1 \leq k \leq n-2.$

$f(u) = 2$

$f(w) = 3$

Now we define edge function as follows

$$f(v_k v_{k+1}) = \left\lfloor \frac{f(v_k)^2 + f(v_{k+1})^2}{2} \right\rfloor Mod\ 2, \quad 1 \leq k \leq n-1$$

$$f(v_1 u) = \left\lfloor \frac{f(v_1)^2 + f(u)^2}{2} \right\rfloor Mod\ 2$$

$$f(v_2 w) = \left\lfloor \frac{f(v_2)^2 + f(w)^2}{2} \right\rfloor Mod\ 2$$

Hence $F(P_n)$ hasa modulo two square mean labeling.

Example 2.8 modulo two square mean labeling of F-Tree $F(P_5)$ is shown in Fig 2.3

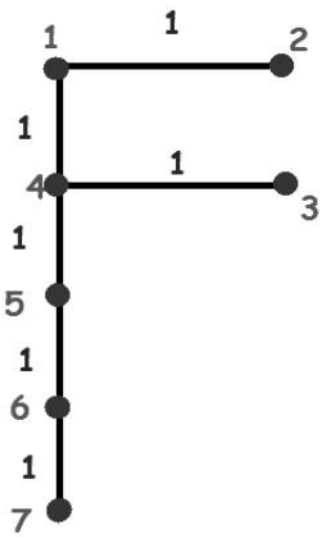


Figure 2.3

2.9 . Algorithm Description

Problem Statement:

Development of C++ Algorithm for F-Tree $F(P_n)$using modulo-2 square mean labeling

Input:

- The Number of Vertices

Output:

- The Labeling of Vertices of v(1),v(2), are assign asv(1)=2,v(2)=3, and Labeling ofVertices of path edges are u(1), u(2),……
- The Labeling of edgese(1), e(2)…… and its Modulo of edges of e(1)=1, e(2)=1……

Development Process

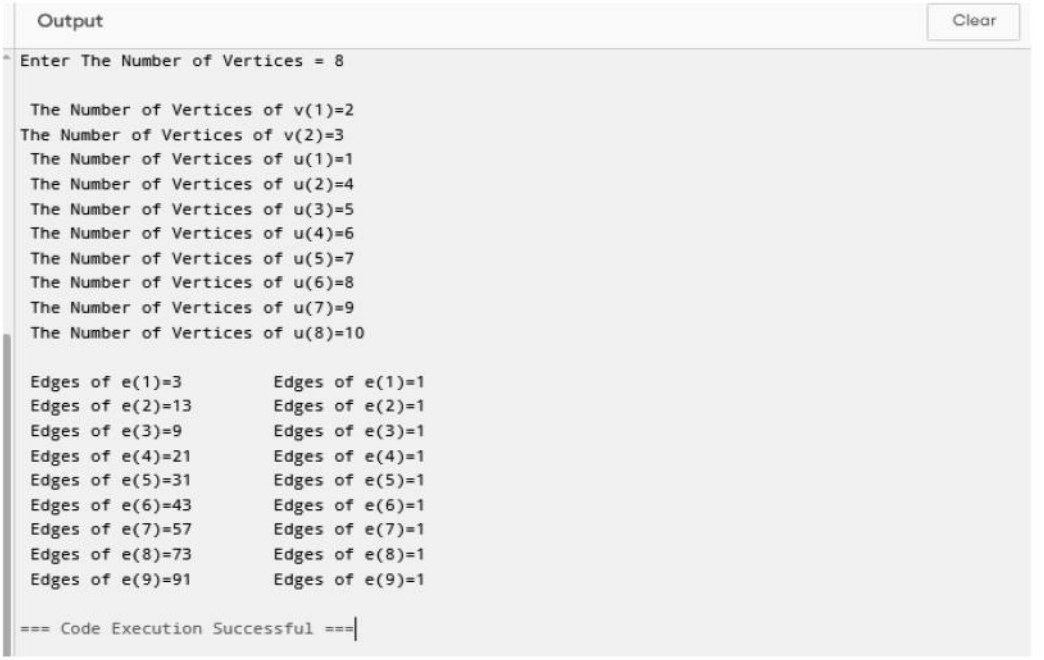
The C++ Algorithm forF-Tree $F(P_n)$employs modulo-2 square mean labeling.

Code Implementation:

```
#include <iostream>
#include <cmath>
int main( )
{
int n,i;
int v[100],u[100];
int e1[100],e2[100];
u[1]=1;
u[2]=4;
v[1]=2;
v[2]=3;
std::cout<< "Enter the Number of Vertices = ";
std::cin>>n;
std::cout<< " \n The Number of Vertices of v (1) ="<<v[1];
std::cout<< " \nThe Number of Vertices of v (2) ="<<v[2];
std::cout<< " \n The Number of Vertices of u (1) ="<<u[1];
std::cout<< " \n The Number of Vertices of u (2) ="<<u[2];
```

```
for (i=1;i<=n-2;i++)
{
u[2+i]=(i+ 4);
std::cout<< " \n The Number of Vertices of u ("<<2+i<<") ="<<u[2+i];
}
std::cout<<"\n";
e1[1]=(u[1]*u[1])+(v[(1)]*v[(1)]);
e2[1]= ceil(  float (e1[1])/2);
std::cout<<"\n Edges of e (1) ="<<e2[1] <<"\t";
std::cout<<"\t Edges of e (1) ="<<e2[1] %2<<"\t";
e1[2]=(u[2]*u[2])+(v[(2)]*v[(2)]);
e2[2]= ceil(  float (e1[2])/2);
std::cout<<"\n Edges of e (2) ="<<e2[2] <<"\t";
std::cout<<"\t Edges of e (2) ="<<e2[2] %2<<"\t";
for (i=1; i<=n-1; i++)
{
e1[i+2]=(u[i]*u[i])+(u[(i+1)]*u[(i+1)]);
e2[i+2]= ceil(  float (e1[i+2])/2);
std::cout<<"\n Edges of e ("<<i+2<<") ="<<e2[i+2] <<"\t";
std::cout<<"\t Edges of e ("<<i+2<<") ="<<e2[i+2] %2<<"\t";
}
return 0;
}
```

OUTPUT: The Output of F-Tree $F(P_8)$has a modulo two square mean labeling.



Explanation:

The algorithm begins by initializing two sets of vertices: one representing the path and the other corresponding to the pendant edges, which together form the labeling structure of the F-Tree $F(P_n)$. The output includes the labeled edges of the F-Tree $F(P_n)$ and its modulo result are equal to 1. This labeling approach ensures greater efficiency and consistency compared to traditional methods.

3 Conclusion

In this paper we have investigated some pathrelated graphslike Hurdle Graph, comb graph($P_n \odot K_1$),F-Tree $F(P_n)$.We have provided C++ algorithm for the theorems.

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