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RESTRAINED PENDANT DOMINATION IN GRAPHS

Abstract: In this article we initiate the study of a variation of standard domination, namely restrained pendant domination. Let $G = (V, E)$ be any graph. A dominating set S in G is called a pendant dominating set if $< S >$ contains at least one pendant node. The minimum cardinality of the pendant dominating set in G is called the pendant domination number of G, denoted by $\gamma_{pe}(G)$. A restrained pendant dominating set is a set $S \subseteq V$ where every node in $V - S$ is adjacent to a node in S as well as another node in $V - S$. The restrained pendant domination number of G, denoted by γ_{rpe}, is the minimum cardinality of a restrained pendant dominating set of G. We determine best possible upper and lower bounds for $\gamma_{rpe}(G)$ and also calculated the exact value for some standard families graphs.

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Keywords and Phrases: Dominating Set, Restrained Dominating Set, Pendant Dominating Set, Restrained Pendant Dominating Set, Pendant Domination Number

1 Introduction

A possible application for minimal restrained dominating sets is any situation in which one group needs to supervise a subordinate group using the minimal number of supervisors possible, but at the same time ensure the supervisors are held accountable by never allowing a supervisor to be alone with subordinate. Hattingh gives an example of this relationship in terms of guards and prisoners. The nodes in S are the guards, while the nodes in $V - S$ are the prisoners. In this way, a guard can supervise every prisoner, but every prisoner is also in view of another prisoner [5]. In application way of restrained pendant dominating set we assign a back up to at least one prisoner. A possible application for independent restrained dominating sets is the location of product distribution centers or hospitals where a certain level of redundancy is desired. In this case, nodes could represent cities. Vertices in S represent cities with a distribution center and edges represent transportation routes between cities. Selecting the cities in which to place distribution centers using an independent restrained dominating set guarantees that every city without a distribution center is at least next to a city with one. It also guarantees that every city without a distribution center has a neighbor that also lacks a distribution center. In case of shortages at one distribution center, every city has access to a different center by going through one of its neighbors. The helm H_n is the graph obtained from wheel W_n by attaching a pendant edge to each rim node. The closed helm CH_n is the graph obtained from helm H_n by joining each pendant node to form a cycle. The web graph $W(t, n)$ is the graph obtained by joining the pendant nodes of a helm to form a cycle and then adding a single pendant edge to each node of this outer cycle. $W(t, n)$ is the generalized web with t cycles each of order n. A firrecracker graph $F_{n,k}$ is formed by

concatenation of n $k-$stars by linking one pendant node of a star to pendant node of next star. $F_{n,k}$ is isomorphic to centepede graph $k = 2$

A dominating set S in G is called a pendant dominating set if $< S >$ contains at least one pendant node. The minimum cardinality of the pendant dominating set in G is called the pendant domination number of G, denoted by $\gamma_{pe}(G)$. A restrained pendant dominating set is a set $S \subseteq V$ where every node in $V - S$ is adjacent to a node in S as well as another node in $V - S$. The restrained pendant domination number of G, denoted by γ_{rpe} is the minimum cardinality of a restrained pendant dominating set of G. For more details about pendant domination parameter refer [6,7,8]

For an example in Figure 1, $S = \{a, b\}$ is a minimum pendant dominating set and the set S is itself a restrained pendant dominating set. Hence $\gamma_{rpe}(G) = 2$.

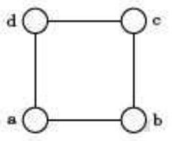


Figure 1: Cycle Graph

Proposition 1.1. [6] Let G be a cycle or a path with n vertices. Then

$$\gamma_{pe}(G) = \begin{cases} \frac{n}{3} + 1, & if\ n \equiv 0(mod3); \\ \left\lceil \frac{n}{3} \right\rceil, & if\ n \equiv 1(mod3); \\ \left\lceil \frac{n}{3} \right\rceil + 1, & if\ n \equiv 2(mod3); \end{cases}$$

Proposition 1.2. For the wheel W_n for $n \geq 3$ $\gamma_r(W_n) = 1$.

Proposition 1.3. If $n \geq 3$ is a positive integer, then $\gamma_{rpe}(K_n) = 2$.

Proposition 1.4. If $n \geq 3$, then $\gamma_{rpe}(K_{1,n}) = n$.

Proposition 1.5. If m and n are integers such that $m, n \geq 2$ then $\gamma_{rpe}(K_{m,n}) = 2$.

Theorem 1.1. Let P_n be a path on n nodes for $n > 4$ and $k \geq 1$ be an integer. Then

$$\gamma_{rpe}(P_n) = \begin{cases} k + 3, & if\ n = 7. \\ k + 2, & if\ n = 3k, n = 3k + 1\ or\ 3k + 2. \end{cases}$$

Proof. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ be a set of nodes of the path graph. Note that the set $S = \{v_1, v_2, v_5, v_6, v_7\}$ will be the restrained pendant dominating set of path graph P_7. Therefore, $\gamma_{rpe}(P_7) = k + 3$.

Suppose $n = 3k$. Note that $S = \{v_1, v_{3i-1}: 0 \leq i \leq k - 1\} \cup \{v_{3k}\}$ is a minimum restrained pendant dominating set with $k + 2$ elements. Thus $\gamma_{rpe}(P_n) \leq k + 2$. Let S be any minimum restrained pendant dominating set of P_n. Then S must be a pendant dominating set. Note that each node in S can dominate a maximum of 3 nodes. Thus to dominate $3k + 1$ nodes, S must have $k + 1$ nodes. Hence to dominate the

remaining one node of P_n, S must have one more node and so $| S | \geq k+2$. Therefore $\gamma_{rpe}(P_n) = k+2$.

Theorem 1.2. Let C_n be a cycle with $n \geq 4$ nodes. Then

$$\gamma_{rpe}(C_n) = \begin{cases} \frac{n}{3}+2, & if\ n \equiv 0(mod3); \\ n - \left\lfloor \frac{n}{3} \right\rfloor, & if\ n \equiv 1(mod3)\ ; \\ n - \left\lfloor \frac{n}{3} \right\rfloor, & if\ n \equiv 2(mod3); \end{cases}$$

Illustration: The helm graph H_4 is shown in figure below where the set of solid nodes is its restrained pendant dominating set of minimum cardinality

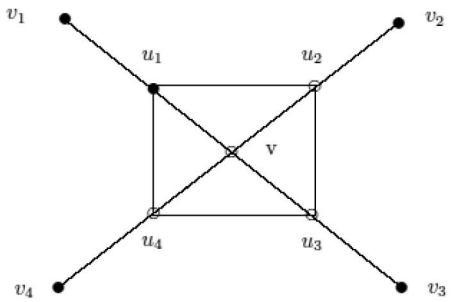


Figure 2. Helm Graph H_4

Theorem 1.3. Let $G \cong H_n$ be a helm graph with $n \geq 3$, then $\gamma_{rpe}(G) = n+1$

Proof. Let $\{v_1, v_2, v_3, \ldots, v_n\}$ be the nodes of degree one, $\{u_1, u_2, \ldots, u_n\}$ be the rim nodes and v be the apex node of maximum degree n of helm graph G with $| V(H_n) | = 2n+1$. The pendant $v_1, v_2, \ldots, v_n$ are mutually non-adjacent which must be in every restrained pendant dominating set. Further the pendant nodes $v_1, v_2, \ldots, v_n$ of H_n are dominate the rim nodes $u_1, u_2, \ldots, u_n$ of H_n. A restrained pendant dominating set S should contain pendant nodes $v_1, v_2, \ldots, v_n$ and the rim node u_1 of H_n. If $u_0 \in S$ the induced subgraph of H_n doesnot contain the pendant node. Therefore, $u_1 \in S$ which implies that $| S | = n+1$. Note that $S = \{v_1, v_2, \ldots, v_n, u_1\}$ is a restrained pendant dominating set with minimum cardinality because removal of any one of the nodes from set S will not dominate all the nodes of H_n. Moreover each node $V - S$ is adjacent to node in S and to a node in $V - S$. Hence $\gamma_{rpe}(H_n) = n+1$.

Theorem 1.4. For the closed helm graph CH_n,for $n \geq 3$

$$\gamma_{rpe}(CH_n) = \begin{cases} 3, & if\ n = 4 \\ \left\lceil \frac{n}{3} \right\rceil + 1, & if\ \ n = 3k+1\ or\ 3k+2 \end{cases}$$

Illustration: The closed helm graph with CH_9 as shown below

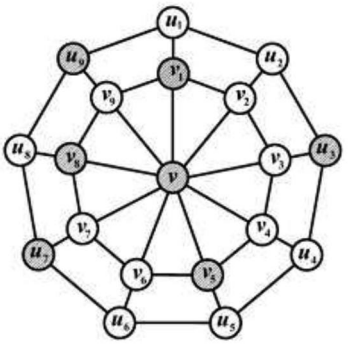


Figure 3. Closed Helm Graph CH_9

Proof. The closed helm CH_5 contains whell graph W_n and outer cycle C_n. Let v indicate the apex of wheel. Also $u_1, u_2, \ldots, u_n$ are the rim nodes of wheel W_n of CH_n and $v_1, v_2, \ldots, v_n$ be the corresponding adjacent nodes of outer cycle of CH_n. So $| V(CH_n) | = 2n + 1$. For $n = 4$, $\Delta(CH_4) = 4$ and $| V(CH_4) | = 9$. It follows that minimum three nodes are required to dominate all the nodes of CH_4. If $S \subseteq V(CH_4)$ is restrained pendant dominating set and induced subgraph of S contains a pendant node, then $| S | = 3$, which is minimum. Therefore $\gamma_{rpe}(CH_4) = 3$.

For $n \geq 4$, $\Delta(CH_4) = n$ and $| V(CH_4) | = n$ and by proposition 1.1. apex node dominates all the nodes of W_n. If $S \subseteq V(CH_n)$ is a restrained pendant dominating set then the set must contain a apex node v. Due to pendant node in induced subgraph of S and adjacency nature of nodes $v_1, v_2, \ldots, v_n$ of outer cycle with corresponding nodes $u_1, u_2, \ldots, u_n$ of W_n. By proposition.1.2. at least nodes are required to dominate all the remaining nodes of outer cycle of CH_n. It follows that $| S | = \left\lceil \frac{n}{3} \right\rceil + 1$.

If possible suppose S' is a restrained pendant dominating set such that $| S' | > | S' | = \left\lceil \frac{n}{3} \right\rceil < | S |$. Now $\Delta(CH_n) = n$ and in order to attain the least cardinality, S' can not contain the nodes where each node among them can dominate distinct n nodes of CH_n. Therefore S' can not be a restrained pendant dominating set of CH_n. This implies that S is a restrained pendant dominating set with minimum cardinality for CH_n. Hence $\gamma_{rpe}(CH_n) = | S | = \left\lceil \frac{n}{3} \right\rceil + 1$.

Theorem 1.5. If G is the Firecracker graph $F_{n,k}$, then

$$\gamma_{rpe}(G) = \begin{cases} n+1, & if\ k = 2 \\ n(k-1), & if\ n, k > 2 \end{cases}$$

Proof. We prove the theorem by following two cases

Case 1. $K = 2$. The firecracker graph is isomorphic to centepede graph, which has n pendant nodes. The set $S = \{v_1, v_2, \ldots, v_n\}$ forms a minimum dominating set and hence $\gamma(G) = n$. Clearly $\gamma_{rpe}(G) \geq n$. But the subgraph induced by S doesn't contains pendant node. Therefore the restrained pendant dominating set of G is

obtained by taking the neighbor of the node v_1 or v_n along with the dominating set. Hence $\gamma_{rpe}(G) = | S | + N(v_1)\ or\ N(v_n) = n + 1$.

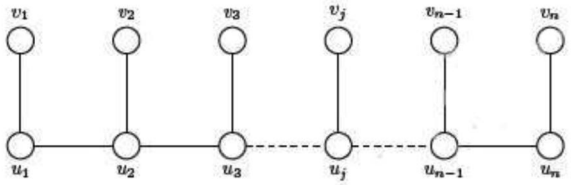


Figure 4. Centepede Graph

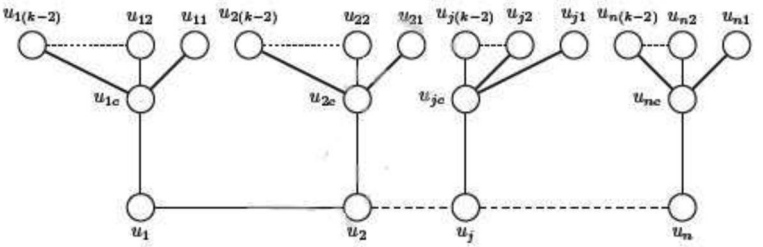


Figure 5. Firecracker Graph

Case 2. $K > 2$

The total number of pendant nodes of G is $n(k-2)$. Then $\gamma_{rpe}(G) \geq n(k-2)$. Let $A = \{u_1, u_2, \ldots, u_{i(k-2)} \ / \ 1 \leq i \leq n\}$ be the set of pendant nodes of the graph G and $B = \{u_{1c}, u_{2c}, \ldots, u_{nc}\}$ be the central nodes of the star. Clearly the set B is a dominating set with minimum cardinality but not a pendant dominating set. Define $C = A \cup B$. forms a minimum pendant dominating set and hence $\gamma_{pe}(G) = n(k-2) + n$. Therefore $\gamma_{rpe}(G) \geq n(k-1)$. But the subgraph induced by C contains a pendant node and every node in $V - C$ is adjacent to a node in C as well as another node in $V - C$. Hence, C is restrained pendant dominating set also.Therefore $\gamma_{rpe}(G) = n(k-1)$.

Theorem 1.6. Let G be a sunflower graph Sf_n $(n \geq 3)$, then $\gamma_{rpe}(Sf_n) = n + 1$

Proof. Let the node v be the apex node of the Sf_n and $v_2, v_4, \ldots, v_{2i}$ for $i = 1,2,3\ldots, n$ be the nodes of degree one in sunflower graph Sf_n with $| Sf_n | = 3n + 1$. The apex node v of sunflower graph dominates $2n$ distinct nodes and nodes of degree one are mutually adjacent. Its is easy to see that any restrained pendant dominating set contain the apex node and pendant nodes and induced subgraph of S contains a pendant node. Therefore $| S | = n + 1$, which is the least cardinality. Hence $\gamma_{ref}(Sf_n) = n + 1$.

Illustration: The sunflower Sf_4 is shown in the below figure where the set of solid nodes $\{v, v_2, v_4, v_6, v_8\}$ is its restrained pendant dominating set with least cardinality.

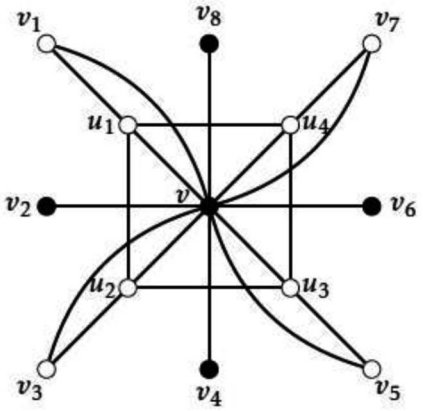


Figure 6. Sunflower Graph Sf_4

Proposition 1.6. Let G be a connected graph of order n. Then $\gamma_{rpe}(G) = n$ if and only if G is a star.

Proof. If G is a connected graph of order n and G is not a star then $\gamma_{rpe} \leq n - 2$. Recall that a leaf in a graph is a node of degree one, while a stem is a node adjacent to a leaf. The next two results will show for which graphs this upper bound is attained.

Theorem 1.7. If T is a tree of order $n \geq 4$ then $\gamma_{rpe}(T) = n - 2$ if and only if T is obtained from P_5 P_6 *or* P_7 by adding zero or more leaves to the stems of the path.

Proof. It is easy to verify that if T is obtained from P_5 or P_6, then $\gamma_{rpe}(G) = n - 2$. If $diam(T) \geq 8$, the T contains an induced path P_8, say $v_1, v_2, \ldots, v_8$. But then $V(T) - \{v_3, v_4, v_5\}$ is a restrained pendant dominating set of size $n - 3$, which is a contradiction. Thus $diam(T) \leq 7$. Furthermore T is not a star and star are the only trees having diameter 2. Consider the following cases.

Case 1. $diam(T) = 4$. Then T has an induced P_5, say $\{v_1, v_2, v_3, v_4, v_5\}$. The set $S = \{v_2, v_5\}$ is dominating set of path graph P_5 but the set S is not a restrained pendant dominating set because the neighbor of the node v_2 cannot have any other neighbor in $V - S$ and the induced subgraph of S doesn't contain the pendant node. Therefore the restrained pendant dominating set is obtained by taking the leaf of the node v_2. Hence $\gamma_{rpe}(P_5) = n - 2$.

Case 2. $diam(T) = 5$. Then T has an induced P_6, say $\{v_1, v_2, v_3, v_4, v_5, v_6\}$. Certainly the nodes v_2 and v_5 can have leaves attached to them. The set $S = \{v_2, v_5\}$ is a dominating set of P_6 but the induced subgraph of S doesn't contain any pendant node and the neighbors of the nodes v_2 and v_5 are not adjacent in any one node in

$V - S$. Therefore the restrained pendant dominating set is obtained taking the leaves attached to nodes v_2 and v_5. Hence, $\gamma_{rpe}(T) = n - 2$

Case 3. $diam(T) = 6$. Then T has an induced P_7, say $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$. If the set $S = \{v_1, v_2, v_3, v_6, v_7\}$ is a restrained pendant dominating set with size $n - 2$. if we remove any one leaves attached to the nodes v_2 and v_6 the nodes v_1 and v_7 doesn't have any neighbors not on the path. Therefore the set S is the minimum restrained pendant dominating set with minimum cardinality. Hence $\gamma_{rpe}(T) = n - 2$

Theorem 1. 8. For any graph G, we have $\gamma_r(G) \leq \gamma_{rpe}(G)$ and $\gamma_{pe}(G) \leq \gamma_{rpe}(G)$.

Proof. Since every restrained pendant dominating set of G is also a restrained dominating set of G, we have $\gamma_r(G) \leq \gamma_{rpe}(G)$. Since every restrained pendant dominating set of G is also a pendant dominating set of G, we have $\gamma_{pe}(G) \leq \gamma_{rpe}(G)$.

Theorem 1.9. The pendant nodes of a graph G belongs to the restrained pendant dominating set

Proof. Let G be a connected graph and S be the restrained pendant dominating set. By the definition of restrained pendant dominating set if each node not in S, then it should be adjacent to a node in S and to a node in $V - S$. But each pendant nodes doesn't belong to the set S, then the nodes are in $V - S$ and that nodes doesn't having any neighbor in $V - S$. Therefore the pendant nodes belong to restrained pendant dominating set S.

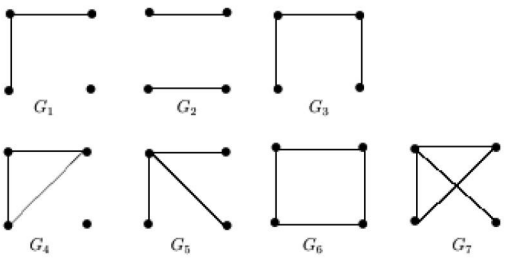
Theorem 1.10. Let G connected graph of order n. Then $\gamma_{rpe}(G) + \Delta(G) \leq 2n - 1$

Proof. For any connected graph G, $\Delta(G) \leq n - 1$. The upper bound of $\gamma_{rpe}(G)$ is found to be n. Therefore $\gamma_{rpe} + \Delta(G) \leq n - 1 + n = 2n - 1$.

Theorem 1.11. Let G be a connected graph order $n \geq 3$ with no leaves. Then $\gamma_{rpe}(G) = 2$ if and only if there exists a nodes of degree $n - 1$.

Proof. Assume that $\gamma_{rpe}(G) = 2$. Let $\{v_1, v_2\}$ be the restrained pendant dominating set of G. Then the node v_1, $(or\ v_2)$ dominates all the other nodes implies that degree of $\{v_1\}$, $(or\ v_2)$ is maximum. Clearly the maximum degree of a graph G is $n - 1$. Thus there exit a node of degree $n - 1$.

Conversely, assume that there exit a nodes of degree $n - 1$. Let u_1 or (u_2) be the nodes which id of degree $n - 1$. Suppose that the set $S = \{u_1, u_2, u_3\}$ is the restrained pendant dominating set, where u_3 is any node of G. Now $deg(u_3) = n - 1$ implies u_3 dominates $V(G)$ but the induced subgraph of S doesn't contain the node of degree one. Therefore the set S doesn't satisfying the condition of restrained pendant dominating set. Hence the set $S' = \{u_1, u_2\}$ is a minimum restrained pendant dominating set .This implies $\gamma_{rpe}(G) = | S' | = 2$.



Theorem 1.12. Let G belongs to any one of the graphs $G_1, G_2, \ldots, G_7$ of order $n = 4$. Then $\gamma_{rpe}(G) + \gamma_{rpe}(\bar{G}) = 5 \; or \; 6$, and $\gamma_{rpe}(G).\gamma_{rpe}(\bar{G}) = 6 \; or \; 9$

Proof. (i) $\gamma_{rpe}(G_1) = 3$. Then $\gamma_{rpe}(\bar{G}_1) = 2$

we get $\gamma_{rpe}(G_1) + \gamma_{rpe}(\bar{G}_1) = 5$ and $\gamma_{rpe}(G_1) \cdot \gamma_{rpe}(\bar{G}_1) = 6$

(ii) $\gamma_{rpe}(G_2) = 3$. Then $\gamma_{rpe}(\bar{G}_2) = 2$.

we get $\gamma_{rpe}(G_2) + \gamma_{rpe}(\bar{G}_2) = 5$ and $\gamma_{rpe}(G_2) \cdot \gamma_{rpe}(\bar{G}_2) = 6$.

(iii) $\gamma_{rpe}(G_3) = 3$. Then $\gamma_{rpe}(\overline{G_3}) = 3$.

we get $\gamma_{rpe}(G_3) + \gamma_{rpe}(\overline{G_3}) = 6$ and $\gamma_{rpe}(G_3) \cdot \gamma_{rpe}(\overline{G_3}) = 9$.

(iv) $\gamma_{rpe}(G_4) = 3$. Then $\gamma_{rpe}(\overline{G_4}) = 3$.

we get $\gamma_{rpe}(G_4) + \gamma_{rpe}(\overline{G_4}) = 6$ and $\gamma_{rpe}(G_4) \cdot \gamma_{rpe}(\overline{G_4}) = 9$.

(v) $\gamma_{rpe}(G_5) = 3$. Then $\gamma_{rpe}(\overline{G_5}) = 3$.

we get $\gamma_{rpe}(G_5) + \gamma_{rpe}(\overline{G_5}) = 6$ and $\gamma_{rpe}(G_5) \cdot \gamma_{rpe}(\overline{G_5}) = 9$.

(vi) $\gamma_{rpe}(G_6) = 2$. Then $\gamma_{rpe}(\overline{G_6}) = 3$.

we get $\gamma_{rpe}(G_6) + \gamma_{rpe}(\overline{G_6}) = 5$ and $\gamma_{rpe}(G_6) \cdot \gamma_{rpe}(\overline{G_6}) = 6$.

(vii) $\gamma_{rpe}(G_7) = 3$. Then $\gamma_{rpe}(\overline{G_7}) = 3$.

we get $\gamma_{rpe}(G_7) + \gamma_{rpe}(\overline{G_7}) = 6$ and $\gamma_{rpe}(G_7) \cdot \gamma_{rpe}(\overline{G_7}) = 9$.

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