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SOME DISTANCE-BASED TOPOLOGICAL INDICES OF DOUBLE STARBARBELL GRAPH AND PETERSENBARBELL GRAPH

Abstract : A topological index is an analytically derived numerical index for the graph structure. In this paper, we study some distance-based topological indices, such as, Wiener index (W), hyper-Wiener index (WW), Harary index (H), Reciprocal Complementary Wiener index (RCW), Wiener Polarity index (W_P), Terminal Wiener index (TW), Reverse Wiener index ($\wedge$) and Reciprocal Reverse Wiener index ($R\wedge$) of Double starbarbell graph $BSB_{r_1,r_2,...,r_{m+n}}$ and Petersenbarbell graph $PB_{m_1,m_2,...,m_5}$.

MSC : 05C09, 05C12, 05C76

Keywords: Topological indices, Wiener index, Double Star graph, Double Starbarbell graph, Petersenbarbell graph.

1 Introduction

In this paper, we consider only finite, undirected, connected and simple graphs. For a graph $G = (V, E)$, the number of vertices and edges will be denoted by $| V(G) |$ and $| E(G) |$ respectively. If $u, v \in V(G)$, length of the shortest distance between u and v in G is denoted by $d_G(u, v)$ and we simply denote it by $d(u, v)$ if there is no ambiguity in the graph under consideration. The eccentricity of a vertex u in a graph G is $e(u) = max\{d(u, v) : v \in V(G)\}$. The radius (resp. diameter) of G is $r = rad(G) = min\{e(v) : v \in V(G)\}$ (resp. $d = diam(G) = max\{e(v) : v \in V(G)\}$). In a graph, a vertex of degree 1 is known as a pendent vertex or terminal node or leaf node or leaf. Definitions which are not seen here can be referred in [4] and [5].

A topological index is an analytically derived numerical index for the graph structure. Indices are graph invariants used to study graph structure. Graph techniques have many applications in various fields

such as Chemistry, Physics, Biology, Computer science, etc. The Wiener index is the distance based topological index introduced by the chemist Harry Wiener in 1947 [26] and also known as the "Wiener number"[7, 9]. Wiener index is widely used based on the chemical applications of graph theory which counts the number of bonds between pairs of atoms and sum the distances between all pairs by generating a distance matrix [20]. The Wiener index is defined by the sum of distances between all unordered pairs of vertices of a graph G,

$$W(G) = \sum_{u,v \in V(G)} d(u,v).$$

The hyper-Wiener index is the generalization of the Wiener index introduced by Milan Randić in 1993 [23] and is defined as follows :

$$WW(G) = \frac{1}{2} \sum_{u,v \in V(G)} [d(u,v) + d(u,v)^2].$$

In [22] Plavšić et. al., and in [16] Ivancine et. al., independently introduced the Harary index, in honor of Frank Harary. For the graph G, the Harary index is defined as the reciprocal of the Wiener index, and denoted by

$$H(G) = \sum_{u,v \in V(G)} \frac{1}{d(u,v)}.$$

In [15, 17] Ivancine et. al., introduced the Reciprocal Complementary Wiener index, denoted by $RCW(G)$ and given by

$$RCW(G) = \sum_{u,v \in V(G)} \frac{1}{d+1-d(u,v)},$$

where d is the diameter of a graph G.

The Wiener Polarity index W_P of a graph G, introduced by Wiener in 1947 [8], is the number of unordered pairs of vertices of G such that the distance between u and v is 3,

$$W_P(G) = |\{(u,v) \mid d(u,v) = 3, u, v \in V(G)\}|.$$

The Terminal Wiener index of a graph G is defined by Gutman et.al., in [14], as the sum of distances between all pairs of pendent vertices of G,

$$TW(G) = \sum_{\substack{u,v \in V(G) \\ deg(u)=deg(v)=1}} d(u,v).$$

The Reverse Wiener index was proposed by Balaban et. al., in 2000 [2] and is defined as follows

$$\wedge(G) = \frac{n(n-1)d}{2} - W(G),$$

where $n = |V(G)|$ and d is the diameter of G.

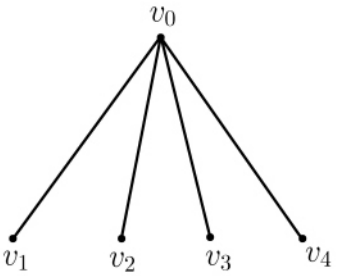
In [18], the Reciprocal Reverse Wiener (RRW) index $R \wedge (G)$ of a connected graph G is defined as

$$R \wedge (G) = \begin{cases} \sum_{u,v \in V(G)} \frac{1}{d - d(u,v)}, & \text{for } 0 < d(u,v) < d, \\ 0, & \text{otherwise.} \end{cases}$$

where d is the diameter of a graph G.

In this paper we calculate $W(G)$, $WW(G)$, $H(G)$, $RCW(G)$, $W_P(G)$, $TW(G)$, $\wedge(G)$ and $R \wedge (G)$ of a Double starbarbell and Petersenbarbell graphs. We mention some concepts and results below, which are required for proving the theorems.Various indices are referred in [10, 11, 12]

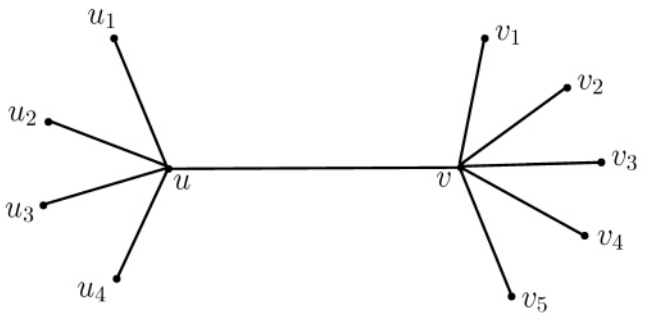
A complete graph on n vertices is denoted by K_n. S_n denotes the star on n vertices in which one vertex is adjacent to all the other vertices, See Figure 1.1 for S_5. Also $S_n \cong K_{1,n-1}$.



Star graph S_5

Figure 1.1

A double star is a graph obtained by inserting an edge joining the centers of $K_{1,n}$ and $K_{1,m}$ for $n, m \geq 2$ and it is denoted by $B_{n,m}$. If $n = m$ in the double star graph, it is called as bistar graph and denoted by $B_{n,n}$, See Figure 1.2 for $B_{4,5}$.



Double star graph $B_{4,5}$

Figure 1.2

Lemma 1.1. [1] *Let K_n and S_n be the complete and star graphs of order n, respectively. Then*
(i) *for $n \geq 1$, $W(K_n) = \binom{n}{2}$*
(ii) *for $n \geq 1$, $W(S_n) = (n-1)^2$.* □

Lemma 1.2. [1] *Let K_n and S_n be the complete and star graphs of order n, respectively. Then*
(i) *for $n \geq 1$, $WW(K_n) = \frac{1}{2}n(n-1)$*
(ii) *for $n \geq 1$, $WW(S_n) = \frac{1}{2}(n-1)(3n-4)$.* □

Lemma 1.3. [25] *Let K_n and S_n be the complete and star graphs of order n, respectively. Then*
(i) *for $n \geq 1$, $H(K_n) = \binom{n}{2}$*
(ii) *for $n \geq 1$, $H(S_n) = \frac{1}{4}(n-1)(n+2)$.* □

Lemma 1.4. [21] *Let K_n and S_n be the complete and star graphs of order n, respectively. Then*
(i) *for $n \geq 1$, $RCW(K_n) = \binom{n}{2}$*
(ii) *for $n \geq 1$, $RCW(S_n) = \frac{1}{2}(n-1)^2$.* □

Lemma 1.5. [8] *Let K_n and S_n be the complete and star graphs of order n, respectively. Then $W_P(K_n) = W_P(S_n) = 0$.* □

Lemma 1.6. [13, 14] *Let K_n and S_n be the complete and star graphs of order n, respectively. Then $TW(K_n) = 0$ and $TW(S_n) = (n-1)(n-2)$.* □

Lemma 1.7. [6] *Let K_n and S_n be the complete and star graphs of order n, respectively. Then*
(i) *for $n \geq 1$, $\wedge(K_n) = 0$*
(ii) *for $n \geq 1$, $\wedge(S_n) = (n-1)$.* □

Lemma 1.8. [27] *Let K_n and S_n be the complete and star graphs of order n, respectively. Then*
(i) *for $n \geq 1$, $R \wedge (K_n) = 0$*
(ii) *for $n \geq 1$, $R \wedge (S_n) = (n-1)$.* □

2 Indices of a Double Star Graph

In this section, we calculate some results for distance-based topological indices $W(G)$, $WW(G)$, $H(G)$, $RCW(G)$, $W_P(G)$, $TW(G)$, $\wedge(G)$ and $R \wedge (G)$, where G is a double star graph as in [19].

Theorem 2.1. *For the double star graph G with $m, n \geq 2$, then*
(i) $W(G) = n(n+2) + m(m+2) + (3mn+1)$.
(ii) $WW(G) = \frac{1}{2}[n(3n+5) + m(3m+5) + 2(6mn+1)]$.
(iii) $H(G) = \frac{1}{12}[3n(n+5) + 3m(m+5) + 4(mn+3)]$.
(iv) $RCW(G) = \frac{1}{12}[n(3n+7) + m(3m+7) + 4(3mn+1)]$.

Proof. Let $G = B_{n,m}$ be the double star graph with $m, n \geq 2$. Let $V(G) = V_1 \cup V_2$, where $V_1 = \{u, u_1, u_2, \ldots, u_n\}$ and $V_2 = \{v, v_1, v_2, \ldots, v_m\}$. For $i, j = 1, 2, \ldots, n$ and $k, \ell = 1, 2, \ldots, m$ the distances between any two vertices in G are given by
$d(u, v) = d(u, u_i) = d(v, v_k) = 1$,
$d(u, v_k) = d(u_i, v) = 2$,
$d(u_i, u_j) = d(v_k, v_\ell) = 2$, $i \neq j$ and $k \neq \ell$,
$d(u_i, v_k) = 3$.
Here $diam(G) = 3$ and the distance between any pair of vertices varies from $1, 2, \ldots, diam(G)$.
The number of 1 distance, pair of vertices is $n + m + 1$.

The number of 2 distance, pair of vertices is $\binom{n}{2} + \binom{m}{2} + n + m$.
The number of 3 distance, pair of vertices is mn.
By using these we derive the following

$$\begin{aligned}
\text{(i)} \quad W(G) &= (n+m+1)1 + [\binom{n}{2} + \binom{m}{2} + n + m]2 + (mn)3 \\
&= n(n+2) + m(m+2) + (3mn+1). \\
\text{(ii)} \quad WW(G) &= \frac{1}{2}[(n+m+1)(1+1^2) + [\binom{n}{2} + \binom{m}{2} + n + m](2+2^2) \\
&\quad + (mn)(3+3^2)] \\
&= \frac{1}{2}[n(3n+5) + m(3m+5) + 2(6mn+1)] \\
\text{(iii)} \quad H(G) &= (n+m+1)\frac{1}{1} + [\binom{n}{2} + \binom{m}{2} + n + m]\frac{1}{2} + (mn)\frac{1}{3} \\
&= \frac{1}{12}[3n(n+5) + 3m(m+5) + 4(mn+3)]. \\
\text{(iv)} \quad RCW(G) &= (n+m+1)\frac{1}{3} + [\binom{n}{2} + \binom{m}{2} + n + m]\frac{1}{2} + (mn)\frac{1}{1} \\
&= \frac{1}{12}[n(3n+7) + m(3m+7) + 4(3mn+1)]. \qquad \square
\end{aligned}$$

Remark 2.1. *In Theorem* 2.1*, when* $m = n = 1$*,* $G \cong P_4$*, the diameter of graph* G *is* 3*. Hence* $W(G)$*,* $WW(G)$*,* $H(G)$ *and* $RCW(G)$ *are valid.* $\square$

Corollary 2.2. *For the double star graph* G *with* $m, n \geq 1$*,* $W_P(G) = mn$*.*

Proof. The number of 3 distance pair of vertices is mn by Theorem 2.1. So, $W_P(G) = mn$. $\square$

Corollary 2.3. *For the double star graph* G *with* $m, n \geq 1$*,* $TW(G) = n(n-1) + m(m-1) + 3mn$*.*

Proof. For $m = n = 1$ in Theorem 2.1, $G \cong P_4$, so $TW(G) = TW(P_4)$. For $m, n \geq 2$ in Theorem 2.1, $TW(G) = \binom{n}{2}2 + \binom{m}{2}2 + (mn)3 = n(n-1) + m(m-1) + 3mn$. $\square$

Lemma 2.4. *For the double star graph* G *with* $m, n \geq 1$*,* $\wedge(G) = \frac{3}{2}(m+n+2)(m+n+1) - [n(n+2) + m(m+2) + (3mn+1)]$*.*

Proof. In Theorem 2.1, $\mid V(G) \mid = m+n+2$, $d=3$ and $W(G) = n(n+2)+m(m+2)+(3mn+1)$. So, $\wedge(G) = \frac{3}{2}(m+n+2)(m+n+1) - [n(n+2)+m(m+2)+(3mn+1)]$. □

Lemma 2.5. *For the double star graph G with $m, n \geq 1$, $R\wedge(G) = \frac{1}{2}[m(m+2)+n(n+2)+1]$.*

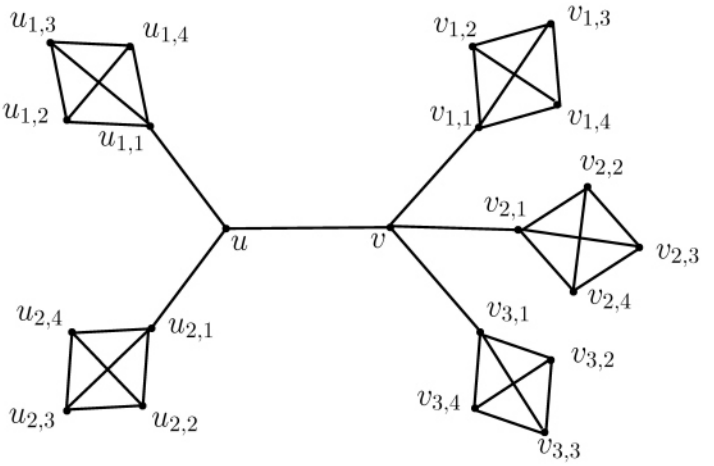
Proof. Here $d=3$, from Theorem 2.1 and the distance between any pair of vertices varies $0 < d(u,v) < d$.
The number of 1 distance pair of vertices is $n+m+1$.
The number of 2 distance pair of vertices is $\binom{n}{2}+\binom{m}{2}+n+m$.
By using these we derive the following.

$$\begin{aligned} R\wedge(G) &= (n+m+1)\frac{1}{2} + [\binom{n}{2}+\binom{m}{2}+n+m]\frac{1}{1} \\ &= \frac{1}{2}[m(m+2)+n(n+2)+1]. \end{aligned}$$ □

3 Indices of a Double Starbarbell Graph

In this section, we introduce double starbarbell graph $BSB_{r_1,r_2,\ldots,r_{m+n}}$ which is similar to starbarbell graph [24] and derive some results for distance-based topological indices $W(G)$, $WW(G)$, $H(G)$, $RCW(G)$, $W_P(G)$, $TW(G)$, $\wedge(G)$ and $R\wedge(G)$, where G is a double starbarbell graph.

Definition 3.1. *The double starbarbell graph $BSB_{r_1,r_2,\ldots,r_{m+n}}$ is a graph obtained from double star graph $B_{n,m}$ and $m+n$ complete graph K_{r_i} by merging one vertex from each K_{r_i} and the i^{th} leaf of $B_{n,m}$, where $r_i \geq 2$, $1 \leq i \leq m+n$ and $m, n \geq 2$, See Figure 3.1 for $BSB_{r_1,r_2,\ldots,r_5}$.*



Double Starbarbell graph $BSB_{4,4,4,4,4}$
where $r_i = 4$, $i = 1, 2, ..., m + n$

Figure 3.1

Theorem 3.2. *For $m, n \geq 1$ and $r \geq 2$ the double starbarbell graph G, in which all the complete graphs are of the same order. Then*

(i) $W(G) = 1 + \frac{m+n}{2}(r^2 + 9r - 4) + mnr(5r - 2)$
$+ r(2r - 1)[n(n - 1) + m(m - 1)]$.

(ii) $WW(G) = \frac{1}{2}[2 + (m + n)(r^2 + 17r - 10)$
$+ [n(n - 1) + m(m - 1)](10r^2 - 8r + 1)$
$+ 2mn(15r^2 - 10r + 1)]$.

(iii) $H(G) = 1 + \frac{m+n}{6}(3r^2 + 2r + 4) + \frac{mn}{30}(6r^2 + 3r + 1)$
$+ \frac{1}{24}[n(n - 1) + m(m - 1)](3r^2 + 2r + 1)$.

(iv) $RCW(G) = \frac{1}{5} + \frac{m+n}{60}(6r^2 + 29r - 8) + \frac{mn}{3}(3r^2 - 3r + 1)$
$+ \frac{1}{24}[n(n - 1) + m(m - 1)](6r^2 - 4r - 1)$.

Proof. Let $G = BSB_{r_1, r_2, ..., r_{m+n}}$ be the double starbarbell graph where each complete graph has the same order with $r_i \geq 2$, $1 \leq i \leq m + n$ and $m, n \geq 1$. Let $r_i = r$, $i = 1, 2, \ldots, m + n$. Let $V(G) = \{u, u_{1,1}, u_{1,2}, \ldots, u_{1,r}, u_{2,1}, u_{2,2}, \ldots, u_{2,r}, \ldots, v_{n,1}, v_{n,2}, \ldots, v_{n,r}, v, v_{1,1}, v_{1,2}, \ldots, v_{1,r}, v_{2,1}, v_{2,2}, \ldots, v_{2,r}, \ldots, v_{n,1}, v_{n,2}, \ldots, v_{m,r}\}$. Then $V(G) = U_1 \cup U_2 \cup \cdots \cup U_n \cup V_1 \cup V_2 \cup \cdots \cup V_m \cup \{u, v\}$ where $U_i = \{u_{i,1}, u_{i,2}, \ldots, u_{i,r}\}$ and $V_j = \{v_{j,1}, v_{j,2}, \ldots, v_{j,r}\}$, $1 \leq i \leq n$ and $1 \leq j \leq m$. For $i, j = 1, 2, \ldots, n$ and $p, q = 1, 2, \ldots, m$ and $k, \ell = 2, 3, \ldots, r$ the distances between any two vertices in G are given by

$d(u,v) = d(u,u_{i,1}) = d(u_{i,1},u_{i,k}) = d(u_{i,k},u_{i,\ell}) = 1$, for $k \neq \ell$,
$d(v,v_{p,1}) = d(v_{p,1},v_{p,k}) = d(v_{p,k},v_{p,\ell}) = 1$, for $k \neq \ell$
$d(u,v_{p,1}) \quad = d(u_{i,1},v) \quad = 2$,
$d(u_{i,1},u_{j,1}) = d(v_{p,1},v_{q,1}) = 2$, for $i \neq j$ and $p \neq q$
$d(u_{i,1},u_{j,k}) = d(v_{p,1},v_{q,k}) = 3$, for $i \neq j$ and $p \neq q$
$d(u,v_{p,k}) \quad = d(u_{i,k},v) \quad = 3$,
$d(u_{i,1},v_{p,k}) = d(u_{j,k},v_{q,1}) = 4$,
$d(u_{i,k},v_{p,k}) = 5$.

Here $diam(G) = 5$ and the distance between any pair of vertices varies from $1,2,\ldots,diam(G)$.

The number of 1 distance, pair of vertices is $1+n+n\binom{r}{2}+m+m\binom{r}{2}$.

The number of 2 distance, pair of vertices is $n(r-1)+m(r-1)+\binom{n}{2}+\binom{m}{2}+n+m$.

The number of 3 distance, pair of vertices is $m(r-1)+2(r-1)\binom{n}{2}+n(r-1)+2(r-1)\binom{m}{2}+mn$.

The number of 4 distance, pair of vertices is $(r-1)^2\binom{n}{2}+(r-1)^2\binom{m}{2}+2(r-1)mn$.

The number of 5 distance pair of vertices is $mn(r-1)^2$.

By using these we derive the following

$$
\begin{aligned}
\text{(i)}\quad W(G) \quad &= [1+n+n\binom{r}{2}+m+m\binom{r}{2}]1+[n(r-1)+m(r-1)\\
&+\binom{n}{2}+\binom{m}{2}+n+m]2+[m(r-1)+2(r-1)\binom{n}{2}\\
&+n(r-1)+2(r-1)\binom{m}{2}+mn]3+[(r-1)^2\binom{n}{2}\\
&+(r-1)^2\binom{m}{2}+2(r-1)mn]4+[mn(r-1)^2]5\\
&= 1+\frac{m+n}{2}(r^2+9r-4)+mnr(5r-2)\\
&+r(2r-1)[n(n-1)+m(m-1)].\\
\text{(ii)}\ WW(G) \quad &= \frac{1}{2}[[1+n+n\binom{r}{2}+m+m\binom{r}{2}](1+1^2)+[n(r-1)\\
&+m(r-1)+\binom{n}{2}+\binom{m}{2}+n+m](2+2^2)+[m(r-1)
\end{aligned}
$$

$$
\begin{aligned}
&+ 2(r-1)\binom{n}{2} + n(r-1) + 2(r-1)\binom{m}{2} + mn](3+3^2)\\
&+ [(r-1)^2\binom{n}{2} + (r-1)^2\binom{m}{2} + 2(r-1)mn](4+4^2)\\
&+ [mn(r-1)^2](5+5^2)]\\
&= \frac{1}{2}[2 + (m+n)(r^2+17r-10) + 2mn(15r^2-10r+1)\\
&+ [n(n-1)+m(m-1)](10r^2-8r+1)].
\end{aligned}
$$

(iii) $$
\begin{aligned}
H(G) &= [1+n+n\binom{r}{2} + m + m\binom{r}{2}]\frac{1}{1}\\
&+ [n(r-1)+m(r-1) + \binom{n}{2} + \binom{m}{2} + n + m]\frac{1}{2}\\
&+ [m(r-1) + 2(r-1)\binom{n}{2} + n(r-1) + 2(r-1)\binom{m}{2} + mn]\frac{1}{3}\\
&+ [(r-1)^2\binom{n}{2} + (r-1)^2\binom{m}{2} + 2(r-1)mn]\frac{1}{4} + [mn(r-1)^2]\frac{1}{5}\\
&= 1 + \frac{m+n}{6}(3r^2+2r+4) + \frac{mn}{30}(6r^2+3r+1)\\
&+ \frac{1}{24}[n(n-1)+m(m-1)](3r^2+2r+1).
\end{aligned}
$$

(iv) $$
\begin{aligned}
RCW(G) &= [1+n+n\binom{r}{2} + m + m\binom{r}{2}]\frac{1}{5} + [n(r-1)+m(r-1)\\
&+ \binom{n}{2} + \binom{m}{2} + n + m]\frac{1}{4} + [m(r-1) + 2(r-1)\binom{n}{2}\\
&+ n(r-1) + 2(r-1)\binom{m}{2} + mn]\frac{1}{3} + [(r-1)^2\binom{n}{2}\\
&+ (r-1)^2\binom{m}{2} + 2(r-1)mn]\frac{1}{2} + [mn(r-1)^2]\frac{1}{1}\\
&= \frac{1}{5} + \frac{m+n}{60}(6r^2+29r-8) + \frac{mn}{3}(3r^2-3r+1)\\
&+ \frac{1}{24}[n(n-1)+m(m-1)](6r^2-4r-1).
\end{aligned}
$$

□

Remark 3.1. *In Theorem* 3.2, *when* $m = n = r = 1$, $G \cong P_4$, *here the diameter of graph* G *is* 3, *hence* $RCW(G)$ *is invalid but* $W(G)$, $WW(G)$ *and* $H(G)$ *are valid.* □

Remark 3.2. *In Theorem* 3.2, *when* $m, n \geq 2$ *and* $r = 1$, $G \cong B_{n,m}$, *here the diameter of graph* G *is* 3, *hence* $RCW(G)$ *is invalid but* $W(G)$, $WW(G)$ *and* $H(G)$ *are valid.* □

Corollary 3.3. *For the double starbarbell graph* G *with* $m, n, r \geq 1$, $W_P(G) = (r-1)[(m+n) + n(n-1) + m(m-1)] + mn$.

Proof. The number of 3 distance pair of vertices is $m(r-1) + 2(r-1)\binom{n}{2} + n(r-1) + 2(r-1)\binom{m}{2} + mn$ by Theorem 3.2. So, $W_P(G) = (r-1)[(m+n) + n(n-1) + m(m-1)] + mn$. □

Corollary 3.4. *For the double starbarbell graph* G *with* $m, n \geq 1$,

$$TW(G) = \begin{cases} TW(B_{n,m}), & \text{if } r = 1 \\ 2[n(n-1) + m(m-1)] + 5mn, & \text{if } r = 2 \end{cases}$$

Proof. In Theorem 3.2, when $r = 1$, $G \cong B_{n,m}$, hence $TW(G) = TW(B_{n,m})$ and same as in Corollary 2.3. In Theorem 3.2, when $r = 2$, hence $TW(G) = [\binom{n}{2} + \binom{m}{2}]4 + 5mn = 2[n(n-1) + m(m-1)] + 5mn$. □

Lemma 3.5. *For the double starbarbell graph* G *with* $m, n \geq 1$ *and* $r \geq 2$, $\wedge(G) = \frac{5}{2}[(m+n)r+2][(m+n)r+1] - [1 + \frac{m+n}{2}(r^2+9r-4) + r(2r-1)[n(n-1) + m(m-1)] + mnr(5r-2)]$.

Proof. In Theorem 3.2, $| V(G) | = (m+n)r+2$, $d = 5$ and $W(G) = 1 + \frac{m+n}{2}(r^2+9r-4) + r(2r-1)[n(n-1) + m(m-1)] + mnr(5r-2)$. So, $\wedge(G) = \frac{5}{2}[(m+n)r+2][(m+n)r+1] - [1 + \frac{m+n}{2}(r^2+9r-4) + r(2r-1)[n(n-1) + m(m-1)] + mnr(5r-2)]$. □

Lemma 3.6. *For the double starbarbell graph* G *with* $m, n \geq 1$ *and* $r \geq 2$, $R \wedge (G) = \frac{1}{4}[1 + 2mn(4r-3)] + \frac{(m+n)}{24}(3r^2 + 17r - 6)$ $+ \frac{1}{6}[n(n-1) + m(m-1)](3r^2 - 3r + 1)$.

Proof. Here $d = 5$, from Theorem 3.2 and the distance between any pair of vertices satisfies $0 < d(u, v) < d$.

The number of 1 distance pair of vertices is $1+n+n\binom{r}{2}+m+m\binom{r}{2}$.
The number of 2 distance, pair of vertices is $n(r-1)+m(r-1)+\binom{n}{2}+\binom{m}{2}+n+m$.
The number of 3 distance, pair of vertices is $m(r-1)+2(r-1)\binom{n}{2}+n(r-1)+2(r-1)\binom{m}{2}+mn$.
The number of 4 distance, pair of vertices is $(r-1)^2\binom{n}{2}+(r-1)^2\binom{m}{2}+2(r-1)mn$.
By using these we derive the following

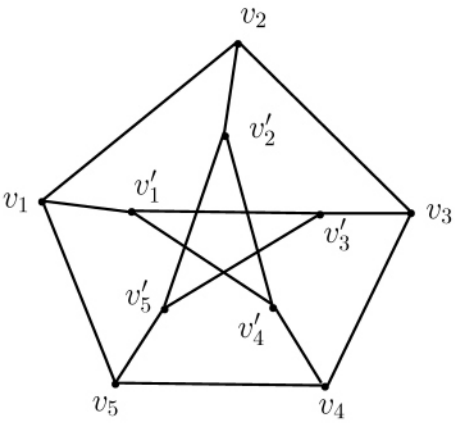
$$\begin{aligned}
R\wedge(G) &= [1+n+n\binom{r}{2}+m+m\binom{r}{2}]\frac{1}{4}\\
&+[n(r-1)+m(r-1)+\binom{n}{2}+\binom{m}{2}+n+m]\frac{1}{3}\\
&+[m(r-1)+2(r-1)\binom{n}{2}+n(r-1)+2(r-1)\binom{m}{2}+mn]\frac{1}{2}\\
&+[(r-1)^2\binom{n}{2}+(r-1)^2\binom{m}{2}+2(r-1)mn]\frac{1}{1}\\
&=\frac{1}{4}[1+2mn(4r-3)]+\frac{(m+n)}{24}(3r^2+17r-6)\\
&+\frac{1}{6}[n(n-1)+m(m-1)](3r^2-3r+1). \qquad \square
\end{aligned}$$

4 Indices of Petersen Graph

In this section, we calculate some results for distance-based topological indices $W(G)$, $WW(G)$, $H(G)$, $RCW(G)$, $W_P(G)$, $TW(G)$, $\wedge(G)$ and $R\wedge(G)$, where G is a Petersen graph.

Theorem 4.1. *For the Petersen graph G,*
(i) $W(G)=5(n+5)$.
(ii) $WW(G)=7(n+5)$.
(iii) $H(G)=2(n+5)$.
(iv) $RCW(G)=\frac{5}{2}(n+5)$.

Proof. Let G be a Petersen graph. (See Figure 4.1).



Petersen graph G

Figure 4.1

Let $V(G) = \{v_1, v_2, \ldots, v_5, v_1', v_2', \ldots, v_5'\}$. For $i, j = 1, 2, \ldots, 5$ the distances between any two vertices in G are given by

$d(v_i, v_i') = 1$,
$d(v_i, v_{i+1}) = 1$, for $i+1$ taken addition modulo 5
$d(v_i, v_{i+2}') = 1$, for $i+2$ taken addition modulo 5
$d(v_i, v_{i+1}') = 2$, for $i+1$ taken addition modulo 5
$d(v_i, v_{i+2}) = 2$, for $i+2$ taken addition modulo 5
$d(v_i, v_j') = 2$, $i \neq j$

Here $diam(G) = 2$ and the distance between any pair of vertices varies from $1, 2, \ldots, diam(G)$.

The number of 1 distance, pair of vertices is $n+5$.

The number of 2 distance, pair of vertices is $2(n+5)$.

By using these we derive the following

(i) $W(G) = [n+5]1 + [2(n+5)]2$
$= 5(n+5)$.

(ii) $WW(G) = \frac{1}{2}[[n+5](1+1^2) + [2(n+5)](2+2^2)]$
$= 7(n+5)$.

(iii) $H(G) = [n+5]\frac{1}{1} + [2(n+5)]\frac{1}{2}$
$= 2(n+5)$.

(iv) $RCW(G) = [n+5]\frac{1}{2} + [2(n+5)]\frac{1}{1}$
$= \frac{5}{2}(n+5)$. □

Corollary 4.2. *For the Petersen graph G, $W_P(G) = 0$.*

Proof. In Theorem 4.1, here $d = 2$, the number of 3 distance pair of vertices is 0. So, $W_P(G) = 0$. □

Corollary 4.3. *For the Petersen graph G, $TW(G) = 0$*

Proof. In Theorem 4.1, there is no pendent vertex, so $TW(G) = 0$. □

Lemma 4.4. *For the Petersen graph G, $\wedge(G) = n^2 - 6n - 25$.*

Proof. In Theorem 4.1, $\mid V(G) \mid = n$, $d = 2$ and $W(G) = 5(n + 5)$. So, $\wedge(G) = \frac{n(n-1)}{2}2 - 5(n + 5) = n^2 - 6n - 25$ □

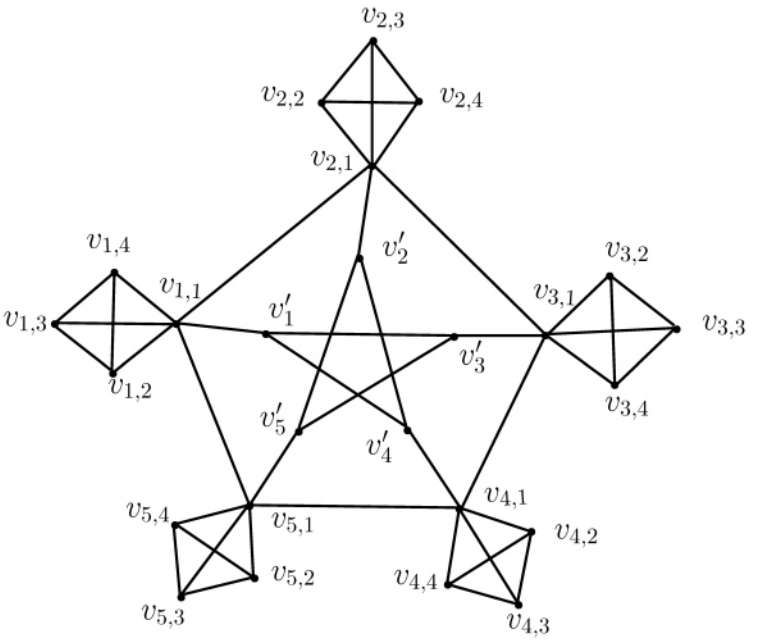
Lemma 4.5. *For the Petersen graph G, $R \wedge (G) = n + 5$.*

Proof. Here $d = 2$, from Theorem 4.1 and the distance between any pair of vertices satisfies $0 < d(u, v) < d$. The number of 1 distance pair of vertices is $n + 5$, hence $R \wedge (G) = n + 5$. □

5 Indices of Petersenbarbell Graph

In this Section, we introduce petersenbarbell graph $PB_{m_1,m_2,\ldots,m_5}$ which is similar to wheelbarbell graph [3] and also derive some results for distance-based topological indices $W(G)$, $WW(G)$, $H(G)$, $RCW(G)$, $W_P(G)$, $TW(G)$, $\wedge(G)$ and $R \wedge (G)$, where G is petersenbarbell graph.

Definition 5.1. *The petersenbarbell graph $PB_{m_1,m_2,\ldots,m_5}$ is a graph obtained from Petersen graph and complete graph K_{m_i} by merging one vertex from each K_{m_i} and the outer 5 vertices of Petersen graph, where $m_i \geq 2$, See Figure 5.1 for $PB_{m_1,m_2,\ldots,m_5}$.*



Petersenbarbell graph $PB_{4,4,4,4,4}$
where $m_i = 4,\ i = 1, 2, ..., 5$
Figure 5.1

Theorem 5.2. *For $m \geq 2$ and $n = 10$ the petersenbarbell graph G, in which each complete graph is of the uniform order. Then*

(i) $W(G) = 5(n+5) + \frac{1}{2}(m-1)[(n-5)(m+48) + 70(m-1)]$.
(ii) $WW(G) = \frac{1}{2}[14(n+5) + (m-1)[(n-5)(m+90) + 160(m-1)]]$.
(iii) $H(G) = 2(n+5) + \frac{1}{12}(m-1)[6(n-5)(m+7) + 35(m-1)]$.
(iv) $RCW(G) = \frac{11}{12}(n+5) + \frac{1}{8}(m-1)[(n-5)(m+32) + 60(m-1)]$.

Proof. Let $G = PB_{m_1,m_2,...,m_5}$ be the petersenbarbell graph, where each complete graph is of the uniform order with $m_i \geq 2$ and $1 \leq i \leq 5$. Let $m_i = m$, $i = 1, 2, \ldots, 5$. Let $V(G) = \{v_{1,1}, v_{1,2}, \ldots, v_{1,m}, v_{2,1}, v_{2,2}, \ldots, v_{2,m}, \ldots, v_{5,1}, v_{5,2}, \ldots, v_{5,m}, v'_1, v'_2, \ldots, v'_5\}$. Then $V(G) = V_1 \cup V_2 \cup, \ldots, \cup V_5 \cup V_t$ where $V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,m}\}$, $1 \leq i \leq 5$ and $V_t = \{v'_1, v'_2, \ldots, v'_5\}$. For $i, j = 1, 2, \ldots, 5$ and $k, \ell = 2, 3, \ldots, m$ the distance between any two vertices in G are given by

$d(v_{i,1}, v_{i,k}) = d(v_{i,1}, v'_i) = d(v_{i,k}, v_{i,\ell}) = 1$, for $k \neq \ell$,
$d(v_i, v'_{i+2}) = 1$, for $i + 2$ taken addition modulo 5
$d(v_{i,1}, v_{i+1,1}) = 1$, for $i + 1$ taken addition modulo 5,

$d(v_{i,k}, v'_i) = d(v_{i,1}, v'_j) = 2,\ i \neq j,$
$d(v_{i,1}, v_{i+2,1}) = 2$, for $i+2$ taken addition modulo 5,
$d(v_i, v'_{i+1}) = 2$, for $i+1$ can taken addition modulo 5,
$d(v_{i,1}, v_{i+1,k}) = 2$, for $i+1$ taken addition modulo 5,
$d(v_{i,k}, v_{i+1,\ell}) = 3$, for $i+1$ taken addition modulo 5,
$d(v_{i,1}, v_{i+2,k}) = 3$, for $i+2$ taken addition modulo 5,
$d(v_{i,k}, v'_j) = 3,\ i \neq j,$
$d(v_{i,k}, v_{i+2,\ell}) = 4$, for $i+2$ taken addition modulo 5.
Here $diam(G) = 4$ and the distance between any pair of vertices varies from $1, 2, \ldots, diam(G)$.
The number of 1 distance, pair of vertices is $(n+5) + (n-5)\binom{m}{2}$.
The number of 2 distance, pair of vertices is $2(n+5)+3(n-5)(m-1)$.
The number of 3 distance, pair of vertices is $6(n-5)(m-1)+5(m-1)^2$.
The number of 4 distance, pair of vertices is $5(m-1)^2$.
By using these we derive the following

$$\begin{aligned}(\text{i}) \quad W(G) &= [(n+5) + (n-5)\binom{m}{2}]1 + [2(n+5) + 3(n-5)(m-1)]2 \\ &\quad + [6(n-5)(m-1) + 5(m-1)^2]3 + [5(m-1)^2]4 \\ &= 5(n+5) + \frac{1}{2}(m-1)[(n-5)(m+48) + 70(m-1)].\end{aligned}$$

$$\begin{aligned}(\text{ii})\ WW(G) &= \frac{1}{2}[[(n+5) + (n-5)\binom{m}{2}](1+1^2) + [2(n+5) \\ &\quad + 3(n-5)(m-1)](2+2^2) + [6(n-5)(m-1) + 5(m-1)^2](3 \\ &\quad + 3^2) + [5(m-1)^2](4+4^2)] \\ &= \frac{1}{2}[14(n+5) + (m-1)[(n-5)(m+90) + 160(m-1)]].\end{aligned}$$

$$\begin{aligned}(\text{iii}) \quad H(G) &= [(n+5) + (n-5)\binom{m}{2}]\frac{1}{1} + [2(n+5) + 3(n-5)(m-1)]\frac{1}{2} \\ &\quad + [6(n-5)(m-1) + 5(m-1)^2]\frac{1}{3} + [5(m-1)^2]\frac{1}{4} \\ &= 2(n+5) + \frac{1}{12}(m-1)[6(n-5)(m+7) + 35(m-1)].\end{aligned}$$

(iv) $RCW(G) = [(n+5) + (n-5)\binom{m}{2}]\frac{1}{4} + [2(n+5) + 3(n-5)(m-1)]\frac{1}{3}$
$+ [6(n-5)(m-1) + 5(m-1)^2]\frac{1}{2} + [5(m-1)^2]\frac{1}{1}$
$= \frac{11}{12}(n+5) + \frac{1}{8}(m-1)[(n-5)(m+32) + 60(m-1)]$. □

Remark 5.1. *In Theorem* 5.2, *when* $m = 1$, $G \cong$ *Petersen graph, here the diameter of graph* G *is* 2 *hence* $RCW(G)$ *is invalid but* $W(G)$, $WW(G)$ *and* $H(G)$ *are valid and same as in Theorem* 4.1. □

Corollary 5.3. *For the petersenbarbell graph* G *with* $m \geq 2$ *and* $n = 10$, $W_P(G) = (m-1)(5m+6n-35)$.

Proof. The number of 3 distance pair of vertices is $6(n-5)(m-1) + 5(m-1)^2$ by Theorem 5.2. So, $W_P(G) = (m-1)(5m+6n-35)$. □

Corollary 5.4. *For the petersenbarbell graph* G *with* $m = 2$ *and* $n = 10$, $TW(G) = 35$

Proof. In Theorem 5.2, when $m = 2$, hence $TW(G) = [5(m-1)^2]3 + [5(m-1)^2]4 = 35(m-1)^2 = 35$. □

Lemma 5.5. *For the petersenbarbell graph* G *with* $m \geq 2$ *and* $n = 10$, $\wedge(G) = 2[(n+5m-5)(n+5m-6)] - [5(n+5) + \frac{1}{2}(m-1)[(n-5)(m+48) + 70(m-1)]]$.

Proof. In Theorem 5.2, $| V(G) | = n + 5(m-1)$, $d = 4$ and $W(G) = 5(n+5) + \frac{1}{2}(m-1)[(n-5)(m+48) + 70(m-1)]$.
Then $\wedge(G) = 2[(n+5m-5)(n+5m-6)] - [5(n+5)$
$+ \frac{1}{2}(m-1)[(n-5)(m+48) + 70(m-1)]]$. □

Lemma 5.6. *For the petersenbarbell graph* G *with* $m \geq 2$ *and* $n = 10$, $R\wedge(G) = \frac{4}{3}(n+5) + \frac{1}{6}(m+9)(m-1)(n-5) + (m-1)(5m+6n-35)$.

Proof. Here $d = 4$, from Theorem 5.2 and the distance between any pair of vertices satisfies $0 < d(u,v) < d$.
The number of 1 distance, pair of vertices is $(n+5) + (n-5)\binom{m}{2}$.
The number of 2 distance, pair of vertices is $2(n+5) + 3(n-5)(m-1)$.

The number of 3 distance, pair of vertices is $6(n-5)(m-1)+5(m-1)^2$. By using these we derive the following

$$\begin{aligned} R \wedge (G) &= [(n+5) + (n-5)\binom{m}{2}]\frac{1}{3} + [2(n+5) + 3(n-5)(m-1)]\frac{1}{2} \\ &\quad + [6(n-5)(m-1) + 5(m-1)^2]\frac{1}{1} \\ &= \frac{4}{3}(n+5) + \frac{1}{6}(m+9)(m-1)(n-5) + (m-1)(5m+6n-35). \end{aligned}$$

□

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