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A FAMILY OF GENERALIZED GIBONACCI SUMS: GRAPH-THEORETIC CONFIRMATION

Abstract: We confirm a generalized sum of a family of gibonacci polynomial squares using graph-theoretic techniques, and its graph-theoretic and Pell consequences.

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1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*.

They can also be defined by the *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 2].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively [2, 6].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n; $b_n = p_n$ or q_n; $\Delta = \sqrt{x^2 + 4}$ and $2\alpha = x + \Delta$ [6, 7].

1.1 Fundamental Gibonacci Identities: Gibonacci polynomials satisfy the following properties [2, 3, 4, 5, 6, 7]:

$$g_{n+k}g_{n-k} - g_n^2 = \begin{cases} (-1)^{n+k+1}f_k^2, & \text{if } g_n = f_n \\ (-1)^{n+k}\Delta^2 f_k^2, & \text{otherwise;} \end{cases} \tag{1}$$

$$g_{n+k+r}g_{n-k} - g_{n+k}g_{n-k+r} = \begin{cases} (-1)^{n+k+1}f_r f_{2k}, & \text{if } g_n = f_n \\ (-1)^{n+k}\Delta^2 f_r f_{2k}, & \text{otherwise;} \end{cases} \tag{2}$$

$$g_{n+k+r}g_{n-k} + g_{n+k}g_{n-k+r} = \begin{cases} \frac{1}{\Delta^2}[2l_{2n+r} - (-1)^{n+k}l_{2k}l_r], & \text{if } g_n = f_n \\ 2l_{2n+r} + (-1)^{n+k}l_{2k}l_r, & \text{otherwise,} \end{cases} \tag{3}$$

where k and r are positive integers. These properties can be confirmed using Binet-like formulas.

Consequently, we have

$$g_{n+k+r}^2 g_{n-k}^2 - g_{n+k}^2 g_{n-k+r}^2 = \begin{cases} \frac{(-1)^{n+k+1}}{\Delta^2}[2l_{2n+r} - (-1)^{n+k}l_{2k}l_r]\, f_{2k}f_r, & \text{if } g_n = f_n \\ (-1)^{n+k}\Delta^2[2l_{2n+r} + (-1)^{n+k}l_{2k}l_r]f_{2k}f_r, & \text{otherwise.} \end{cases} \tag{4}$$

Again, in the interest of brevity and convenience, we now let

$$A = 2l_{(2(2pn+t-p)k+r} - (-1)^{tk}l_{2pk}l_r; \text{ and } B = 2l_{(2(2pn+t-p)k+r} + (-1)^{tk}l_{2pk}l_r.$$

It follows identities (1) and (4) that

$$g_{(2pn+t)k}g_{(2pn+t-2p)k} - g^2_{(2pn+t-p)k} = \begin{cases}(-1)_{tk+1}f^2_{pk}, & \text{if } g_n = f_n\\ (-1)^{tk}\Delta^2 f^2_{pk}, & \text{otherwise;}\end{cases} \tag{5}$$

$$g^2_{(2\ pn+t)k+r}g^2_{(2\ pn+t-2p)k} - g^2_{(2pn+t)k}g^2_{(2pn+t-2p)k+r} = \begin{cases}\frac{(-1)^{tk+1}}{\Delta^2}Af_{2pk}f_r, & \text{if } g_n = f_n\\ (-1)^{tk}\Delta^2 Bf_{2pk}f_r, & \text{otherwise,}\end{cases} \tag{6}$$

respectively, where $k, p, r,$ and t are positive integers and $t \le 2p$ [6].

2. A Telescoping Gibonacci Sum

Using recursion, we established the following telescoping sum in [6]. In the interest of brevity, we omit its proof here.

Lemma 1: *Let k, p, r, t , and λ be positive integers, where $t \le 2p$. Then*

$$\sum_{n-1}^{\infty}\left[\frac{g^{\lambda}_{(2pn+t-2p)k+r}}{g^{\lambda}_{(2pn+t-2p)k}} - \frac{g^{\lambda}_{(2pn+t)k+r}}{g^{\lambda}_{(2pn+t)k}}\right] = \frac{g^{\lambda}_{tk+r}}{g^{\lambda}_{tk}} - \alpha^{\lambda r}. \tag{7}$$

3. A Family of Gibonacci Sums

This lemma, coupled with identities (5) and (6), played a major role in the development of the following theorem. To present it in a concise fashion, we now let:

$$\mu = \begin{cases}1, & if\ g_n = f_n\\ \Delta^2, & otherwise;\end{cases} \qquad \mu^* = \begin{cases}\frac{1}{\Delta^2}, & if\ g_n = f_n\\ \Delta^2, & otherwise;\end{cases}$$

and $$\nu = \begin{cases}-1, & if\ g_n = f_n\\ 1, & otherwise.\end{cases}$$

These tools served as building blocks in the development of the theorem [6].

Theorem 1: *Let $k, p, r,$ and t be positive integers, where $t \le 2p$. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk}\mu^*[2l_{2(2pn+t-p)k+r} + (-1)^{tk}\nu l_{2pk}l_r]f_{2pk}f_r}{[g^2_{(pn+t-p)k} + (-1)^{tk}\mu\nu f^2_{pk}]^2} = \frac{g^2_{tk+r}}{g^2_{tk}} - \alpha^{2r}. \qquad (8)$$

The objective of our discourse is to confirm this result using graph-theoretic techniques. To this end, we now present the needed tools.

4. Graph-Theoretic Tools

Consider the Fibonacci digraph in Figure 1 with vertices v_1 and v_2, where a weight is assigned to each edge [2, 5]. It follows from its *weighted adjacency matrix* $Q = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$ that

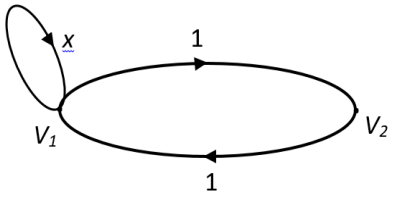


Figure 1: Weighted Fibonacci Digraph

$$Q^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix},$$

where $n \ge 1$ [2, 3, 4, 5]. We extend the exponent n to 0, which is consistent with the *Cassini-like formula* $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$, where $f_{-1} = 1$ [2, 5].

A *walk* from vertex v_i to vertex v_j is a sequence $v_i - e_i - v_{i+1} - \cdots - v_{j-1} - e_{j-1} - v_j$ of vertices v_k and edges e_k, where edge e_k is incident with vertices v_k and v_{k+1}. The walk is *closed* if $v_i = v_j$; and *open*, otherwise. The *length* of a walk is the number of edges in the walk. The *weight* of a walk is the product of the weights of the edges along the walk.

The ijth entry of Q^n gives the sum of the weights of all walks of length n from v_i to v_j in the weighted digraph, where $1 \le i, j \le n$ [2, 3, 4]. Consequently, the sum of the weights of closed walks of length n originating at v_1 in the digraph is f_{n+1} and that of those originating at v_2 is f_{n-1}. So, the sum of the weights of all closed walks of length n in the digraph is $f_{n+1} + f_{n-1} = l_n$ [2, 5].

Let A and B denote sets of walks of varying lengths originating at a vertex v. Then the sum of the weights of the elements (a, b) in the product set $A \times B$ is defined as the product of the sums of weights from each component [3, 4]. This definition can be extended to any finite number of component sets. In particular, let $A, B, C,$ and D denote the sets of walks of varying lengths originating at a vertex v, respectively. Then the sum of the weights of the elements (a, b, c, d) in the product set $A \times B \times C \times D$ is the product of the sums of weights from each component [3, 4].

We now make an interesting observation. Let $A = \{u\}$ and $B = \{v\}$, where u denotes the closed walk $v_1 - v_1$ and v denotes the closed walk $v_1 - v_2 - v_1$. The weight of the element (u, u) in $A \times A$ is x^2, and that in $B \times B$ is 1. Consequently, the sum w of the weights of the elements in $C^* = (A \times A) \cup (B \times B) \cup (B \times B) \cup (B \times B) \cup (B \times B)$ is given by $w = x^2 + 4 = \Delta^2$.

These tools play a major role in the discourse. With them at our finger tips, we are now ready for our pursuit of the graph-theoretic confirmation.

5. Graph-Theoretic Confirmation

Let T_n^* denote the set of closed walks of length n in the digraph originating at v_1, and U_n^* the set of all closed walks of the same length n in the digraph. Correspondingly, let T_n denote the sum of the weights of all elements in T_n^*, and U_n that of those in U_n^*. Clearly, $T_n = f_{n+1}$ and $U_n = f_{n+1} + f_{n-1} = l_n$ [2, 5]. With this brief background, we now begin the proof of the gibonacci sum (8) in two cases, where $k, p, r, t \ge 1$ and $t \le 2p$.

Proof: Case 1: Suppose $g_n = f_n$. The sum of the weights of the elements in the product set $T^*_{2pk-1} \times T^*_{r-1}$ is $T_{2pk-1}T_{r-1} = f_{2pk}f_r$; the sum of those in $T^*_{(2pn+t-p)k-1} \times T^*_{(2pn+t-p)k-1}$ is $T^2_{(2pn+t-p)k-1} = f^2_{(2pn+t-p)k}$; and that of those in $T^*_{pk-1} \times T^*_{pk-1}$ is $T^2_{pk-1} = f^2_{pk}$.

We now let

$$S_n = \frac{(-1)^{tk+1}[2U_{2(2pn+t-p)k+r} - (-1)^{tk}U_{2pk}U_r]T_{2pk-1}T_{r-1}}{w[T^2_{(2pn+t-p)k} - (-1)^{tk}T^2_{pk-1}]^2}$$

$$= \frac{(-1)^{tk+1}[2l_{2(2pn+t-p)k+r} - (-1)^{tk}l_{2pk}l_r]f_{2pk}f_r}{\Delta^2[f^2_{(2pn+t-p)k} - (-1)^{tk}f^2_{pk}]^2}$$

With identities (3) and (4), and Lemma 1, this yields

$$\frac{(-1)^{tk+1}[Af_{2pk}f_r}{\Delta^2[f^2_{(2pn+t-p)k} - (-1)^{tk}f^2_{pk}]^2} = \frac{f^2_{(2pn+t)k+r}f^2_{(2pn+t-2p)k} - f^2_{(2pn+t)k}f^2_{(2pn+t-2p)k+r}}{f^2_{(2pn+t)k}f^2_{(2pn+t-2p)k}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk}[Af_{2pk}f_r}{\Delta^2[f^2_{(2pn+t-p)k} - (-1)^{tk}f^2_{pk}]^2} = \sum_{n=1}^{\infty}\left[\frac{f^2_{(2pn+t-2p)k+r}}{f^2_{(2pn+t-2p)k}} - \frac{f^2_{(2pn+t)k+r}}{f^2_{(2pn+t)k}}\right]$$

$$= \frac{f^2_{tk+r}}{f^2_{tk}} - \alpha^{2r}. \tag{9}$$

We now turn to the next case.

Case 2: Let $g_n = l_n$. Recall that the sum w of the weights of the elements in C^* is given by $w = x^2 + 4 = \Delta^2$, and that of the elements in the product set $C^* \times T^*_{2k-1} \times T^*_{r-1}$ is given by $wT_{2pk-1}T_{r-1} = \Delta^2 f_{2pk}f_r$. The sum of the weights of the elements in the product set $U^*_{(2pn+t-p)k} \times U^*_{(2pn+t-p)k}$ is $U^2_{(2pn+t-p)k} = l^2_{(2pn+t-p)k}$; and that of those in $T^*_{2pk-1} \times T^*_{pk-1}$ is $T^2_{2pk-1} = f^2_{2pk}$.

As above, we now let

$$S_n = \frac{(-1)^{tk+1} w[2U_{2(2pn+t-p)k+r} + (-1)^{tk} U_{2pk} U_r] T_{2pk-1} T_{r-1}}{[U^2_{(2pn+t-p)k} + (-1)^{tk} w T^2_{pk-1}]^2}$$

$$= \frac{(-1)^{tk+1} \Delta^2 [2l_{2(2pn+t-p)k+r} + (-1)^{tk} l_{2pk} l_r] f_{2pk} f_r}{[l^2_{(2pn+t-p)k} + (-1)^{tk} \Delta^2 f^2_{pk}]^2}.$$

It then follows by identities (3) and (4), and Lemma 1 that

$$\frac{(-1)^{tk+1} \Delta^2 B f_{2pk} f_r}{[l^2_{(2pn+t-p)k} + (-1)^{tk} \Delta^2 f^2_{pk}]^2} = \frac{l^2_{(2pn+t)k} l^2_{(2pn+t-2p)k+r} - l^2_{(2pn+t)k+r} l^2_{(2pn+t-2p)k}}{l^2_{(2pn+t)k} l^2_{(2pn+t-2p)k}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk+1} \Delta^2 B f_{2pk} f_r}{[l^2_{(2pn+t-p)k} - (-1)^{tk} \Delta^2 f^2_{pk}]^2} = \sum_{n=1}^{\infty} \left[\frac{l^2_{(2pn+t-2p)k+r}}{l^2_{(2pn+t-2p)k}} - \frac{l^2_{(2pn+t)k+r}}{l^2_{(2pn+t)k}} \right]$$

$$= \frac{l^2_{tk+r}}{l^2_{tk}} - \alpha^{2r}. \tag{10}$$

This equation, coupled with equation (9), yields Theorem 1, as desired. □

Interestingly, equation (9) can be rewritten in terms of graph-theoretic tools.

To realize this goal, we define $T_0 = 1$, $U_0 = 2$; $H_n = T_n$ or U_n;

$$\mu = \begin{cases} 1, & if\ H_n = T_n \\ w, & otherwise; \end{cases} \qquad \mu^* = \begin{cases} \frac{1}{w}, & if\ H_n = T_n \\ w, & otherwise; \end{cases}$$

$$\nu = \begin{cases} -1, & if\ H_n = T_n \\ 1, & otherwise; \end{cases} \qquad \nu^* = \begin{cases} 1 & if\ H_n = T_n \\ -1 & otherwise; \end{cases}$$

$$\nu' = \begin{cases} -1, & if\ H_n = T_n \\ 0, & otherwise. \end{cases}$$

With these new tools, and integers k, p, r, and t as before, we now present the graph-theoretic version of equation (8):

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk}\mu^*\nu^*[2U_{2(2pn+t-p)k+r} + (-1)^{tk}\nu U_{2pk}U_r]T_{2pk-1}T_{r-1}}{[H^2_{(2pn+t-p)k-\nu} + (-1)^{tk}\mu\nu T^2_{pk-1}]^2} = \frac{H^2_{tk+r+\nu'}}{H^2_{tk+\nu'}} - \alpha^{2r}. \tag{11}$$

Next, we turn to the Pell implications of the graph-theoretic techniques.

6. Pell Consequence

With the gibonacci-Pell relationship $b_n(x) = g_n(2x)$, we can construct the graph-theoretic proof of the Pell version of Theorem 1 independently by changing the weight of the loop at v_1 from x to $2x$. We encourage gibonacci enthusiasts to explore this path.

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