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INVESTIGATING THE EXISTENCE OF THE EXPONENTIAL DIOPHANTINE RECTANGLES OVER STAR AND PRONIC NUMBERS

Abstract: This paper is focused on collecting a different sort of rectangle called Exponential Diophantine Rectangles over Star and Pronic Numbers. We demonstrated that there is only one Exponential Diophantine Rectangle over the *Star numbers* and no Exponential Diophantine rectangles over the Pronic numbers. Python programming is provided for the existence of such rectangles.

Keywords: Binomial Expansion, Catalan's Conjecture, Diophantine Equation, Exponential Diophantine Equation, Star numbers, Pronic numbers, Rectangles.

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1. Introduction

An exponential Diophantine equation is a special type of Diophantine equation where the variables exist in exponents. Many authors solved the different forms of exponential Diophantine equations. In particular, William Sobredo Gayo, Jr. and Jerico Bravo Bacani [12] solved the exponential Diophantine equation of the form $M_p^x + (M_q + 1)^y = z^2$ and Mahalakshmi, M. *et al*. [5], [6], and [7] solved various Diophantine equations to collect various geometrical shapes, including peble triangles and almost equilateral triangles.

We define and collect the exponential Diophantine rectangle over Special numbers (ED Rectangles over Special numbers), inspired by the above. In this paper we deal only with two types of special numbers especially star and Pronic numbers. After the introduction basis preliminaries provided. In section (3), the definition

of ED Rectangles over Star numbers (S_m) and the lemmas needed for the main theorem, while subsections establish theorems for the existence of solutions to the Exponential Diophantie equations. Python code is displayed for the existence of such rectangles. Section (3) provides the definition theorems and Python programming for the existence of exponential Diopahantine Rectangles over Pronic numbers.

2. Preliminaries

This section contains basic definitions and lemmas required for this article.

Lemma 2.1 (*Catalan's Conjecture*)**:** $(3, 2, 2, 3)$ *is the unique solution for the exponential Diophantine equation* $a^x - b^y = 1$, *where* $a, b, x, y \in \mathbb{Z}$ *such that* $\min\{a, b, x, y\} \geq 2$.

Definition 2.1 (*Binomial Expansion*)**:** *For* $x \in \mathbb{Z}$ *and* $n \in \mathbb{N}$, *the expansion for* $(1 + x)^n$ *is* $1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$ *and for* $(1 - x)^n$ *is* $1 - nx + \frac{n(n-1)}{2!}x^2 - \ldots$.

In general, $(x + y)^n = \sum_{k=0}^{n} nC_k x^{n-k} y^k = \sum_{k=0}^{n} nC_k x^k y^{n-k}$.

Definition 2.2: *The* m^{th} *Star number* (S_m) *is given by* $S_m = 6m^2 - 6m + 1$, *for* $m \in \mathbb{N}$.

Example 1: $S_1 = 1$, $S_{10} = 541$.

Definition 2.3: *The* m^{th} *Pronic numbers* (P_m) *are of the form* $P_m = m^2 + m$, *for all* $m \in \mathbb{N}$:

Definition 2.4: *An Exponential Diophantine rectangle is defined as a rectangle with the length* (l) *and breadth* (b) *as* $(l, b) = (rp + (r-1)(x+y), q(r-1) + (r+2)(x+y))$ *where* $p, q, r \in \mathbb{N}$ *and* $x; y$ *are non-negative integers such that*

$$p^x \pm q^x = r^y \tag{1}$$

3. Exponential Diophantine Rectangles over Star Numbers

This section defines Exponential Diophantine Rectangles over S_m, provides some lemmas for solving exponential Diophantine equations, and is divided into two subsections that examine some theorems. The existence of Exponential Diophantine Rectangles over S_m is proved by python programming with certain limits.

Definition 3.1 (*Exponential Diophantine Rectangles over* S_m): *An Exponential Diophantine rectangle over star numbers* (S_m) *is defined as a rectangle with the length* (l) *and breadth* (b) *as* $(l, b) = ((m+1)S_{m+1} + m(x+y),\ mS_m + (m+1)(x+y))$ *where* $m \in \mathbb{N}$ *and* x, y *are non-negative integers such that*

$$S_{m+1}^x + S_m^x = (m+1)^y \tag{2}$$

or

$$S_{m+1}^x - S_m^x = (m+1)^y \tag{3}$$

Notation: Exponential Diophantine Rectangles over star numbers - ED Rectangles over S_m.

Lemma 3.1: *The inequality* $(1+n)^x > 2$ *holds for all* $n, x > 1$.

Proof: Let us show this by using induction hypothesis on n. Now, for $n = 2$, the inequality becomes $3^x > 2$. This is true for $x > 1$. Now, assume that the inequality $(1+n)^y > 2$ holds for $n = k$. That is, $(k+1)^y > 2$ for $k > 1$. We have to show that the given inequality holds for $n = k+1$. We know that $(k+2) > (k+1)$ implies $(k+2)^y > 2$, for all $k, y > 1$. □

Lemma 3.2: *The equation* $b^4 - 6b^2 + 4b^3 + 4b - 1 = 0$ *has no positive integer solution for* $b > 1$.

Proof: The equation $b^4 - 6b^2 + 4b^3 + 4b - 1 \equiv 2, 3 \pmod 4$, which is an absurd one. □

Lemma 3.3: *If* $y > 2$, *the inequality* $(1 + n)^y > 4n$ *holds for all* $n > 1$.

Lemma 3.4: *The equation* $m^{12} + 12m^{11} + 66m^{10} + 220m^9 + 495m^8 + 792m^7 + 924m^6 + 792m^5 + 495m^4 + 220m^3 + 66m^2 + 1 = 0$ *has no solution* $\forall m \in \mathbb{N}$.

Proof: *The equation* $m^{12} + 12m^{11} + 66m^{10} + 220m^9 + 495m^8 + 792m^7 + 924m^6 + 792m^5 + 495m^4 + 220m^3 + 66m^2 + 1 \quad 0 \pmod{10}$

Hence it has no solution. □

3.1 The exponential Diophantine equation $S_{m+1}^x + S_m^x = (m + 1)^y$:

The existence of the solution to the equation $S_{m+1}^x + S_m^x = (m + 1)^y$ is discussed here.

Theorem 3.1: *The exponential Diophantine equation* $S_{m+1}^x + S_m^x = (m + 1)^y$ *has only one solution* $(x, y, m) = (0, 1, 1)$ *for all* $x, y \in \mathbb{Z}^+ \cup \{0\}$ *and* $m \in \mathbb{N}$.

Proof: Consider the exponential Diophantine equation $S_{m+1}^x + S_m^x = (m + 1)^y$ and solve this in various possibilities.

Possibility 1: For $x = 0$ and $y = 0$, the equation (2) has no solution.

Possibility 2: If $x = 0$ and $y = 1$, then we obtain $m = 1$ from the exponential Diophantine equation.

Possibility 3: If $x = 1$ and $y = 0$, the equation becomes $12m^2 = -1$ which has no solution as $m \in \mathbb{N}$.

Possibility 4: For $x = y = 1$, it reduces to a quadratic equation $12m^2 - m - 1 = 0$. On solving this we obtain $m = \frac{8}{24} \notin \mathbb{N}$.

Possibility 5: When $x = 0$ and $y > 1$, the equation becomes $(m + 1)^y = 2$. This is not possible by lemma (3.3).

Possibility 6: For $x > 1$ and $y = 0$, the equation becomes $(6m^2 + 6m + 1)^x + (6m^2 - 6m + 1)^x = 1$. Here $S_m^x + S_{x+1}^m \equiv 0 \pmod 2$ which is an absurd one.

Possibility 7: For $x = 1$ and $y > 1$, we get $12m^2 + 2 = (m + 1)^y$. By using definition (2.1) and equating the coefficient of m^2, we get the value $y = 4$. Putting the value of y in $12m^2 + 2 = (m + 1)^y$ implies $(m + 1)^4 = 12m^2 + 2$. By lemma (3.2), it has no solutions.

Possibility 8: Now $x > 1$ and $y = 1$, the equation changes into $S_m^x + S_{m+1}^x - m - 1 = 0$. For $m = 1$, it reduces to $13^x = 1$, which is an impossible one as $x > 1$. Now, $m > 1$ for, $S_m + S_{m+1} < m + 1 \Rightarrow 12m^2 - m + 1 < 0$. This is not possible as $m > 1$.

Possibility 9: Here $x, y > 1$ and $m = 1$, the equation has no solution by lemma (2.1). Now for $x, y, m > 1$, this possibility fails by using definition (2.1).

Hence, there is only one solution for the Diophantine equation $S_{m+1}^x + S_m^x = (m + 1)^y$ (i.e., $(x, y, m) = (0, 1, 1)$). □

3.2 The exponential Diophantine equation $S_{m+1}^x - S_m^x = (m+1)^y$: This subsection provides the theorem for determining the solution for the equation $S_{m+1}^x + S_m^x = (m+1)^y$.

Theorem 3.2: *No integral solution exists for the exponential Diophantine equation* $S_{m+1}^x - S_m^x = (m+1)^y \ \forall x,\ y \in \mathbb{Z}+ \cup \{0\}$ *and* $m \in \mathbb{N}$.

Proof: Consider the exponential Diophantine equation $S_{m+1}^x + S_m^x = (m+1)^y$ and solve this in various possibilities.

Possibility 1: For $x = 0$ and $y = 0$, this possibility fails.

Possibility 2: $x = 0$ and $y = 1$, the equation becomes $m = -1$, which contradicts.

Possibility 3: Here $x = y = 1$, then on solving the above equation, the value of m obtained as $\frac{1}{11} \notin \mathbb{N}$.

Possibility 4: For $x = 1$ and $y = 0$, we obtain $m \notin \mathbb{N}$.

Possibility 5: Now $x = 0$ and $y > 1$, then from the equation we obtain $(m+1)^y = 0$, as $m \geq 1$. This possibility fails.

Possibility 6: When $x > 1$ and $y = 0$, the equation becomes $S_{m+1}^x - S_{m=1}^x$. But $S_{m+1}^x - S_m^x \equiv 1(\mathrm{mod}\ 2)$. This is not possible.

Possibility 7: For $x = 1$ and $y > 1$, now by the definition (2.1) we get $y = 12$. If $y = 12$, then this possibility fails by lemma (3.4).

Possibility 8: When $y = 1$ and $x > 1$, the equation reduces into $m + 1 > S_{m+1} - S_m$ which implies $11m < 1$.

Possibility 9: $x > 1$ and $y > 1$, This possibility also fails by Binomial Expansion.

Therefore the given equation has no solution. □

Theorem 3.3: $(27, 1)$ *is the only one ED Rectangle over* S_m .

Proof: By the theorem (3.1) and (3.2), there exists only one (x, y, z) and so there only one ED Rectangle exists over S_m . □

3.3 Python Programing for Existence of ED Rectangles over S_m : In this section, we provided the python programming for the existence and non existence of the ED Rectangles over S_m

```
# ED rectangle over Star number
import math
def rectangle ():
 print ('x\ty\tn\tSm\tSn\t(l,b)')
   for x in range (0, m + 1) :
    for y in range (0, m + 1) :
     for n in range (1, m + 1) :
     Sm = 6* n **2 + 6* n + 1
     Sn = 6* n **2 - 6* n + 1
     l = Sm *(n + 1) + n*(x + y)
     b = n*Sn + (n -1) *(x + y)
     if (Sm)**x + ( Sn)**x == ( n + 1) **y :
     print (x,'\t', y,'\t', n,'\t', Sm ,'\t', Sn ,'\t' ,(l, b))
m = int ( input (" Enter the maximum range :"))
# m is the maximum range
rectangle ()
```

Coding 1: Calculating the solution for $S_m^x + S_{m+1}^x = (m + 1)^y$

```
Enter the maximum range:200
x       y       n       Sm      Sn      (l,b)
0       1       1       13      1       (27, 1)
>>>
```

Figure 1: Output: Coding 1

```
# ED rectangle over Star number
import math
def rectangle ():
 print ('x\ty\tn\tSm\tSn\t(l,b)')
  for x in range (0,m + 1) :
   for y in range (0,m + 1) :
    for n in range (1,m + 1) :
     Sm = 6* n **2 + 6* n + 1
     Sn = 6* n **2 - 6* n + 1
     l = Sm *(n + 1) + n*(x + y)
     b = n*Sn + (n -1) *(x + y)
     if (Sm)**x - ( Sn)**x == ( n + 1) **y :
     print (x,'\t',y,'\t',n,'\t', Sm ,'\t',Sn ,'\t' ,(l,b))
m = int ( input (" Enter the maximum range :"))
#m is the maximum range
rectangle ()
```

Coding 2: Calculating the solution for $S_m^x - S_{m+1}^x = (m+1)^y$

```
Enter the maximum range:100
x       y       n       Sm      Sn      (l,b)
>>>
```

Figure 2: Output: Coding 2

4. Exponential Diophantine Rectangles over Pronic Numbers

This section includes defintion of Exponential Diophantine Rectangles over P_m and also contains three subsections. First two subsections provide the theorems for solving the exponential Diopahntine equations. In the final subsection the python programming is provided for the existence of Exponential Diophantine Rectangles over P_m within a specific limit.

Definition 4.1: *An Exponential Diophantine rectangle over Pronic numbers* P_m *is defined as a rectangle with the length* (l) *and breadth* (b) *as* $(l,b) = (P_{m+1}(m+1) + m(x+y),\ mP_m + (m-1)(x+y))$ *where* $m \in \mathbb{N}$ *and* x, y *are non-negative integers such that*

$$P_{m+1}^x + P_m^x = (m+1)^y \tag{4}$$

or

$$P_{m+1}^x - P_m^x = (m+1)^y \tag{5}$$

Notation - Exponential Diophantine Rectangles over Pronic numbers- ED Rectangles over P_m .

4.1 The exponential Diophantine equation $P_{m+1}^x + P_m^x = (m+1)^y$: This subsections contains the theorem for finding the solution for the exponential Diophantine equation $P_{m+1}^x + P_m^x = (m+1)^y$.

Theorem 4.1: *The only solution for the Exponential Diophantine equation* $P_{m+1}^x + P_m^x = (m+1)^y$ *are* $(x, y, m) \in \{(0,1,1)\}$ *with* $m \in \mathbb{N}$ *and* $x, y \in \mathbb{Z}^+ \cup \{0\}$.

Proof: Consider the exponential Diophantine equation $P_{m+1}^x + P_m^x = (m+1)^y$ and solve this in various possibilities.

Possibility 1: Whenever $x = y = 0$, there is no possibility.

Possibility 2: Here $x = 0$ and $y = 1$, we obtain $m = 1$.

Possibility 3: Suppose $x = 1$ and $y = 0$, the equation 4 reduced to $12m^2 = 1$ which is an impossible one.

Possibility 4: Now $x = y = 1$, the equation (4) reduces to a quadratic equation $2m^2 + 3m + 1 = 0$ and it is not solvabe over $\mathbb{N}$.

Possibility 5: $x = 0$ and $y > 1$ the equation (4) reduces $(m+1)^y = 2$. For $m = 1$, it becomes $y = 1$ which is contradiction to our assumption. For $m > 1, (m+1)^y$ is always greater than 2. This possibility fails.

Possibility 6: For $y = 0$ and $x > 1$, the equation (4) reduces to $P_{m=1}^{x} + P_m^x = 1$. Thus, $P_{m+1}^{x} + P_m^x \equiv 0 \pmod 2$. We get a contradiction.

Possibility 7: Whenever $x = 1$ and $y > 1$, the equation reduced into $2m^2 + 4m + 2 = (m+1)^y$. By using Binomial Expansion we get $y = 2$. Now for $y = 2$, the equation reduces into a quadratic equation $2m + 1 = 0$ and it is not solvable.

Possibility 8: Here $y = 1$ and $x > 1$, we have $m + 1 > P_{m+1} + P_m$ implies $1 > m^2 + 3m + 2$. This is an impossible one.

Possibility 9: When $x, y > 1$ Now for $m = 1$, we obtain an impossible one. If $m > 1$ and by using the definition 2.1, the possibility fails.

$\therefore$ The given equation has only one solution. □

4.2 The exponential Diophantine equation $P_{m+1}^{x} - P_m^x = (m+1)^y$: This subsection discusses about the solution for the equation $P_{m+1}^{x} - P_m^x = (m+1)^y$.

Theorem 4.2: *There is no solution exists for the Exponential Diophantine equation* $P_{m+1}^{x} - P_m^x = (m+1)^y$ *with* $m \in \mathbb{N}$ *and* $x, y \in \mathbb{Z}^+ \cup \{0\}$.

Proof: Consider the equation $P_{m+1}^{x} - P_m^x = (m+1)^y$ and deal with different possibilities.

Possibility 1: Whenever $x = y = 0$, this is one fails.

Possibility 2: Suppose $x = 0$ and $y = 1$, we obtain $m = -1$. This is a contradiction.

Possibility 3: If $x = 1$ and $y = 0$, then $m \notin \mathbb{N}$.

Possibility 4: For $x = y = 1$, we obtain $m = -1$.

Possibility 5: When $x = 0$ and $y > 1$, the equation becomes $(m + 1)^y = 0$. This is impossible.

Possibility 6: Now $y = 0$ and $x > 1$, we deal with two possibilities. For $m = 1$, we obtain a contradiction. For $m > 1$, then by lemma (2.1) P_{m+1} and P_m are obtained as 3 and 2 respectively, which contradicts.

Possibility 7: For $x = 1$ and $y > 1$, by using Definition (2.1) we obtain $y = 2$. This is an absurd one.

Possibility 8: Here $x > 1$ and $y = 1$, the equation becomes $m + 1 > 2m + 2$ (not possible).

Possibility 9: $x, y > 1$, When $m = 1$, we get an impossible one and $m > 1$ this possibility fails (by Definition 2.1)

Hence, the Diophantine equation $P_{m+1}^x - P_m^x = (m + 1)^y$ has no solution. □

Theorem 4.3: *There is no ED Rectangles over Pronic numbers exists.*

Proof: By the above two theorems (4.1) and (4.2), we get the side of ED rectangle over P_m is (13, 0), this is not possible and conclude that there is no ED rectangles over Pronic numbers.

4.3 Python Programing for Existence of ED Rectangles over P_m : The Python programming for the existence and nonexistence of the exponential Diophantine equation solution is provided in this part; however, the ED rectangle over P_m , does not exist.

```
#ED rectangle over Pm
import math
def rectangle ():
 print ('x\ty\tm\tPm\tPn\t(l,b)')
  for x in range (0, n + 1) :
  for y in range (0, n + 1) :
```

```
    for m in range (1, n + 1) :
    Pm = (m + 1) *(m + 2)
    Pn = m*(m - 1)
    l = Pm *(m + 1) +m*(x + y)
    b = m*Pn + (m - 1) *(x + y)
    if (Pm)**x + ( Pn)**x ==( m + 1) **y :
      print (x,'\t',y,'\t',m,'\t', Pm ,'\t',Pn ,'\t' ,(l,b))
n = int ( input (" Enter the maximum range :"))
#n is the maximum range
rectangle ()
```

Coding 3: Calculating the solution for $P_{m+1}^{x} + P_{m}^{x} = (m+1)^{y}$

```
Enter the maximum range:200
x       y       m       Pm      Pn      (l,b)
0        1       1       6       0       (13, 0)
>>>
```

Figure 3: Output: Coding 3

```
#ED rectangle over Pm
import math
def rectangle ():
 print ('x\ty\tm\tPm\tPn\t(l,b)')
   for x in range (0,n +1) :
    for y in range (0,n +1) :
     for m in range (1,n +1) :
      Pm =(m +1) *(m +2)
      Pn=m*(m -1)
      l=Pm *(m +1) +m*(x+y)
      b=m*Pn +(m -1) *(x+y)
      if (Pm)**x -( Pn)**x == ( m +1) **y :
        print (x,'\t',y,'\t',m,'\t', Pm ,'\t',Pn ,'\t' ,(l,b))
n = int ( input (" Enter the maximum range :"))
#n is the maximum range
rectangle ()
```

Coding 4: Calculating the solution for $P_{m+1}^{x} - P_{m}^{x} = (m+1)^{y}$

```
Enter the maximum range:100
x       y       m       Pm      Pn      (l,b)
>>>
```

Figure 4: Output: Coding 4

Since the sides of the rectangles are positive, but we have $l = 13$ and $b = 0$ and therefore does not exists ED rectangle over P_m .

5. Conclusion

Finally we infer that there exists only one ED Rectangle over Star numbers and there is no ED Rectangles over Pronic numbers. In the future, this could be employed in cryptographic concepts like it helps to develop efficient algorithms. Additionally, it can be applied to furnishings design (making tables, chairs, etc.), building construction, graphic design (such as creating logos), etc. One can also work on these topics using various types of equations.

REFERENCES

[1] Abdelkader Hamtat (2023): An Exponential Diophantine equation on Triangular numbers, *Mathematica Applicanda*, Vol. 51(1), pp. 99-107.

[2] Kannan, J. and Manju Somanath (2023): Fundamental perceptions in contemporary number theory, Nova Science Publisher, New York.

[3] Kannan, J, Manju Somanath and Raja, K. (2019): On the class of solutions for the hyperbolic Diophantine equation, *International Journal of Applied Mathematics*, Vol. 32(3), pp. 443-449.

[4] Kaleeswari, K., Kannan, J. and Narasimman, G. (2022): Exponential Diophantine Equations Involving Isolated primes, *Advances and Applications in Mathematical Sciences*, Vol. 22(1), pp. 169-177.

[5] Mahalakshmi. M, Kannan. J and Narasimman, G. (2022): Certain sequels on almost equilateral triangle, *Advances and Applications in Mathematical Sciences*, Vol. 22(1), pp. 149-157.

[6] Mahalakshmi, M., Kannan, J., Deepshika, A., and Kaleeswari, K. (2023): 2-Peble triangles over figurate numbers, *Indian Journal of Science and Technology*, Vol. 16(44), pp. 4108-4113.

[7] Mahalakshmi, M., Kannan, J., Deepshika, A., and Kaleeswari, K. (2023): Existence and Non-Existence of Exponential Diophantine triangles over triangular numbers, *Indian Journal of Science and Technology*, Vol. 16(41), pp. 3599-3604.

[8] Manju Somanath, Kannan, J. and Raja, K. (2017): Exponential Diophantine equation in three variables $7^x + 7^{2y} = z^2$, *International Journal of Engineering Research Online*, Vol. 5(4), pp. 91-93.

[9] Manju Somanath, Raja, K., Kannan, J. and Nivetha, S. (2020): Exponential Diophantine equation in three unknowns, *Advances and Applications in Mathematical Sciences*, Vol. 19(11), pp. 1113-1118.

[10] Titu Andreescu, Dorin Andrica and Ion Cucurezeanu (2010): An introduction to Diophantine equations: a problem-based approach, Birkhauser Boston.

[11] Telang, S. G. (1996): Number Theory, Tata McGraw-Hill Publishing Company Limited, New York.

[12] William Sobredo Gayo. Jr and Jerico Bravo Bacani, (2021): On the Diophantine equation $M_p^x + (M_q + 1)^y = z^2$, *European Journal of Pure and Applied Mathematics*, Vol. 14(2), pp. 396-403.

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