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DUALITY CRITERIA INVOLVING (p, φ ,d)-INVEXITY AND (p, φ, d) -PSEUODINVEXITY IN INTERVAL -VALUED MULTIOBJECTIVE OPTIMIZATION PROBLEMS

Abstract: In this present research work, study of duality associated with a special class of multiobjective optimization that include the interval valued components is delt. We define (ρ,φ,d)-Invexity and (ρ,φ,d-Pseudoinvexity, which are connected with an interval valued multiple integral functional. For such class of variational problems, we write dual problem associated with primal problem. We prove weak, strong and converse duality theorems for this type of variational problems. A brief comparison with existed methods have been done to show the importance of this research work. Additionally, numerical examples have been displayed at the appropriate places to support the results which shows the significance of our Study.

Keywords: Multiobjective Optimization, (ρ, φ, d)-Invexity and (ρ, φ, d)-Pseudoinvexity, Duality.

Mathematics Subject Classification (2010) No.: 58E17, 65K05, 90C46, 90C29, 26B25, 49K20, 49N15.

1. Introduction and Literature Review

Mathematical optimization problems involving multiple objective functions that must be optimized simultaneously fall under the purview of multi-objective optimization, also known as Pareto optimization or multi-objective programming, vector optimization, multicriteria optimization, or multiattribute optimization. Several scientific domains, such as engineering, economics, and logistics, have used multi-

objective vector optimization to make optimal judgments when there are trade-offs between two or more competing objectives. Multi-objective optimization issues with two or three objectives include things like optimizing performance while limiting fuel consumption and vehicle emissions, and minimizing cost while maximizing comfort while purchasing an automobile. There may be more than three objectives in practical tasks.

The application of duality theory to more general classes of functions has grown as a result of its success in mathematical programming. Kumar et al. [1] have considered multiobjective semi-infinite variational problem (MSVP) and generalised the concept of inveity. Kumar *et al*. [2] defined certain conditions on the functionals of multi-objective fractional variational problem in order that it becomes F-Kuhn Tucker pseudo invex or F-Fritz John pseudo invex. Bhardwaj and Ram [3] established the relationships between a class of interval-valued vector optimization problems and interval-valued vector variational-like inequality problems of both Stampacchia and Minty kinds in terms of convexificators.

Upadhyay *et al*. [4] dealt with a certain class of multiobjective semi-infinite programming problems with switching constraints (in short, MSIPSC) in the framework of Hadamard manifolds. Sahay and Bhatia [5] introduced new classes of higher order generalized strong invex functions under non-differentiable settings.

Soni *et al*. [6] discussed optimization problems with multiobjective functions and their applications in engineering field.

Zalmai [7] established global semiparametric sufficient efficiency results under various generalized $(\mathcal{F}, b, \phi, \rho, \theta)$-univexity assumptions for a multiobjective fractional subset programming problem. Hachimi and Aghezzaf [8] generalized a fairly large number of sufficient optimality conditions and duality results previously obtained for multiobjective variational problems. Treanţă [9] introduced necessary efficiency conditions for a class of multi-time vector fractional variational problems with nonlinear equality and inequality constraints involving higher-order partial derivatives. Treanţă [10] introduced a generalised condition on the functionals involved in a multidimensional vector control problem and prove that a (strongly) *b*-V-KT-pseudoinvex multidimensional control problem is characterized so that all Kuhn-Tucker points are efficient solutions. Kim [11] formulated duality for nondifferentiable multiobjective variational problems and established the weak, strong, and converse duality theorems under generalized (F, ρ)-convexity assumptions. Gulati *et al*. [12] obtained Fritz John and Kuhn-Tucker type necessary optimality conditions for a Pareto optimal (efficient) solution of a multiobjective control problem are by first reducing the multiobjective control

problem to a system of single objective control problems, and then using already established optimality conditions. Nahak and Nanda [13] presented the sufficient optimality criteria for a class of multiobjective variational control problems under the V-invexity assumption. They also proved duality results under a variety of V-invexity assumptions.

Antczak and Jiménez [14] generalized the notion of B-(p, r)-invexity and proved sufficient optimality conditions under the assumptions that the functions constituting them are B-(p, r)-invex. Antczak [15] extended the notions of (Φ, ρ)-invexity and generalized (Φ, ρ)-invexity to the continuous case and we use these concepts to establish sufficient optimality conditions for the considered class of nonconvex multiobjective variational control problems and established several mixed duality results are under (Φ, ρ)-invexity. Khazafi *et al*. [16] introduced the classes of (B, ρ)-type I and generalized (B, ρ)-type I, and derived various sufficient optimality conditions and mixed type duality results for multiobjective control problems under (B, ρ)-type I and generalized (B, ρ)-type I assumptions. Zhang *et al*. [17] extended the vector-valued G-invex functions to multiobjective variational control problems, by using this concept, a number of sufficient optimality results and Mond-Weir type duality results were obtained for multiobjective variational control programming problem. Treanţă and Arana [18] defined a Kuhn-Tucker (KT)-pseudoinvex multidimensional control problem and introduced a new condition on the functions, which were involved in a multidimensional control problem proved that a KT-pseudoinvex multidimensional control problem is characterized such that a KT point is an optimal solution. Mititelu [19] established necessary conditions for normal efficient solutions of a class of multiobjective fractional variational problem (MFP) with nonlinear equality and inequality constraints using a parametric approach to relate efficient solutions of fractional problems and a non-fractional problem and established the sufficiency of these conditions for efficiency solutions in problem (MFP) using the (ρ, b)-quasiinvexity notion.

Mititelu and Treanţă [20] formulated and proved necessary and sufficient optimality conditions in multiobjective control problems which involve multiple integral and under (ρ, b)-quasiinvexity assumptions, sufficient efficiency conditions for a feasible solution were derived. Treanţă and Mititelu [21] introduced several results of duality for a class of multiobjective fractional control problems involving multiple integrals and under (ρ, b)-quasiinvexity assumptions, they formulated and prove weak, strong and converse duality results. Treanţă [22] formulated and proved efficiency conditions for the considered uncertain variational control problem and established sufficiency of Karush-Kuhn-Tucker conditions under some invexity and (ρ, b)-quasiinvexity assumptions of the involved functionals. Treanţă [23] formulated and proved weak, strong, and converse duality results for the considered class of variational control problems by using the new notion of (ρ,ψ,d)-quasiinvexity

associated with an interval-valued multiple-integral functional. Treanţă [24] investigated some connections between an LU-optimal solution of a variational control problem governed by interval-valued multiple integral functional and a saddle-point associated with an LU-Lagrange functional corresponding to a modified interval-valued variational control problem.

In contrast to earlier studies, the current work addresses the duality study related to a novel class of multiobjective optimization problems that involve interval-valued ratio vector components. When taken into account simultaneously, these three emphasized components are completely novel in the relevant literature. Additionally, numerical example is given to show how useful the conclusions drawn in the study are.

The following table compares our study with the available literature in this field

Research Article	**Mutliobjective Optimization**	**Invexity and Pseudoinvexity**	**Inverval Valued Components**	**Duality Criteria**
Kumar *et al*. [1]	Yes	Generalised Invexity	No	No
Bhardwaj *et al*. [3]	No	Generalised Approximate Invexity	Yes	No
Upadhyay *et al*. [4]	Yes	No	No	Yes
Hachimi and Aghezzaf [8]	Yes	No	No	Yes
Kim [11]	Yes	No	No	Yes
Gulati *et al*. [12]	Yes	No	No	Yes
Nahak and Nanda [13]	Yes	V-Invexity	No	Yes
Antczak and Jiménez [14]	Yes	B-(p, r)-Invexity	No	Yes
Antczak [15]	Yes	No	No	Yes
Khazafi *et al*. [16]	Yes	Yes	No	No
Mititelu [19]	Yes	No	No	No
Treanţă and Mititelu [21]	Yes	(ρ, b)-Q uasiinvexity	No	Yes
Treanţă [23]	Yes	(ρ, ψ, d)-Quasiinvexity	Yes	Yes
Treanţă [24]	Yes	(p, b, d)-Invexity	Yes	No
Our Proposed Paper	**Yes**	**Both (ρ,φ,d)-Invexity and (ρ, φ,d)-Pseudoinvexity**	**Yes**	**Yes**

In the field of multiobjective optimization, somewhere invexity or pseudoinvexity were discussed, somewhere mutliobjective optimization with interval valued components were discussed, while somewhere duality results were discussed. To the best of our knowledge, all four components simultaneously with (ρ,φ,d)-Invexity and (ρ, φ, d)-Pseudoinvexity were not discussed, so there was a research gap in this field.

The structure of the paper is as follows: The problem formulation, preliminary mathematical tools, and notations are included in Section 2 of this article. The key findings are presented in Section 3 of this document. Results for Mond-Weir weak, strong, and converse duality are developed and demonstrated for the recently introduced category of multiobjective optimization problems. The paper is finally concluded in Section 4.

2. The formulation of Problem and Notations

This part presents the definitions, notations, and preliminary findings that will be utilized in the follow-up. Given this, we take into account:

Let us assume **Ω** be a compact domain which is a subset of Euclidean space $\mathbb{R}^m$ and a point in this compact domain **Ω** is represented by $t = (t^\alpha)$ where $\alpha = 1, 2 \ldots\ldots m$.

Now, following continuous differentiable functions are defined

$X = (X_\alpha^i) : \Omega \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^{mn}$ where $i = 1, 2 \ldots\ldots n$ and $\alpha = 1, 2 \ldots\ldots m$

$Y = (Y_1, Y_2 \ldots\ldots Y_q) = (Y_\beta) : \Omega \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^q$ where $\beta = 1, 2 \ldots\ldots q$

It is assumed that the functions that are continuously differentiable

$X_\alpha = (X_\alpha^i) : \Omega \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^{mn}$ where $i = 1, 2 \ldots\ldots n$ and $\alpha = 1, 2 \ldots\ldots m$

Satisfy the complete integrability conditions (closeness conditions)

$$D_\tau X_\alpha^i = D_\alpha X_\tau^i \text{ where } \alpha \neq \tau, \quad \alpha, \tau = 1, 2 \ldots\ldots m \text{ and } i = 1, 2 \ldots\ldots n$$

Where D_τ represent the total derivative operator.

If we consider any two vectors $d = (d_1, d_2.....d_s)$ and $e = (e_1, e_2.....e_s)$ in $\mathbb{R}^s$, then following partial ordering is used

$$d = e \Leftrightarrow d_r = e_r, \qquad d \leq e \Leftrightarrow d_r \leq e_r,$$

$$d < e \Leftrightarrow d_r < e_r, \qquad d \preceq e \Leftrightarrow d_r \leq e_r, d_r \neq e_r, r = 1, 2.....s$$

Now let us assume that K is the set of all closed and bounded real intervals, we represent a closed and bounded interval by $F = [f^L, f^U]$, where f^L and f^U are the lower and upper bounds of F, respectively. The interval operations covered in this paper can be carried out in the following ways:

(1) $F = G \Rightarrow f^L = g^L$ and $f^U = g^U$;

(2) if $f^L = f^U = f$ then $F = [f, f] = f$;

(3) $F + G = [f^l + g^L, f^U + g^U]$;

(4) $-F = -[f^L, f^U] = [-f^L, -f^U]$;

(5) For any $h \in R, h + F = \{h + f^L, h + f^U]$;

(6) For any $h \in R$ and $h \geq 0, hF = [hf^L, hf^U]$;

(7) For any $h \in R$ and $h < 0, hF = [hf^U, hf^L]$;

(8) $F - G = [f^L - g^L, f^U - g^U]$;

(9) $F/G = [f^L / g^L, f^U / g^U]$, where $g^L, g^U > 0$.

Now we have some following definitions

Definition 1: If F and G are two closed and bounded real intervals, i.e. $F, G \in K$, then we have

$$F \leq G \Leftrightarrow f^L \leq g^U \text{ and } f^U \leq g^U$$

Definition 2: If F and G are two closed and bounded real intervals, i.e. $F, G \in K$, then we have

$$F < G \Leftrightarrow f^L < g^U \text{ and } f^U < g^U$$

Definition 3: Interval valued Functions

If we define a function f from $\Omega \times \mathbb{R}^n \times \mathbb{R}^k$ to K, i.e. $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^k \to K$ such that

$$f(t, b(t), c(t)) = [f^L(t, b(t), c(t)), f^U(t, b(t), c(t))], \text{ where } t \in \Omega$$

Where both $f^L(t, b(t), c(t))$ and $f^U(t, b(t), c(t))$ are real valued functions and the condition $f^L(t, b(t), c(t)) \leq f^U(t, b(t), c(t)) \forall t \in \Omega$ is satisfied , then f is said to be an interval valued function.

The following (per Mititelu and Treantă [19], and Treantă [21]) was used to formulate and demonstrate the primary findings of this work, now we are going to introduce (ρ, φ, d)-Invexity and (ρ, φ, d)-Pseudoinvexity with the help of functional which is interval valued multiple integral.

For this first we consider an interval-valued function which is continuously differentiable

$$h : \Omega \times R^n \times R^{mn} \times R^k \to K \text{ such that}$$

$$h = h(t, b(t), b_\alpha(t), c(t)) = [h^L(t, b(t), b_\alpha(t), c(t)), h^U(t, b(t), b_\alpha(t), c(t))]$$

Where $b_\alpha(t)$ represents partial derivative of $b(t)$ with respect to t^α i.e. $b_\alpha(t) = \dfrac{\partial b}{\partial t^\alpha}(t)$.

Now for any $b \in B$ and $c \in C$, we define following interval-valued multiple integral functional:

$H : B \times C \to K$ such that

$$H(b, c) = \int_\Omega^{\cdot} h(t, b(t), b_\alpha(t), c(t))dt$$

$$= [\int_\Omega^{\cdot} h^L(t, b(t), b_\alpha(t), c(t))dt, \int_\Omega^{\cdot} h^U(t, b(t), b_\alpha(t), c(t))dt]$$

If ρ is a real number and $\varphi : B \times C \times B \times C \to [0, \infty)$ be a positive functional and $(d(b, c), (b^0, c^0))$ is a real valued function defined on $(B \times C)^2$.

Definition 4: (ρ, φ,d)-Invexity and (ρ, φ,d)-Pseudoinvexity

(i) Now if there exists a functional such that

$$\xi : \Omega \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n \text{ such that}$$

$\xi = \xi(t, b(t), c(t), b^0(t), c^0(t)) = (\xi_i(t, b(t), c(t), b^0(t), c^0(t)))$ where $i = 1, 2 \ldots..n$, of the C^1 class functional with $\xi(t, b(t), c(t), b^0(t), c^0(t)) = 0,\ \forall t \in \Omega,\ \xi|_{\partial\Omega} = 0$, and another functional such that

$$\eta : \Omega \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^K \quad \text{such that}$$

$\eta = \eta(t, b(t), c(t), b^0(t), c^0(t)) = (\eta_j(t, b(t), c(t), b^0(t), c^0(t)))$ where $j = 1, 2 \ldots..k$, of the C^0 class function with $\eta(t, b(t), c(t), b^0(t), c^0(t)) = 0,\ \forall t \in \Omega,\ \eta|_{\partial\Omega} = 0$ such that for each $(b, c) \in B \times C$:

$$H(b, c) \leq H(b^0, c^0)$$

$$\Rightarrow \varphi(b, c, b^0, c^0)\int_\Omega^{\cdot} [h_b^L(t, b^0(t), b_\alpha^0(t), c^0(t)), h_b^U(t, b^0(t), b_\alpha^0(t), c^0(t))]\xi dt$$

$$+\varphi(b, c, b^0, c^0)\int_\Omega^{\cdot} [h_{b_\alpha}^L(t, b^0(t), b_\alpha^0(t), c^0(t)), h_{b_\alpha}^U(t, b^0(t), b_\alpha^0(t), c^0(t))]D_\alpha\xi dt$$

$$+\varphi(b, c, b^0, c^0)\int_\Omega^{\cdot} [h_c^L(t, b^0(t), b_\alpha^0(t), c^0(t)), h_c^U(t, b^0(t), b_\alpha^0(t), c^0(t))]\eta dt$$

$$+\rho\varphi(b, c, b^0, c^0)d^2((b, c), (b^0, c^0)) \leq 0$$

Or in other words

$$\varphi(b, c, b^0, c^0)\int_\Omega^{\cdot} [h_b^L(t, b^0(t), b_\alpha^0(t), c^0(t)), h_b^U(t, b^0(t), b_\alpha^0(t), c^0(t))]\xi dt$$

$$+\varphi(b, c, b^0, c^0)\int_\Omega^{\cdot} [h_{b_\alpha}^L(t, b^0(t), b_\alpha^0(t), c^0(t)), h_{b_\alpha}^U(t, b^0(t), b_\alpha^0(t), c^0(t))]D_\alpha\xi dt$$

$$+\varphi(b, c, b^0, c^0)\int_\Omega^{\cdot} [h_c^L(t, b^0(t), b_\alpha^0(t), c^0(t)), h_c^U(t, b^0(t), b_\alpha^0(t), c^0(t))]\eta dt$$

$$+\rho\varphi(b, c, b^0, c^0)d^2((b, c), (b^0, c^0)) > 0 \Rightarrow H(b, c) > H(b^0, c^0)$$

In this case H is called as (ρ, φ, d)-Invex at point $(b^0, c^0)\epsilon B \times C$ with respect to ξ and η.

(ii) Now if there exists a functional such that

$$\xi : \Omega \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n \quad \text{such that}$$

$\xi = \xi(t, b(t), c(t), b^0(t), c^0(t)) = (\xi_i(t, b(t), c(t), b^0(t), c^0(t)))$ where $i = 1, 2 \ldots..n$, of the C^1 class functional with $\xi(t, b(t), c(t), b^0(t), c^0(t)) = 0,\ \forall t \in \Omega,\ \xi|_{\partial\Omega} = 0$, and another functional such that

$$\eta : \Omega \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^K \quad \text{such that}$$

$\eta = \eta(t, b(t), c(t), b^0(t), c^0(t)) = (\eta_j(t, b(t), c(t), b^0(t), c^0(t)))$ where $j = 1, 2 \ldots..k$, of the C^0 class function with $\eta(t, b(t), c(t), b^0(t), c^0(t)) = 0,\ \forall t \in \Omega,\ \eta|_{\partial\Omega} = 0$ such that for each $(b, c) \neq (b^0, c^0) \in B \times C$:

$$H(b, c) < H(b^0, c^0)$$

$$\Rightarrow \varphi(b, c, b^0, c^0)\int_\Omega^{\cdot} [h_b^L(t, b^0(t), b_\alpha^0(t), c^0(t)), h_b^U(t, b^0(t), b_\alpha^0(t), c^0(t))]\xi dt$$

$$+\varphi(b, c, b^0, c^0)\int_\Omega^{\cdot} [h_{b_\alpha}^L(t, b^0(t), b_\alpha^0(t), c^0(t)), h_{b_\alpha}^U(t, b^0(t), b_\alpha^0(t), c^0(t))]D_\alpha\xi dt$$

$$+\varphi(b, c, b^0, c^0)\int_\Omega^{\cdot} [h_c^L(t, b^0(t), b_\alpha^0(t), c^0(t)), h_c^U(t, b^0(t), b_\alpha^0(t), c^0(t))]\eta dt$$

$$+\rho\varphi(b, c, b^0, c^0)d^2((b, c), (b^0, c^0)) < 0$$

Or in other words

$$\varphi(b, c, b^0, c^0)\int_\Omega^{\cdot} [h_b^L(t, b^0(t), b_\alpha^0(t), c^0(t)), h_b^U(t, b^0(t), b_\alpha^0(t), c^0(t))]\xi dt$$

$$+\varphi(b, c, b^0, c^0)\int_\Omega^{\cdot} [h_{b_\alpha}^L(t, b^0(t), b_\alpha^0(t), c^0(t)), h_{b_\alpha}^U(t, b^0(t), b_\alpha^0(t), c^0(t))]D_\alpha\xi dt$$

$$+\varphi(b, c, b^0, c^0)\int_\Omega^{\cdot} [h_c^L(t, b^0(t), b_\alpha^0(t), c^0(t)), h_c^U(t, b^0(t), b_\alpha^0(t), c^0(t))]\eta dt$$

$$+\rho\varphi(b, c, b^0, c^0)d^2((b, c), (b^0, c^0)) \geq 0 \Rightarrow H(b, c) \geq H(b^0, c^0)$$

In this case H is called as (ρ, φ, d)-pseudoinvex at point $(b^0, c^0)\epsilon B \times C$ with respect to ξ and η.

Definition 5: Now if we consider a vector valued continuously differentiable function h such that

$$h : \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn} \times \mathbb{R}^k \times \rightarrow \mathbb{K}^p \quad \text{such that}$$

$$h = h(t, b(t), b_\alpha(t), c(t)) \quad \text{where} \quad r = 1, 2 \ldots\ldots p$$

$$= ([h_1^L(t, b(t), b_\alpha(t), c(t)), h_1^U(t, b(t), b_\alpha(t), c(t))]$$
$$\ldots\ldots\ldots\ldots\ldots[h_p^L(t, b(t), b_\alpha(t), c(t)), h_p^U(t, b(t), b_\alpha(t), c(t))])$$

Now we define vector multiple integral functional H with the help of above continuously differentiable function

$$H : B \times C \rightarrow K^p \text{ such that}$$

$$H(b, c) = \int_\Omega^{\cdot} h(t, b(t), b_\alpha(t), c(t))dt$$

$$\left(\left[\int_\Omega^{\cdot} h_1^L(t, b(t), b_\alpha(t), c(t))dt, \int_\Omega^{\cdot} h_1^U(t, b(t), b_\alpha(t), c(t))dt\right], \ldots\right.$$
$$\left.\left[\int_\Omega^{\cdot} h_p^L(t, b(t), b_\alpha(t), c(t))dt, \int_\Omega^{\cdot} h_p^U(t, b(t), b_\alpha(t), c(t))dt\right]\right)$$

Now this vector valued multiple integral functional H is said to be (ρ, φ,d)-Invex or (ρ, φ,d)-Pseudoinvex at point $(b^0, c^0)\epsilon B \times C$ with respect to ξ and η if each of the interval valued component of the vector is (ρ, φ,d)-Invex or (ρ, φ, d)-Pseudoinvex respectively at point $(b^0, c^0)\epsilon B \times C$ with respect to ξ and η.

Now consider a vector valued continuous differentiable function g such that

$$g = (g_1, g_2 \ldots\ldots g_p) \quad \text{where} \quad g_r : \Omega \times R^n \times R^k \times K^p, \qquad r = 1, 2 \ldots\ldots p$$

We may now design a new class of multiobjective variational control problems with interval-valued components that we refer to as Primal Problems

(abbreviated PP for short)

$$\min_{(b,c)} \left\{ G(b, c) = \left(\int_{\Omega}^{\cdot} g_1(t, b(t), c(t)dt, \ldots\ldots\ldots \int_{\Omega}^{\cdot} g_p(t, b(t), c(t)dt \right) \right\}$$

subject to

$$\frac{\partial b^i}{\partial t^\alpha}(t) = X_\alpha^i(t, b(t), c(t), \ i = 1, 2\ldots\ldots n \text{ and } \alpha = 1, 2\ldots\ldots\ldots m \text{ and } t \in \Omega \tag{1}$$

$$Y(t, b(t), c(t)) \leq 0, \quad t \in \Omega \tag{2}$$

$$b(t)|_{\partial\Omega} = \psi(t) = given \tag{3}$$

Now for $r = 1, 2\ldots\ldots\ldots p$ we have

$$\int_{\Omega}^{\cdot} g_r(t, b(t), c(t)dt = [\int_{\Omega}^{\cdot} g_r^L(t, b(t), c(t)dt, g_r^U(t, b(t), c(t)dt]$$

or $$G_r(b, c) = [G_r^L(b, c), G_r^U(b, c)]$$

The set of all feasible solutions in primal problem is defined by

$D = \{(b, c) |\ b \in B \text{ and } c \in C\}$ satisfying equations (1), (2) and (3).

Definition 6: A feasible solution $(b^0, c^0) \in D$ in primal problem is said to be an LU-optimal solution if there does not exist any $(b, c) \in D$ such that $G(b, c) < G(b^0, c^0)$.

Constrained by certain qualification assumptions, if $(b^0, c^0) \in D$ is an LU-Optimal solution of the variational control, then Treantă [21] and Mititelu and Treantă [19] can be considered. According to this there exists piecewise smooth functions θ, μ and λ, with $\theta(t) = (\theta^L(t), \theta^U(t)), \mu(t) = (\mu^\beta(t))$ and $\lambda(t) = \lambda_i^\alpha(t)$ such that

$$\theta_l^r \frac{\partial g_r^l}{\partial b^i}(t, b^0(t), c^0(t)) + \lambda_i^\alpha(t) \frac{\partial X_\alpha^i}{\partial b^i}(t, b^0(t), c^0(t))$$

$$+ \ \mu^{\beta}(t)\frac{\partial Y_{\beta}}{\partial b^{i}}(t, b^{0}(t), c^{0}(t)) + \frac{\partial \lambda_{i}^{\alpha}}{\partial t^{\alpha}}(t) = 0. \tag{4}$$

Where $i = 1, 2.....n$ and $l = L, U$.

$$\theta_{l}^{r}\frac{\partial g_{r}^{l}}{\partial c^{j}}(t, b^{0}(t), c^{0}(t)) + \lambda_{i}^{\alpha}(t)\frac{\partial X_{\alpha}^{i}}{\partial c^{j}}(t, b^{0}(t), c^{0}(t)) + \mu^{\beta}(t)\frac{\partial Y_{\beta}}{\partial c^{j}}(t, b^{0}(t), c^{0}(t)) = 0. \tag{5}$$

Where $j = 1, 2.....k$ and $l = L, U$.

$$\text{And} \ \ \mu^{\beta}(t)Y_{\beta}(t, b^{0}(t), c^{0}(t)) = 0 \ \text{(no summation)} \ \theta(t), \mu(t) \geq 0 \tag{6}$$

for all $t \in \Omega$ except at the point of discontinuities.

Definition 7: For the primal problem an LU-Optimal solution $(b^{0}, c^{0}) \in D$ is called an normal LU-optimal solution if above necessary LU-optimality conditions in equation (4) to (6) are satisfied.

3. Dual problem associated with Primal problem

Suppose that the set $P = \{1, 2.....q\}$ is partitioned into the set $\{P_1, P_2, ...P_s\}$, where $s < q$. Using the same notations as in Section 2, we relate the next multiobjective variational control problem with interval-valued vector components, known as the Dual Problem (DP), to the above primal problem for $(a, u) \in B \times C$:

$$\min_{(a,u)} \left\{ G(a, u) = \left(\int_{\Omega}^{\cdot} g_1(t, a(t), u(t)dt, ...\int_{\Omega}^{\cdot} g_p(t, a(t), u(t)dt \right) \right\}$$

subject to

$$\theta_{l}^{r}\frac{\partial g_{r}^{l}}{\partial a^{i}}(t, a(t), u(t)) + \lambda_{i}^{\alpha}(t)\frac{\partial X_{\alpha}^{i}}{\partial a^{i}}(t, a(t), u(t)) + \mu^{\beta}(t)\frac{\partial Y_{\beta}}{\partial a^{i}}(t, a(t), u(t)) + \frac{\partial \lambda_{i}^{\alpha}}{\partial t^{\alpha}}(t) = 0. \tag{7}$$

Where $i = 1, 2.....n$ and $l = L, U$

$$\theta_l^r \frac{\partial g_r^l}{\partial u^j}(t, a(t), u(t)) + \lambda_i^\alpha(t) \frac{\partial X_\alpha^i}{\partial u^j}(t, a(t), u(t)) + \mu^\beta(t) \frac{\partial Y_\beta}{\partial u^j}(t, a(t), u(t)) = 0. \quad (8)$$

Where $j = 1, 2.....k$ and $l = L, U$.

$$\lambda_i^\alpha(t) \left[X_\alpha^i(t, a(t), u(t)0 - \frac{\partial b^i}{\partial t^\alpha}(t) \right] \geq 0. \quad (9)$$

And $\mu^{P\theta}(t) Y_{P_\theta}(t, a(t), u(t)0 = 0$ where $\theta = 1, 2.....s$ (10)

Where $\theta = (\theta_l^r) \geq 0, \mu(t) = (\mu^\beta(t)) \geq 0$, $a(t)|_{\partial\Omega} = \psi(t) = given$ $l = L, U$. (11)

And the expression is $\mu^{P\theta}(t) Y_{P_\theta}(t, a(t), u(t))$ is

$$\mu^{P\theta}(t) Y_{P_\theta}(t, a(t), u(t)) = \sum_{\beta \epsilon P_\theta} \mu^\beta(t) Y_\beta(t, a(t), u(t))$$

In this section, we prove that, under (ρ, φ, d)-Invexity hypotheses, the multiobjective optimization problems with interval-valued components, Primal Problem and Dual Problem, are a Mond-Weir dual pair. Moreover, keep in mind that γ is the collection of all feasible solutions related to dual problem.

Now we formulate and establish the initial duality result, which is also known as weak duality.

Weak Duality theorem-For any multiobjective variational problem with interval-valued components (Primal Problem), let $(b, c) \in D$ be a feasible solution; similarly, let $(a, u, \theta, \lambda, \mu) \in \gamma$ be a feasible solution for the multiobjective variational problem with interval-valued components (Dual Problem). Furthermore, keep in mind that the following prerequisites are satisfied:

(i) For each r, the functional

$\mathcal{G}_{r,l}^{a,u}(b, c) = \int_{\Omega}^{\cdot} g_r(t, b(t), c(t)dt \qquad r = 1, 2.....p$ and $l = L, U$ is (ρ^1, φ, d)-Invex at (a, u) with regard to ξ and η or in other words, each interval-valued multiple-integral functional

$\mathcal{G}_{r}^{a,u}(b, c) = [\mathcal{G}_{r,L}^{a,u}(b, c), \mathcal{G}_{r,U}^{a,u}(b, c)]$, $r = 1, 2.....p$ is (ρ^1, φ, d)-Invex at (a, u) with regard to ξ and η.

(ii) The functional $X(b, c) = \int_{\Omega}^{\cdot} \lambda_i^{\alpha}(t)\left[X_{\alpha}^{i}(t, b(t), c(t), -\frac{\partial b^i}{\partial t^{\alpha}}(t)\right] dt$ is (ρ^2,φ, d)-Invex at (a, u) with regard to ξ and η.

(iii) Each functional

$Y_{\theta}(b, c) = \int_{\Omega}^{\cdot} \mu^{Q\theta}(t) Y_{Q\theta}(t, b(t), c(t)dt \qquad \theta = 1, 2.....s$ is $(\rho_{\theta}^{3}, \varphi, d)$-Invex at (a, u) with respect to ξ and η.

(iv) With regard to ξ *and* η, at least one of the functionals provided in (i) to (iii) is (ρ, φ, d)-Pseudoinvex at (a, u), where $\rho = \rho_r^1, \rho^2$ and ρ_{θ}^{3}.

(v) For the given

$\theta_l^r \rho_r^1 + \rho^2 + \sum_{\theta=1}^{s} \rho_{\theta}^{3} \geq 0$ where ρ_r^1, ρ^2 and $\rho_{\theta}^{3} \in \mathbb{R}$.

Then, supremum of dual problem is less than or equal to the infimum of primal problem.

Proof: The values of primal problem at $(b, c) \in D$ and dual problem at $(a, u, \theta, \lambda, \mu) \in \gamma$ are denoted by $\delta(b, c)$ and $\pi(a, u, \theta, \lambda, \mu)$ respectively. Contrast to the result, if possible, suppose that $\delta(b, c) \leq \pi(a, u, \theta, \lambda, \mu)$.

Now, take into consideration the following non-empty set for $r = 1, 2.....p, l = L, U$ and $\theta = 1, 2, ...s$:

$$S = \{(b, c) \in B \times C | \mathcal{g}_{r,l}^{a,u}(b, c) \leq \mathcal{g}_{r,l}^{b,c}(a, u), X(b, c) \leq X(a, u), Y_\theta(b, c) \leq Y_\theta(a, u)\}$$

Now by using above (i) for $(b, c) \in S$ and $r = 1, 2 \ldots\ldots p$ and $l = L, U$, we have

$$\mathcal{g}_{r,l}^{a,u}(b, c) \leq \mathcal{g}_{r,l}^{b,c}(a, u) \Rightarrow$$

$$\varphi(b, c, a, u)\int_\Omega^{\cdot} (g_r^l)_a(t, a(t), u(t)\xi dt + \varphi(b, c, a, u)\int_\Omega^{\cdot} (g_r^l)_u(t, a(t), u(t)\eta dt$$

$$\leq -\rho_r^1 \varphi(b, c, a, u) d^2((b, c)(a, u))$$

Now we multiply this by $\theta = \theta_l^r \geq 0$ where $l = L, U$ and take summation over $r = 1, 2 \ldots\ldots p$, we get the following

$$\varphi(b, c, a, u)\int_\Omega^{\cdot} \theta_l^r (g_r^l)_a(t, a(t), u(t)\xi dt + \varphi(b, c, a, u)\int_\Omega^{\cdot} \theta_l^r (g_r^l)_u(t, a(t), u(t)\eta dt$$

$$\leq -\theta_l^r \rho_r^1 \varphi(b, c, a, u) d^2((b, c)(a, u)). \tag{12}$$

Now since for each $(b, c) \in S$, the inequality $X(b, c) \leq X(a, u)$ satisfies, now according to (ii), we have the following

$$\varphi(b, c, a, u)\int_\Omega^{\cdot} [\lambda_i^\alpha(t)(X_\alpha^i)_a(t, a(t), u(t))\xi - \lambda^\alpha(t) D_\alpha \xi + \lambda_i^\alpha(t)(X_\alpha^i)_u(t, a(t), u(t))\eta] dt$$
$$\leq -\rho^2 \varphi(b, c, a, u) d^2((b, c)(a, u)). \tag{13}$$

Similarly, for each $(b, c) \in S$, the inequality $Y_\theta(b, c) \leq Y_\theta(a, u)$ for $\theta = 1, 2 \ldots\ldots s$ exists, now using (iii), we have the following

$$\varphi(b, c, a, u)\int_\Omega^{\cdot} [\mu^{Q\theta}(t)(Y_{Q\theta})_a(t, a(t), u(t))\xi + \mu^{Q\theta}(t)(Y_{Q\theta})_u(t, a(t), u(t))\eta] dt$$
$$\leq -\rho_\theta^3 \varphi(b, c, a, u) d^2((b, c)(a, u)).$$

Now, taking the summation over $\theta = 1, 2 \ldots\ldots s$, we have

$$\varphi(b, c, a, u)\int_{\Omega}^{\cdot}[\mu^{\beta}(t)(Y_{\beta})_{a}(t, a(t), u(t))\xi + \mu^{\beta}(t)(Y_{\beta})_{u}(t, a(t), u(t))\eta]dt$$

$$\leq -\sum_{\theta=1}^{s}\rho_{\theta}^{3}\varphi(b, c, a, u)d^{2}((b, c)(a, u)) \tag{14}$$

Now, adding equations (12),(13) and (14) and taking condition (iv) under consideration, we have

$$\varphi(b, c, a, u)\int_{\Omega}^{\cdot}\theta_{l}^{r}(g_{r}^{l})_{a}(t, a(t), u(t))\xi dt$$

$$+\varphi(b, c, a, u)\int_{\Omega}^{\cdot}[\lambda_{i}^{\alpha}(t)(X_{\alpha}^{i})_{a}(t, a(t), u(t)) + \mu^{\beta}(t)(Y_{\beta})_{a}(t, a(t), u(t))]dt$$

$$+\varphi(b, c, a, u)\int_{\Omega}^{\cdot}\theta_{l}^{r}(g_{r}^{l})_{u}(t, a(t), u(t))\eta dt$$

$$+\varphi(b, c, a, u)\int_{\Omega}^{\cdot}[\lambda_{i}^{\alpha}(t)(X_{\alpha}^{i})_{u}(t, a(t), u(t)) + \mu^{\beta}(t)(Y_{\beta})_{u}(t, a(t), u(t))]\eta dt$$

$$-\varphi(b, c, a, u)\int_{\Omega}^{\cdot}[\lambda^{\alpha}(t)D_{\alpha}\xi]dt < -\left(\theta_{l}^{r}\rho_{r}^{1} + \rho^{2} + \sum_{\theta=1}^{s}\rho_{\theta}^{3}\right)\varphi(b, c, a, u)d^{2}((b, c), (a, u))$$

Where $l = L, U$.

Since, $\varphi(b, c, a, u) > 0$, using this, we have the following

$$\int_{\Omega}^{\cdot}\theta_{l}^{r}(g_{r}^{l})_{q}(t, a(t), u(t))\xi dt$$

$$+\int_{\Omega}^{\cdot}[\lambda_{i}^{\alpha}(t)(X_{\alpha}^{i})_{a}(t, a(t), u(t)) + \mu^{\beta}(t)(Y_{\beta})_{a}(t, a(t), u(t))]\xi dt$$

$$+\int_{\Omega}^{\cdot}\theta_{l}^{r}(g_{r}^{l})_{u}(t, a(t), u(t))\eta dt$$

$$+\int_{\Omega}^{\cdot}[\lambda_{i}^{\alpha}(t)(X_{\alpha}^{i})_{u}(t, a(t), u(t)) + \mu^{\beta}(t)(Y_{\beta})_{u}(t, a(t), u(t))]\eta dt$$

$$-\int_{\Omega}^{\cdot}[\lambda^{\alpha}(t)D_{\alpha}\xi]dt < -\left(\theta_{l}^{r}\rho_{r}^{1} + \rho^{2} + \sum_{\theta=1}^{s}\rho_{\theta}^{3}\right)\varphi(b, c, a, u)d^{2}((b, c), (a, u)).$$

Where $l = L, U$.

Now using the constraints (7) and (8) of dual problem, we have

$$-\int_{\Omega}^{\cdot}[\xi D_{\alpha}\lambda^{\alpha}(t)dt - \int_{\Omega}^{\cdot}[\lambda^{\alpha}(t)D_{\alpha}\xi]dt + 0$$

$$< -\left(\theta_l^r\rho_r^1 + \rho^2 + \sum_{\theta=1}^{s}\rho_{\theta}^3\right)\varphi(b, c, a, u)d^2((b, c), (a, u))$$

Where $l = L, U$.

By direct formula of derivative, we know that

$$D_{\alpha}[\xi\lambda^{\alpha}(t)] = \lambda^{\alpha}(t)D_{\alpha}\xi + \xi D_{\alpha}\lambda^{\alpha}(t)$$

$$\xi D_{\alpha}\lambda^{\alpha}(t) = D_{\alpha}[\xi\lambda^{\alpha}(t)] - \lambda^{\alpha}(t)D_{\alpha}\xi$$

Now applying integral over the region Ω , we have

$$\int_{\Omega}^{\cdot}\xi D_{\alpha}\lambda^{\alpha}(t)dt = \int_{\Omega}^{\cdot}D_{\alpha}[\xi\lambda^{\alpha}(t)]dt - \int_{\Omega}^{\cdot}[\lambda^{\alpha}(t)D_{\alpha}\xi]dt$$

Using the condition $\xi|_{\partial\Omega} = 0$ and applying the flow-divergence formula, we get

$$\int_{\Omega}^{\cdot}D_{\alpha}[\xi\lambda^{\alpha}(t)dt = \int_{\partial\Omega}^{\cdot}[\xi\lambda^{\alpha}(t)\vec{n}\,d\sigma = 0$$

Where $\vec{n} = (n)_{\alpha}$ where $\alpha = 1, 2 \ldots\ldots m$, is the unit normal vector to the hyper surface $\partial\Omega$, now it follows that

$$\int_{\Omega}^{\cdot}\xi D_{\alpha}\lambda^{\alpha}(t)dt = \int_{\Omega}^{\cdot}[\lambda^{\alpha}(t)D_{\alpha}\xi]dt$$

or

$$-\int_{\Omega}^{\cdot}\xi D_{\alpha}\lambda^{\alpha}(t)dt - \int_{\Omega}^{\cdot}[\lambda^{\alpha}(t)D_{\alpha}\xi]dt = 0.$$

Therefore, we have

$$0 < -\left(\theta_l^r\rho_r^1 + \rho^2 + \sum_{\theta=1}^{s}\rho_{\theta}^3\right)\varphi(b, c, a, u)d^2((b, c), (a, u)).$$

Where $l = L, U$.

Now applying the condition (v) and $d^2((b, c), (a, u)) \geq 0$, we get a contradiction. Therefore, supremum of dual problem is less than or equal to the infimum of primal problem.

The following outcome proves a strong duality between the two multiobjective optimization problems with interval-valued components under consideration.

Strong Duality theorem-If we consider the same (ρ, φ, d)-Invexity hypotheses mentioned in above weak duality theorem, if $(b^0, c^0) \in D$ is a normal LU-optimal solution of the given primal problem, then $\exists\ \theta^0, \mu^0(t)$ and $\lambda^0(t)$ such that $(b^0, c^0, \theta^0, \lambda^0, \mu^0) \in \gamma$ is an LU-optimal solution of the dual problem, and the values of corresponding objective functions are equal.

Proof: Consider that $(b^0, c^0) \in D$ is a normal LU-optimal solution of the primal problem, the necessary LU-optimality conditions mentioned in equations (4) to (6) involve that $\exists\ \theta^0, \mu^0(t)$ and $\lambda^0(t)$ such that $(b^0, c^0, \theta^0, \lambda^0, \mu^0) \in \gamma$ is an feasible solution for dual problem.

$$\frac{\partial b^{0i}}{\partial t^{\alpha}}(t) = X^i_{\alpha}(t, b^0(t), c^0(t)) \text{ for } i = 1, 2, ...n,\ \alpha = 1, 2.....m\ \ t \in \Omega$$

Now by equation (6)

$\mu^{\beta}(t)Y_{\beta}(t, b^0, (t), c^0(t)0 = 0$, (summation is taken over β) and $t \in \Omega$.

Therefore, the value of objective function of dual problem has the same value of objective function of primal problem. Hence by weak duality theorem $(b^0, c^0, \theta^0, \lambda^0, \mu^0) \in \gamma$ is an LU-optimal solution of dual problem.

A converse duality conclusion related to considered multiobjective optimization problems with interval-valued components is formulated in the following theorem.

Converse Duality theorem-Assume that the LU-optimal solution of dual problem is $(b^0, c^0, \theta^0, \lambda^0, \mu^0) \in \gamma$. Furthermore, presumptively the following circumstances hold true:

(i) $(\overline{b}, \overline{c}) \in D$ is a normal LU-optimal solution of the given primal problem.

(ii) For $(b^0, c^0, \theta^0, \lambda^0, \mu^0)$, the hypotheses of weak duality theorem are met.

Consequently, the corresponding objective values are equal and $(\overline{b}, \overline{c}) = (b^0, c^0)$.

Proof: In contrast to the outcome, let's assume that $(\overline{b}, \overline{c}) \neq (b^0, c^0)$ and that (b^0, c^0) is not a normal LU-optimal solution of primal problem. According to Treantă and Mititelu and Treantă, since $(\overline{b}, \overline{c}) \in D$ is a normal LU-optimal solution of primal problem, there exist $\overline{\theta}, \overline{u}(t)$ and $\overline{\lambda}(t)$, satisfying equations (4) to (6) and definition of normal LU-optimal solution. Consequently

$$\overline{\lambda}_i^{\alpha}(t)\left[X_{\alpha}^i(t, \overline{b}(t), \overline{c}(t) - \frac{\partial \overline{b}^i}{\partial t^{\alpha}}(t)\right] \geq 0,$$

$$\overline{\mu}^{Q\theta}(t) Y_{Q\theta}(t, \overline{b}(t), \overline{c}(t) \geq 0, \quad \theta = 1, 2.....s$$

where $(\overline{b}, \overline{c}, \overline{\theta}, \overline{\lambda}, \overline{\mu}) \in \gamma$ as a result. Additionally, $\delta(\overline{b}, \overline{c}) = (\overline{b}, \overline{c}, \overline{\theta}, \overline{\lambda}, \overline{\mu}) \in \gamma$ is present. We obtain $\delta(\overline{b}, \overline{c}) \geq \pi(b^0, c^0, \theta^0, \lambda^0, \mu^0)$ in accordance with weak duality theorem, or $\pi(\overline{b}, \overline{c}, \overline{\theta}, \overline{\lambda}, \overline{\mu}) \geq \pi(b^0, c^0, \theta^0, \lambda^0, \mu^0)$. The maximal LU-optimality of $(b^0, c^0, \theta^0, \lambda^0, \mu^0)$ is in conflict with this. As a result, the corresponding objective values are identical and $(\overline{b}, \overline{c}) = (b^0, c^0)$.

Illustrative instance: The following two-dimensional interval-valued variational control problem is taken into consideration:

$$\min_{(b,c)} \int_{\Omega_{(0.3)}}^{\cdot} g(t, b(t), c(t), dt$$

$$= \left[\int_{\Omega_{(0.3)}}^{\cdot} (c^2(t) - 8c(t) + 16) dt^1 dt^2, \int_{\Omega_{(0.3)}}^{\cdot} (c^2(t) dt^1 dt^2 \right],$$

Subject to

$$\frac{\partial b}{\partial t^1}(t) = \frac{\partial b}{\partial t^2}(t) = 3 - c(t) \quad \text{where } t = (t^1, t^2) \in \Omega_{(0.3)}$$

$$81 - b^2(t) \leq 0 \quad \text{where } t = (t^1, t^2) \in \Omega_{(0.3)}$$

$$b(0) = b(0,0) = 6\,, \qquad b(3) = b(3,3) = 8$$

where $t_0 = (t_0^1, t_0^2) = (0,0)$ and $t_1 = (t_1^1, t_1^2) = 33$ in $\mathbb{R}^2$ are the diagonally opposed points that fix the square $b : \Omega_{(0.3)} \to \mathbb{R}.$, $c : \Omega_{(0.3)} \to \left[-\frac{8}{3}, \frac{8}{3}\right]$ and $\Omega_{(t_0^1, t_0^2)} = \Omega_{(0.3)}$.

Furthermore, we consider that in the examined variational control problem in which affine state functions are the only ones that interest us. It is possible to demonstrate by direct computation that the feasible point

$$b^0(t) = \frac{1}{3}(t^1 + t^2) + 6\,, \qquad c^0(t) = \frac{8}{3}\,, \qquad t = (t^1, t^2) \in \Omega_{(0.3)}$$

is a normal LU-optimal solution with $\lambda = (\lambda^1, \bar{\lambda}^2) = (1, \frac{5}{3}), \theta = (\theta^L, \theta^U) = (1,1)$ and $\mu = 0$ for the optimization problem under consideration. Moreover, the $(\rho, 1, 0)$-invexity (with $\rho \in \mathbb{R}$) of the functionals involved (refer to weak duality theorem) at (b^0, c^0) with regard to ξ and η may be easily verified as follows: Given by $\xi, \eta : \Omega_{(0.3)} \times (\mathbb{R} \times \mathbb{R})^2 \to \mathbb{R}$

$$\xi(t, b(t), c(t), b^0(t), c^0(t) = \begin{cases} b(t) - b^0(t), & t \in \operatorname{int}(\Omega_{(0.3)}) \\ 0, & t \in \partial\Omega_{(0.3)} \end{cases}$$

$$\eta(t, b(t), c(t), b^0(t), c^0(t) = \begin{cases} c(t) - c^0(t), & t \in \operatorname{int}(\Omega_{(0.3)}) \\ 0, & t \in \partial\Omega_{(0.3)} \end{cases}$$

where $\mathrm{int}(\Omega_{(0.3)})$ and $\partial(\Omega_{(0.3)})$ represent interior region and boundary of $\Omega_{(0.3)}$ respectively.

Therefore, by strong duality theorem $\left(\frac{1}{3}(t^1 + t^2) + 6, \frac{8}{3}, (1,1), (1, \frac{5}{3}), 0\right)$ will be an LU-optimal solution for the dual problem mentioned below

$$\max_{(a,u)} \int_{\Omega_{(0.3)}} g(t, a(t), u(t), dt$$

$$= \left[\int_{\Omega_{(0.3)}} (u^2(t) - 8u(t) + 16)dt^1dt^2, \int_{\Omega_{(0.3)}} u^2(t)dt^1dt^2\right]$$

subject to

$$-2\mu(t)a(t) + \frac{\partial\lambda^1}{\partial t^1}(t) = \frac{\partial\lambda^2}{\partial t^2}(t) = 0 \qquad \text{where } t = (t^1, t^2) \in \Omega_{(0.3)}$$

$$2\theta^L u(t) - 8\theta^L + 2\theta^U u(t) - \lambda^1(t) - \lambda^2(t) = 0 \qquad \text{where } t = (t^1, t^2) \in \Omega_{(0.3)}$$

$$\lambda^1(t)\left(3 - u(t) - \frac{\partial a}{\partial t^1}(t)\right) + \lambda^2(t)\left(3 - u(t) - \frac{\partial b}{\partial t^2}(t)\right) \geq 0$$

$$\text{where } t = (t^1, t^2) \in \Omega_{(0.3)}$$

$$\mu(t)(81 - a^2(t)) \geq 0, \qquad \text{where } t = (t^1, t^2) \in \Omega_{(0.3)}$$

$$\theta = \theta^L, \theta^U \geq [0,0], \quad \mu(t) \geq 0, \quad a(0) = a(0,0) = 6, \quad b(3) = b(3,3) = 8.$$

and the values of objective of both primal and dual problem are equal.

4. Conclusions

In this paper we have formulated and proved Mond-Weir weak, strong, and converse duality theorems for a completely new concept of multiobjective optimization problems having interval-valued components, based on the completely new notion of (ρ, φ, d)-Invexity and (ρ, φ, d)-Pseudoinvexity related with an interval-valued multiple-integral functional. Considering the relevance of interval analysis and duality theory to optimization and control, this work constitutes a significant contribution for applied sciences researchers and engineers.

Future Scope

This paper can be extended from numerous points of view for additional exploration. In this paper we have studied for one parameter t, which can be generalised for two or three parameters. On the other hand, here, we have studied Both (ρ, φ, d)-Invexity and (ρ, φ, d)-Pseuodinvexity for multiobjective optimization, which can also be studied for fractional programming or Inverse optimization. So, this study has great future scope.

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Conflict of Interest

All authors declare that they have no conflicts of interest.

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