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GENERALIZED TOTALLY POSITIVE FUNCTIONS AND GENERALIZED TOTALLY BOUNDED FUNCTIONS IN A PARTICULAR ECT-SPACE

Abstract

In this paper, we give an example of a generalized totally positive function and a generalized totally bounded function in the ECT - space generated by $\{\frac{1}{x+1}, \frac{1}{x+2}, \ldots \frac{1}{x+n}\}$.

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1 Introduction

Totally positive functions and totally bounded functions on $[-1, 1]$ have been introduced and studied by Alan L.Horwitz and Lee A.Rubel in [2]. A generalization of these functions using generalized Lagrange interpolants is introduced in [6, 7] and some properties of these functions are also studied there. In this paper, we give examples of these generalized functions in a particular ECT-space.

2 Preliminaries

The chapters 2 and 9 of [5] are chiefly referred for the definitions and results given in this section and these results are necessary to prove our main theorem in the fourth section. In the first two sections of this paper, I denotes the closed bounded interval $[a, b]$ of $(-\infty, \infty)$.

Definition 2.1. *Let u_1, u_2, ..., u_m be real valued functions defined on $I = [a, b]$ and let $x_1 \leq x_2 \leq \ldots \leq x_m$ be points in I. The collocation matrix associated with $\{u_i\}_1^m$ and $\{x_i\}_1^m$ is denoted by $M \begin{pmatrix} x_1, & \ldots, & x_{m-1}, & x_m \\ u_1, & \ldots, & u_{m-1}, & u_m \end{pmatrix}$ and is defined by*

$$M \begin{pmatrix} x_1, & \ldots, & x_{m-1}, & x_m \\ u_1, & \ldots, & u_{m-1}, & u_m \end{pmatrix} = [D^{(d_i)} u_j(x_i)]_{i,j=1}^m$$

where $d_i = \mathcal{N}\{j : j < i, x_j = x_i\}$, $i = 1, 2, \ldots m$, provided the d_i^{th} derivative of u_j exists at the points x_i, $i, j = 1, 2, \ldots, m$ and $\mathcal{N}(A)$ denotes the number of elements in the set A.

Definition 2.2. *In Definition 2.1, if all the points coincide , that is, if* $x_1 = x_2 = \cdots = x_m$ *all equal to* x_0*(say), then the determinant of the collocation matrix is called the Wronskian and is denoted by* $W(u_1, u_2, \ldots, u_m)(x_0)$*. That is,*

$$W(u_1, u_2, \ldots, u_m)(x_0) = \det \begin{bmatrix} u_1(x_0) & \ldots & u_{m-1}(x_0) & u_m(x_0) \\ u_1'(x_0) & \ldots & u_{m-1}'(x_0) & u_m'(x_0) \\ \vdots & \ddots & \vdots & \vdots \\ u_1^{(m-1)}(x_0) & \ldots & u_{m-1}^{(m-1)}(x_0) & u_m^{(m-1)}(x_0) \end{bmatrix}$$

Definition 2.3. *Let* $U_m = \{u_i\}_{i=1}^m$ *be any collection of functions in* $C^{m-1}[I]$*, the space of all* $(m-1)$*-times continuously differentiable functions on* I*.* U_m *is called an extended Tchebycheff system (ET-system) on* I *if the determinants associated with the collocation matrices* $M \begin{pmatrix} x_1, & \ldots, & x_{m-1}, & x_m \\ u_1, & \ldots, & u_{m-1}, & u_m \end{pmatrix}$ *are positive for all* $x_1 \leq x_2 \leq \ldots \leq x_m$ *in* I*.*

Let $\{u_1, u_2, \ldots\}$ *be any finite or infinite sequence of functions in* I*. If for each* k*,* $\{u_1, \ldots, u_k\}$ *forms an ET-system on* I*, then* $\{u_1, u_2, \ldots\}$ *is called an extended complete Tchebycheff system (ECT-system) on* I*.*

Remark 2.1. *The Determinant associated with the matrix* $M \begin{pmatrix} x_1, & \ldots, & x_{m-1}, & x_m \\ u_1, & \ldots, & u_{m-1}, & u_m \end{pmatrix}$ *is denoted by* $D \begin{pmatrix} x_1, & \ldots, & x_{m-1}, & x_m \\ u_1, & \ldots, & u_{m-1}, & u_m \end{pmatrix}$*. The determinant of the collocation matrix arising from an ECT-system* $U_m = \{u_1, u_2, \ldots u_m\}$ *is denoted by* $D_{U_m}(x_1, x_2, \ldots, x_m)$*.*

Definition 2.4. *A subspace of* $C(I)$*, finite or infinite dimensional, is called an Extended complete Tchebycheff space (ECT-space) if it has an ordered basis which is an ECT-system.*

Elements of an ECT-space are called generalized polynomials.

Definition 2.5. *Let* $U_m = \{u_1, u_2, \ldots u_m\}$ *be an ECT-system on* I*, and let* f *be a sufficiently differentiable function defined on* I*. Associated with any* m *points* x_1*,* x_2*, ...,* x_m *in* I*, not necessarily distinct, we define a function on* I *as follows:*

$$D \begin{pmatrix} x_1, & \ldots, & x_m; & x \\ u_1, & \ldots, & u_m, & f \end{pmatrix} = \det \begin{bmatrix} u_1^{(d_1)}(x_1) & \ldots & u_m^{(d_1)}(x_1) & f^{(d_1)}(x_1) \\ \vdots & \ddots & \vdots & \vdots \\ u_1^{(d_i)}(x_i) & \ldots & u_m^{(d_i)}(x_i) & f^{(d_i)}(x_i) \\ \vdots & \ddots & \vdots & \vdots \\ u_1^{(d_m)}(x_m) & \ldots & u_m^{(d_m)}(x_m) & f^{(d_m)}(x_m) \\ u_1(x) & \ldots & u_m(x) & f(x) \end{bmatrix}$$

where $d_i = \mathcal{N}\{j : j < i,\ x_j = x_i\}$*,* $i = 1, 2, \ldots m$*, provided the* $d_i{}^{th}$ *derivative of* u_j *exists at the points* x_i*,* $i, j = 1, 2, \ldots, m$*.*

Remark 2.2. *If* $U_{m+1} = \{u_1, u_2, \ldots u_m, u_{m+1}\}$ *is an ECT-system on* $I = [a, b]$*, then the function in Definition 2.5 with* f *replaced by* u_{m+1} *is denoted by* $D_{U_{m+1}}(x_1, \ldots, x_m; x)$

Example 1. *A well-known example of an infinite ECT-system on any interval* $I = [a, b]$ *is* $P = \{1, x, x^2, \ldots\}$. *For each* n, $P_n = \{1, x, \ldots, x^{n-1}\}$ *is an ECT-system forming a basis for* $\mathcal{P}_n$, *the space of all polynomials of degree atmost* $n - 1$.

Remark 2.3 (cf.[5], p.30). *When* $x_1, \ldots, x_n$ *are distinct points in* I, *the Vandermonde determinant,* $D_{P_n}(x_1, \ldots, x_n)$, *has the following value:*

$$D_{P_n}(x_1, x_2, \ldots, x_n) = \prod_{\substack{i,j=1 \\ i>j}}^{n} (x_i - x_j)$$

An equivalent condition for a set of functions to form an ECT-system is given in the next result.

Theorem 2.1 (cf.[5], p.363). *A set of functions* u_1, u_2, …, u_m *in* $C^{m-1}[I]$ *form an ECT-system if and only if their Wronskian determinants are positive for all* $x \in I$. *That is,* $W(u_1, u_2, \ldots, u_k)(x) > 0$ *for all* $x \in I$, $k = 1, 2, \ldots, m$.

Definition 2.6. *Suppose* $U_m = \{u_1, u_2, \ldots, u_m\}$ *is an ECT−system in* I. *Given any sufficiently differentiable function* f *defined on* I, *its* $(m-1)$*th order divided difference with respect to* U_m *is defined by*

$$[x_1, \ldots, x_{m-1}, x_m]_{U_m} f = \frac{D\begin{pmatrix} x_1 & , \ldots , & x_{m-1}, & x_m \\ u_1 & , \ldots , & u_{m-1}, & f \end{pmatrix}}{D\begin{pmatrix} x_1 & , \ldots , & x_{m-1}, & x_m \\ u_1 & , \ldots , & u_{m-1}, & u_m \end{pmatrix}}$$

where $x_1 \leq x_2 \leq \ldots \leq x_m$ *are given points in* I

If $\mathcal{U}_m$ is an ECT-space on $I = [a, b]$ and if $x_1 < \ldots < x_m$ are prescribed points in I, then for any given function f defined on I, there corresponds a unique generalized polynomial in $\mathcal{U}_m$ which interpolates to f at the points $x_1 < \ldots < x_m$ (cf. [5]). In [8], we have derived an explicit expression for this unique generalized polynomial p_m in $\mathcal{U}_m$ and also an expression for the error $f - p_m$.

Theorem 2.2 (cf [8]). *Let* $\mathcal{U}_m$ *be an ECT-space on* $I = [a, b]$ *and let* x_1, x_2, …, x_m *be distinct points in* I. *Let* f *be any function defined on* I, *which is sufficiently differentiable.*

(i) Then an explicit expression for the unique generalized polynomial p_m *in* $\mathcal{U}_m$ *satisfying the conditions*

$$p_m(x_i) = f(x_i) \qquad i = 1, 2, \ldots, m.$$

is given by

$$p_m(x) = [x_1]_{U_1} f . D_{U_1}(x) + [x_1, x_2]_{U_2} f . \frac{D_{U_2}(x_1; x)}{D_{U_1}(x_1)} + \cdots + [x_1, \ldots, x_m]_{U_m} f . \frac{D_{U_m}(x_1, \ldots, x_{m-1}; x)}{D_{U_{m-1}}(x_1, \ldots, x_{m-2}, x_{m-1})}.$$

where $U_m = \{u_1, u_2, \ldots, u_m\}$ *is an ECT-system forming a basis for* $\mathcal{U}_m$ *and* $U_k = \{u_1, u_2, \ldots, u_k\}$, $k = 1, 2, \ldots, m$

(ii) The error is given by

$$f(x) - p_m(x) = [x_1, \ldots, x_m; x]_{U_{m+1}} f. \; \frac{D_{U_{m+1}}(x_1, \ldots, x_m; x)}{D_{U_m}(x_1, \ldots, x_m)}.$$

where $U_{m+1} = \{u_1, u_2, \ldots, u_m, u_{m+1}\}$ is an ECT-system on I containing U_m.

3 Generalizations and associated results

Throughout this section, $\mathcal{U}$ denotes an infinite dimensional ECT-space on $I = [a, b]$ with an ordered basis $U = \{u_1, u_2, \ldots\}$ which is an ECT-system on I. For each m, $\mathcal{U}_m$ is the ECT-space spanned by $U_m = \{u_1, u_2, \ldots, u_m\}$.

Definition 3.1. *Let f be a real valued function defined on I. A generalized polynomial p is called a generalized Lagrange interpolant to f from $\mathcal{U}$ if for some m, there exists distinct points x_1, x_2, …, x_m in I such that p is the unique generalized polynomial in $\mathcal{U}_m$ interpolating to f at x_1, x_2, …, x_m . We denote this generalized Lagrange interpolant to f by $L(x_1, \ldots, x_m; f)$*

The collection of all generalized Lagrange interpolants to f from $\mathcal{U}$ is denoted by $\mathcal{GL}_{\mathcal{U}}(f)$.

Definition 3.2. *Let f be a sufficiently differentiable function defined on I. A generalized polynomial q is called a generalized Taylor interpolant to f from $\mathcal{U}$ if for some m, there exists a point x_0 in I such that q is the unique generalized polynomial in $\mathcal{U}_m$ interpolating to f and its $m-1$ derivatives at x_0.*

The collection of all generalized Taylor interpolants to f from $\mathcal{U}$ is denoted by $\mathcal{GT}_{\mathcal{U}}(f)$.

Definition 3.3. *A function f defined on I is said to be generalized totally positive on I if $p > 0$ on I for all $p \in \mathcal{GL}_{\mathcal{U}}(f)$.*

The collection of all generalized totally positive functions on I is denoted by $\mathcal{GTP}_{\mathcal{U}}$.

Definition 3.4. *A real valued function f defined on I is said to be generalized totally bounded on I if there exists an M such that*

$$|p(x)| \leq M$$

for all $p \in \mathcal{GL}_{\mathcal{U}}(f)$ and for all $x \in I$.

The collection of all generalized totally bounded functions on I is denoted by $\mathcal{GTB}_{\mathcal{U}}$.

4 Main Results

Let $I = [a, b]$ be any sub interval of $[0, \infty)$ and let $0 < s_1 < s_2 < \cdots < s_n < \cdots$ be an infinite sequence of real numbers. Consider the function $u_i(x) = \frac{1}{s_i + x}$ for $i = 1, 2, \ldots,$ defined on I. This is a collection of infinitely differentiable functions arising from the Cauchy kernel and therefore forms a complete Tchebycheff system on I. (cf. [4], p.no.11). First we will show that the collection of functions $\{u_i\}$ forms an ECT-system on I .

Proposition 4.1. *Let $I = [a, b]$ and let $0 < s_1 < s_2 < \cdots < s_n$ be an increasing sequence of real numbers. Let $u_i(x) = \frac{1}{s_i+x}$ for $i = 1, 2, \ldots, n$. Then $U_n = \{u_1, u_2, \ldots, u_n\}$ form an ECT-system on I.*

Proof. First we will show that the wronskian determinants $W(u_1, u_2, \ldots, u_k)(x)$ are positive for all x in I and for $k = 1, 2, \ldots n$. Fix $x \in I$. Also fix k.

Now

$$W(u_1, u_2, \ldots, u_k)(x) = \det \begin{bmatrix} \frac{1}{s_1+x} & \frac{1}{s_2+x} & \cdots & \frac{1}{s_k+x} \\ \frac{-1}{(s_1+x)^2} & \frac{-1}{(s_2+x)^2} & \cdots & \frac{-1}{(s_k+x)^2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{(-1)^{k-1}(k-1)!}{(s_1+x)^k} & \frac{(-1)^{k-1}(k-1)!}{(s_2+x)^k} & \cdots & \frac{(-1)^{k-1}(k-1)!}{(s_k+x)^k} \end{bmatrix}$$

$$= \frac{(-1)^{1+2+\ldots(k-1)}.1!.2!.\ldots(k-1)!}{(s_1+x)(s_2+x)\ldots(s_k+x)} \cdot \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \frac{1}{s_1+x} & \frac{1}{s_2+x} & \cdots & \frac{1}{s_k+x} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{(s_1+x)^{k-1}} & \frac{1}{(s_2+x)^{k-1}} & \cdots & \frac{1}{(s_k+x)^{k-1}} \end{vmatrix}$$

$$= \frac{(-1)^{\frac{k(k-1)}{2}}.1!.2!.\ldots(k-1)!}{(s_1+x)(s_2+x)\ldots(s_k+x)} \begin{vmatrix} 1 & 1 & \ldots & 1 \\ y_1 & y_2 & \cdots & y_k \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{k-1} & y_2^{k-1} & \ldots & y_k^{k-1} \end{vmatrix}$$

where $y_i = \frac{1}{s_i+x}$. By using the value of the Vandermonde determinant given in Remark 2.3, we have

$$W(u_1, u_2, \ldots, u_k)(x) = \frac{(-1)^{\frac{k(k-1)}{2}}.1!.2!.\ldots(k-1)!}{(s_1+x)(s_2+x)\ldots(s_k+x)} \cdot \prod_{\substack{i,j=1 \\ i>j}}^{k} (y_i - y_j)$$

Now

$$y_i - y_j = \frac{1}{s_i+x} - \frac{1}{s_j+x} = \frac{-(s_i - s_j)}{(s_i+x)(s_j+x)}$$

Thus

$$W(u_1,u_2,\ldots,u_k)(x) = \frac{(-1)^{\frac{k(k-1)}{2}}.1!.2!.\ldots(k-1)!}{(s_1+x)(s_2+x)\ldots(s_k+x)}.(-1)^{(k-1)+(k-2)+\cdots+1}.$$
$$\prod_{\substack{i,j=1\\ i>j}}^{k}\frac{s_i-s_j}{(s_i+x)(s_j+x)}$$
$$= \frac{(-1)^{\frac{k(k-1)}{2}}.1!.\ldots(k-1)!}{(s_1+x)\ldots(s_k+x)}.(-1)^{\frac{k(k-1)}{2}}.\prod_{\substack{i,j=1\\ i>j}}^{k}\frac{s_i-s_j}{(s_i+x)(s_j+x)}$$
$$= \frac{1!.\ldots(k-1)!}{(s_1+x)\ldots(s_k+x)}.\prod_{\substack{i,j=1\\ i>j}}^{k}\frac{s_i-s_j}{(s_i+x)(s_j+x)} > 0$$

Therefore, by Theorem 2.1, $\{u_1,u_2,\ldots,u_n\}$ forms an ECT-system on I.

□

From here onwards, in this section, we will restrict our discussion to $J=[0,1]$. From the previous theorem, the collection of functions $\{\frac{1}{1+x},\frac{1}{2+x},\ldots\frac{1}{n+x}\}$ form an ECT-system on J.

Notation 1. *We use the notation $\mathcal{R}^*$ to denote the ECT-space on $J=[0,1]$ spanned by $R^*=\{\frac{1}{1+x},\frac{1}{2+x},\ldots\frac{1}{n+x},\ldots\}$. Also the notation $\mathcal{R}_n^*$ is used to denote the ECT-space spanned by $R_n^*=\{\frac{1}{1+x},\frac{1}{2+x},\ldots\frac{1}{n+x}\}$ on J.*

We next consider the function f_s defined on $[0,1]$ by

$$f_s(x)=\frac{1}{x+s} \tag{4.1}$$

For this function to be well defined on $[0,1]$ it is necessary and sufficient that s is a real number outside the interval $[-1,0]$. Here we first obtain an expression for the error function

$$f_s-p_n$$

where p_n is the unique generalized polynomial in $\mathcal{R}_n^*$ interpolating to f at n distinct points $x_1, x_2, \ldots, x_n$ in $[0,1]$. For this we need the value of Cauchy determinant which is given by the following lemma(cf.[3]).

Lemma 4.1 ([3], p.268). *If $a_i+b_j\neq 0$ for $i,\ j=1,\ 2,\ \ldots,\ n$ and if*

$$D_n=\begin{vmatrix} \frac{1}{a_1+b_1} & \frac{1}{a_1+b_2} & \cdots & \frac{1}{a_1+b_n}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{1}{a_n+b_1} & \frac{1}{a_n+b_2} & \cdots & \frac{1}{a_n+b_n}\end{vmatrix}$$

then

$$D_n=\frac{\prod_{i>j}^{n}(a_i-a_j)(b_i-b_j)}{\prod_{i,j=1}^{n}(a_i+b_j)}$$

Proposition 4.2. *For any real number s, s not in $[-1,0]$, let f_s be defined on $[0,1]$ as in equation 4.1. Let p_n be the unique generalized polynomial in $\mathcal{R}_n^*$ interpolating to f at n distinct points $x_1, x_2, \ldots, x_n$ in $J=[0,1]$. Then for all x in $[0,1]$,*

$$f_s(x)-p_n(x)=\frac{\prod_{k=1}^{n}(x-x_k).\prod_{k=1}^{n}(s-k)}{\prod_{k=1}^{n}(x_k+s)\prod_{k=1}^{n}(x+k)}.\frac{1}{x+s}$$

Proof. Given that p_n is the generalized polynomial in $\mathcal{R}_n^*$ interpolating to f_s at the points $x_1, x_2, \ldots, x_n$ in $[0,1]$. . By (ii) of Theorem 2.2, the error of interpolation is given by

$$f_s(x)-p_n(x)=\frac{D\begin{pmatrix} x_1 & ,\ldots, & x_n; & x\\ u_1 & ,\ldots, & u_n, & f\end{pmatrix}}{D\begin{pmatrix} x_1 & ,\ldots, & x_{n-1}, & x_n\\ u_1 & ,\ldots, & u_{n-1}, & u_n\end{pmatrix}}$$

$$=\frac{\det\begin{bmatrix} \frac{1}{x_1+1} & \cdots & \frac{1}{x_1+n} & \frac{1}{x_1+s}\\ \vdots & \vdots & \vdots & \vdots\\ \frac{1}{x_n+1} & \cdots & \frac{1}{x_n+n} & \frac{1}{x_n+s}\\ \frac{1}{x+1} & \cdots & \frac{1}{x+n} & \frac{1}{x+s}\end{bmatrix}}{\det\begin{bmatrix} \frac{1}{x_1+1} & \cdots & \frac{1}{x_1+n}\\ \vdots & \vdots & \vdots\\ \frac{1}{x_n+1} & \cdots & \frac{1}{x_n+n}\end{bmatrix}}$$

$$=\frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_1+s)(x_2+s)\ldots(x_n+s)}.\frac{(s-1)\ldots(s-n)}{(x+1)\ldots(x+n)}.\frac{1}{x+s}$$

$$=\frac{\prod_{k=1}^{n}(x-x_k).\prod_{k=1}^{n}(s-k)}{\prod_{k=1}^{n}(x_k+s)\prod_{k=1}^{n}(x+k)}.\frac{1}{x+s}$$

□

We now prove that $f_s \in \mathcal{GTB}_{\mathcal{R}^*}$ whenever $s<-2$ or $s\geq 1$.

Theorem 4.1. *The function f_s has the following properties.*

(i) f_s belongs to $\mathcal{GTB}_{\mathcal{R}^*}$, if $s\geq 1$ or $s<-2$.

(ii) f_s belongs to $\mathcal{GTP}_{\mathcal{R}^*}$, if $s\geq 1$

Proof. Let $x_1, \ldots, x_n$ be any n distinct points in $[0,1]$ and let p_n be the unique generalized polynomial in $\mathcal{R}_n^*$ interpolating to f_s at $x_1, \ldots, x_n$. Then by Proposition 4.2, for all x in $[0,1]$,

$$f_s(x)-p_n(x)=\frac{\prod_{k=1}^{n}(x-x_k).\prod_{k=1}^{n}(s-k)}{\prod_{k=1}^{n}(x_k+s)\prod_{k=1}^{n}(x+k)}.\frac{1}{x+s}$$

For any $x\in[0,1]$,

$$|x-x_k|\leq 1 \quad \text{and} \quad |x+k|=x+k\geq k.$$

Consequently,

$$\frac{\prod_{k=1}^{n}|x-x_k|}{\prod_{k=1}^{n}|x+k|} \leq \frac{1}{\prod_{k=1}^{n}k} = \frac{1}{n!}.$$

Therefore, for $0 \leq x \leq 1$,

$$|f_s(x) - p_n(x)| \leq \frac{1}{n!}.\frac{\prod_{k=1}^{n}|s-k|}{\prod_{k=1}^{n}|x_k+s|}.\frac{1}{|x+s|} \tag{4.2}$$

We now consider the cases $s \geq 1$ and $s < -2$ separately.

Case(i) : $s \geq 1$

Then

$$|s + x_k| = s + x_k \geq s$$

Therefore,

$$\prod_{k=1}^{n}|s+x_k| \geq s^n \tag{4.3}$$

From inequalities 4.2 and 4.3,

$$\begin{aligned}|f_s(x) - p_n(x)| &\leq \frac{1}{n!}.\frac{\prod_{k=1}^{n}|s-k|}{s^n}.\frac{1}{|x+s|} \\ &= \prod_{k=1}^{n}\frac{|s-k|}{ks}.\frac{1}{|x+s|} \\ &= \frac{1}{|x+s|}.\prod_{k=1}^{n}\left|\frac{1}{k}-\frac{1}{s}\right|\end{aligned}$$

Now $\frac{1}{k} \in (0,1)$ and $\frac{1}{s} \in (0,1]$. Therefore,

$$\left|\frac{1}{k}-\frac{1}{s}\right| < 1 \qquad (k = 1, \ldots n).$$

and so

$$\prod_{k=1}^{n}\left|\frac{1}{k}-\frac{1}{s}\right| < 1.$$

Consequently,

$$|f_s(x) - p_n(x)| < \frac{1}{|x+s|} = \frac{1}{x+s}$$

That is,

$$\left|\frac{1}{x+s} - p_n(x)\right| < \frac{1}{x+s}, \quad 0 \leq x \leq 1.$$

Equivalently, for $0 \leq x \leq 1$,

$$\frac{1}{x+s} - p_n(x) < \frac{1}{x+s} \quad \text{and} \quad p_n(x) - \frac{1}{x+s} < \frac{1}{x+s}.$$

Therefore, for $0 \leq x \leq 1$,

$$0 < p_n(x) < \frac{2}{x+s} \leq 2.$$

Since p_n is an arbitrary generalized Largange interpolant to f_s from $\mathcal{R}^*$, it follows that f_s is generalized totally positive and generalized totally bounded with respect to $\mathcal{R}^*$, if $s \geq 1$. That is,

$$f_s \in \mathcal{G}TP_{\mathcal{R}^*}\ , \quad \text{if} \quad s \geq 1 \tag{4.4}$$

and

$$f_s \in \mathcal{G}TB_{\mathcal{R}^*}\ , \quad \text{if} \quad s \geq 1 \tag{4.5}$$

Case(ii) : $s \leq -2$

Now $s \leq -2$ implies $|s| > 2$. Choose a positive integer m such that $m \leq |s| < m+1$ .Thus for $k = 1, 2, \ldots, n$, $|s-k| = |s| + k < m + 1 + k$. Therefore,

$$\begin{aligned}\prod_{k=1}^{n} |s-k| &< \prod_{k=1}^{n} (m+1+k) \\ &= (m+2)(m+3)\ldots(m+1+n) \\ &= \frac{(m+n+1)!}{(m+1)!}\end{aligned} \tag{4.6}$$

Again,

$$\prod_{k=1}^{n} |x_k + s| \geq \prod_{k=1}^{n} (|s| - x_k) \geq \prod_{k=1}^{n} (|s|-1)^n. \tag{4.7}$$

Hence, from inequalities 4.2, 4.6 and 4.7, since $0 \leq x \leq 1$, we have

$$\begin{aligned}|f_s(x) - p_n(x)| &< \frac{1}{n!}\frac{(m+n+1)!}{(m+1)!}\frac{1}{(|s|-1)^n}\frac{1}{x+s} \\ &= \binom{m+n+1}{m+1}\left(\frac{1}{(|s|-1)}\right)^n \frac{1}{x+s} \\ &= \binom{m+n+1}{m+1}\left(\frac{1}{(|s|-1)}\right)^{n+1}\end{aligned} \tag{4.8}$$

Now

$$\binom{m+n+1}{m+1}.\left(\frac{|s|}{2}-1\right)^{m+1} < \left(1+\left(\frac{|s|}{2}-1\right)\right)^{m+n+1} = \left(\frac{|s|}{2}\right)^{m+n+1}$$

Therefore,

$$\begin{aligned}\binom{m+n+1}{m+1} &= \frac{(\frac{|s|}{2})^{m+n+1}}{\frac{|s|}{2}-1)^{m+1}} \\ &= \left(\frac{|s|}{|s|-2}\right)^{m+1}.\left(\frac{|s|}{2}\right)^n\end{aligned} \tag{4.9}$$

From inequalities 4.8 and 4.9,

$$\begin{aligned}|f_s(x) - p_n(x)| &\leq \left(\frac{|s|}{|s|-2}\right)^{m+1}.\left(\frac{|s|}{2}\right)^n \frac{1}{(|s|-1)^{n+1}} \\ &\leq \left(\frac{|s|}{|s|-2}\right)^{m+1}.\left(\frac{|s|/2}{|s|-1}\right)^n \\ &\leq K\left(\frac{|s|}{2(|s|-1)}\right)^n\end{aligned} \tag{4.10}$$

where $K = \left(\frac{|s|}{|s|-2}\right)^{m+1}$. Since $|s| > 2$, we have

$$\frac{|s|}{2(|s|-1)} < 1.$$

Therefore, from inequality 4.10, we see that

$$|f_s(x) - p_n(x)| \leq K, \qquad \text{where} \qquad K = \left(\frac{|s|}{|s|-2}\right)^{m+1}.$$

Thus

$$\begin{aligned} |p_n(x)| \leq |f_s(x)| + K &= \frac{1}{|x+s|} + K \\ &\leq \frac{1}{|s|-1} + K < 1 + K \end{aligned}$$

$$|p_n(x)| \leq 1 + K, \qquad \text{for all} \quad x \in [0,1] \tag{4.11}$$

Estimate 4.11 holds for all generalized Lagrange interpolants to f_s. Therefore,

$$f_s \in \mathcal{G}TB_{\mathcal{R}^*}\ , \quad \text{for} \quad s < -2 \tag{4.12}$$

Thus $f_s \in \mathcal{G}TP_{\mathcal{R}^*}$, if $s \geq 1$ or $s < -2$. Further, we have seen that

$$f_s \in \mathcal{G}TP_{\mathcal{R}^*}\ , \quad \text{if} \quad s \geq 1.$$

The proof of Theorem 4.1 is now complete.

□

Now we will prove the converse of the above theorem.

Theorem 4.2. *A necessary condition for f_s to belong to $\mathcal{G}TB_{\mathcal{R}^*}$ is that s lies outside the interval $[-2, 1)$.*

Proof. Now $[-2,1)$ is the union of the intervals $(0,1)$, $[-1,0]$ and $[-2,-1)$. We prove separately that if s is in any of these intervals, then f_s does not belong to $\mathcal{G}TB_{\mathcal{R}^*}$

Case (i) : s is in $(0,1)$.

In this case consider the generalized Lagrange interpolant p_n to f_s from $\mathcal{G}TB_{\mathcal{R}^*}$ interpolating to f_s at the n points $x_k = \frac{k}{(n+1)^2}$, $k = 1, 2, \ldots, n$. By Proposition 4.2, for all x in $[0,1]$,

$$f_s(x) - p_n(x) = \frac{\prod_{k=1}^{n}(x - x_k)}{\prod_{k=1}^{n}(x+k)} . \frac{\prod_{k=1}^{n}(s-k)}{\prod_{k=1}^{n}(x_k+s)} . \frac{1}{x+s}.$$

Therefore,

$$f_s(1) - p_n(1) = \frac{\prod_{k=1}^{n}(1 - x_k)}{\prod_{k=1}^{n}(1+k)} . \frac{\prod_{k=1}^{n}(s-k)}{\prod_{k=1}^{n}(x_k+s)} . \frac{1}{x+s}.$$

Since $0 < s < 1$, in this case,

$$|f_s(1) - p_n(1)| = \frac{\prod_{k=1}^{n}\left(1 - \frac{k}{(n+1)^2}\right)}{(n+1)!} \cdot \frac{\prod_{k=1}^{n}(k-s)}{\prod_{k=1}^{n}(x_k+s)} \cdot \frac{1}{1+s}.$$
$$> \frac{\left(1 - \frac{1}{(n+1)}\right)^n}{(n+1)!} \cdot \frac{(n-1)!}{(s+\frac{1}{n+1})^n} \cdot \frac{1}{1+s}.$$

Now

$$\left(1 - \frac{1}{(n+1)}\right)^n = \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(1+\frac{1}{n}\right)^n} > \frac{1}{e}$$

Also

$$\frac{(n-1)!}{(n+1)!.(1+s)} = \frac{1}{n(n+1)(1+s)} > \frac{1}{2n^2}.$$

Therefore,

$$|f_s(1) - p_n(1)| > \frac{1}{2e} \cdot \frac{1}{n^2\left(s+\frac{1}{n+1}\right)^n}$$

Choose s_1, s_2 such that $s < s_1 < s_2 < 1$. Since $0 < s < 1$, $s + \frac{1}{n+1} < s_1$ for all large n. Therefore, for large n,

$$n^{\frac{2}{n}}(s + \frac{1}{n+1}) < n^{\frac{2}{n}} s_1 < s_2,$$

since $n^{\frac{2}{n}} \to 1$ as $n \to \infty$. Therefore, for large n,

$$|f_s(1) - p_n(1)| > \frac{1}{2e} \cdot \frac{1}{s_2^n} \to \infty \quad \text{as } n \to \infty.$$

It follows that , in this case,

$$f_s \notin \mathcal{GTB}_{\mathcal{R}^*}.$$

Case (ii) : s is in $[-1, 0]$.

In this case, f_s is not even defined on $[0, 1]$. Hence trivially, for $-1 \leq s \leq 0$

$$f_s \notin \mathcal{GTB}_{\mathcal{R}^*}.$$

Case (iii) : $-2 \leq s < -1$

In this case, consider the generalized Lagrange interpolant q_n to f_s from $\mathcal{R}^*$ interpolating to f_s at the points

$$x'_k = \frac{n}{n+1} + \frac{k}{(n+1)^2}, \quad k = 1, 2, \ldots, n.$$

Then, using Proposition 4.2,

$$|f_s(0) - q_n(0)| = \frac{\prod_{k=1}^{n}|x'_k|}{\prod_{k=1}^{n} k} \cdot \frac{\prod_{k=1}^{n}|s-k|}{\prod_{k=1}^{n}|x_k+s|} \cdot \frac{1}{|s|}.$$

Now

$$\prod_{k=1}^{n}|x'_k| > \prod_{k=1}^{n}\frac{n}{n+1} = \left(\frac{n}{n+1}\right)^n = \frac{1}{(1+\frac{1}{n})^n} > \frac{1}{e}$$

Also since s is negative, $s-k<0$. Thus $|s-k|=-s+k=|s|+k>1+k$, since $1<|s|\leq 2$. Therefore,

$$|f_s(0)-q_n(0)|>\frac{1}{e}.\frac{(n+1)!}{n!}.\frac{1}{\prod_{k=1}^{n}|s+\frac{n}{n+1}+\frac{k}{(n+1)^2}|}.\frac{1}{2},$$

since $|s|<2$. Now

$$\frac{n}{n+1}+\frac{k}{(n+1)^2}\leq\frac{n}{n+1}+\frac{n}{(n+1)^2}<\frac{n}{n+1}+\frac{n+1}{(n+1)^2}=1.$$

Also $s<-1$. Therefore,

$$s+\frac{n}{n+1}+\frac{k}{(n+1)^2}<0.$$

Hence

$$\begin{aligned}\left|s+\frac{n}{n+1}+\frac{k}{(n+1)^2}\right|&=-s-\frac{n}{n+1}-\frac{k}{(n+1)^2}\\&=|s|-\left(\frac{n}{n+1}+\frac{k}{(n+1)^2}\right)\\&=(|s|-1)+\frac{1}{n+1}+\frac{k}{(n+1)^2}\\&<|s|-1+\frac{1}{n+1}\end{aligned}$$

Consequently,

$$\begin{aligned}|f_s(0)-q_n(0)|&>\frac{n+1}{2e}.\left(|s|-1+\frac{1}{n+1}\right)^{-n}\\&\geq\frac{n+1}{2e}.\left(1+\frac{1}{n+1}\right)^{-n},\quad\text{since}\quad|s|-1\leq 1\\&>\frac{n+1}{2e}.\frac{1}{\left(1+\frac{1}{n+1}\right)^{n}}\\&>\frac{n+1}{2e}.\frac{1}{e}=\frac{n+1}{2e^2}\to\infty\quad\text{as}\quad n\to\infty.\end{aligned}$$

Thus $f_s\notin\mathcal{GTB}_{\mathcal{R}^*}$ in this case also. From cases (i), (ii) and (iii), it follows that

$$f_s\notin\mathcal{GTB}_{\mathcal{R}^*},\quad\text{if}\quad s\in[-2,1).$$

Equivalently,

$$f_s\in\mathcal{GTB}_{\mathcal{R}^*}\implies s\notin[-2,1).$$

□

Combining the above two theorems, we get the following theorem.

Theorem 4.3. *f_s belongs to $\mathcal{GTB}_{\mathcal{R}^*}$ if and only if s is a real number not belonging to $[-2,1)$.*

Proof. From part (i) of Theorem 4.1 and Theorem 4.2, it follows that

$$f_s\in\mathcal{GTB}_{\mathcal{R}^*}\iff s\notin[-2,1).$$

□

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