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Chirag Barasara[1]
and
Palak Prajapati[2]

ANTIMAGIC LABELING OF LINE GRAPH OF SOME GRAPHS

Abstract: Motivated from the study of magic square, Hartsfield and Ringel defined antimagic labeling as a bijection $f : E(G) \to \{1, 2, 3, \dots, |E(G)\}$ such that $\forall u, v \in V(G), uf \neq v$, sum of $f(e)$ for all *e* incident to u is different from sum of $f(e)$ for all e incident to v. In this paper, we discussed antimagic labeling of the line graph of armed crown, double comb, ladder, wheel and tadpole.

Keywords: Graph Labeling, Antimagic Labeling, Graph Operation, Line Graph.

Mathematics Subject Classification (2020) No.: 05C78, 05C76.

1. Introduction

All the graphs considered in this paper are simple, finite, connected and undirected. A graph $G = (V(G), E(G))$ with q edges is said to be antimagic, if there exist a bijective labeling *f* from edge set of G to $1, 2, 3, \dots, q$ such that the sums of the labels of the edges incident to each vertex is distinct. Hartsfield and Ringel [12] in 'Pearls in graph theory' introduced antimagic labeling and conjecture that 'every connected graph different from K_2 is antimagic'.

Many authors have tried to attack antimagic conjecture, Alon *et al.* [1] have derived conditions on degree of a vertices for graph to be antimagic. Arumugam

et al. [2] have shown that various pyramid graphs are antimagic graphs. Cheng [6] has proved that Cartesian products of two or more regular graphs are antimagic. Joseph and Kureethara[15] have investigated that Cartesian product of wheel graph and path graph is antimagic. Bača *et al.* [3] as well as Wang *et al* .[24] have discussed antimagic labeling for some join graphs. Latchoumanane and Varadhan [17] have studied antimagicness for tensor product of wheel and star. Lozano *et al.* [18] have proved antimagic labeling of caterpillars. Sethuraman and Shermily [21] have verified binomial tree and Fibonacci tree are antimagic. Barasara and Prajapati [4, 5] have obtained antimagic labeling of some degree splitting graphs as well as for some snake graphs. Although researchers applied various techniques, still antimagic conjecture remains open.

A detailed survey on antimagic labeling can be found in Jin and Tu [14]. While survey on graph labeling is carried out by Gallian [10].

In this paper, we study antimagic labeling in the context of line graph operation.

2. Preliminaries

Definition 2.1 ([7]): *The line graph* $L(G)$ *of a graph* G *is the graph whose vertex set is* $E(G)$ *and two vertices are adjacent in* $L(G)$ *whenever they are incident in* G .

Definition 2.2 ([23]): *The armed crown* AC_n *is a graph in which path* P_2 *is attached at each vertex of cycle* C_n *by an edge.*

Definition 2.3 ([19]): *The* Cartesian product of graphs G_1 *and* G_2 *denoted by* $G_1 \square G_2$ *is the graph with vertex set* $V(G_1) \times V(G_2) = \{(u, v) / u \in V(G_1)$ *and* $v \in V(G_2)\}$ *and* (u, v) *is adjacent to* (u', v') *if and only if either* $u = u'$ *and* $vv' \in E(G_2)$ *or* $v = v'$ *and* $uu' \in E(G_1)$.

Definition 2.4 ([11]): *The ladder graph* L_n *is defined as* $L_n = P_n \square K_2$.

Definition 2.5 ([9]): *Let G and H be two graphs. The corona product of G and H, denoted by $G \odot H$, is obtained by taking one copy of G and $|V(G)|$ copies of H, and by joining each vertex of the i^{th} copies of H to the i^{th} vertex of G, for $1, 2, 3, \ldots, |V(G)|$.*

Definition 2.6 ([13]): *Let P_n be a path graph with n vertices. The double comb graph is defined as $P_n \odot 2K_1$.*

Definition 2.7 ([22]): *The graph obtained by joining cycle C_n to a path P_m with an edge is called tadpole graph. It is denoted by $T(n, m)$.*

Preposition 2.1 ([1]): *If G has $n \geqslant 4$ vertices and $\Delta(G) \geqslant n - 2$ then G is antimagic.*

Preposition 2.2 (Exercise in [12]): *The cycle C_n is antimagic.*

3. Main Results

Theorem 3.1: *The armed crown graph AC_n is an antimagic graph.*

Proof: Let AC_n be an armed crown graph with $V(AC_n) = \{v_i, v_i', v_i'' / i = 1, 2, \ldots, n\}$ and $E(AC_n) = \{v_i v_{i+1} / i = 1, 2, \ldots, n-1\} \cup \{v_1 v_n\} \cup \{v_i v_i' / i = 1, 2, \ldots, n\} \cup \{v_i' v_i'' / i = 1, 2, \ldots, n\}$.

Then $|V(AC_n)| = 3n$ and $|E(AC_n)| = 3n$.

We define $f : E(AC_n) \to \{1, 2, \ldots, 3n\}$ as follows.

$f(v_1 v_n) = 2n + 1$,

$f(v_i v_{i+1}) = 3n - i + 1$; For $1 \leqslant i \leqslant n - 1$,

$f(v_i v_i') = n + i;$ For $1 \leqslant i \leqslant n$,

$f(v_i' v_i'') = i;$ For $1 \leqslant i \leqslant n$.

Above define edge labeling function will generate distinct vertex labels for all the vertices of AC_n. Thus, f is an antimagic labeling.

Hence, the armed crown graph AC_n is an antimagic graph. □

Illustration 3.1: *The graph AC_8 and its antimagic labeling is shown in* Figure 1.

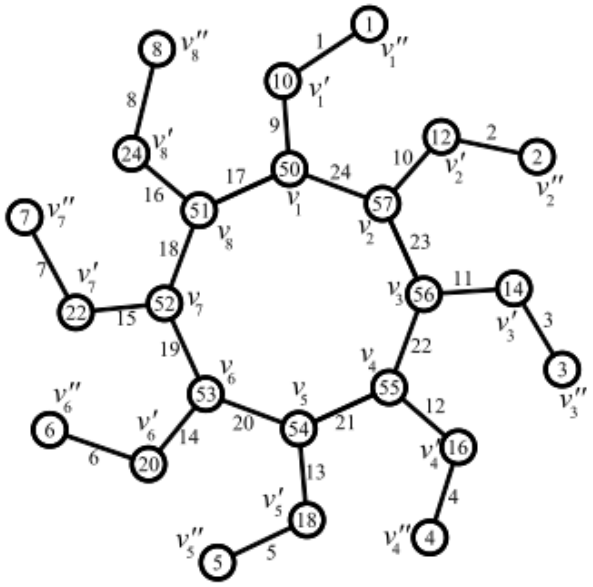


Figure 1: AC_8 and its antimagic labeling.

Theorem 3.2: *The graph $L(AC_n)$ is an antimagic graph.*

Proof: Let AC_n be an armed crown graph with $V(AC_n) = \{v_i, v_i', v_i'' / i = 1, 2, \ldots, n\}$ and $E(AC_n) = \{e_i = v_i v_{i+1} / i = 1, 2, \ldots, n-1\} \cup \{e_n = v_1 v_n\} \cup \{e_i' = v_i v_i' / i = 1, 2, \ldots, n\} \cup \{e_i'' = v_i' v_i'' / i = 1, 2, \ldots, n\}$. To construct $L(AC_n)$,

let the vertices corresponding to e_i be x_i, e_i' be x_i' and e_i'' be x_i'' for each i. Then $|V(L(AC_n)| = 3n$ and $|E(L(AC_n)| = 4n$.

We define $f : E(L(AC_n)) \to \{1, 2, \ldots, 4n\}$ as follows.

$$
\begin{aligned}
f(x_i'x_i'') &= i; && \text{For } 1 \leqslant i \leqslant n,\\
f(x_nx_1') &= n+1,\\
f(x_ix_{i+1}') &= n+1+i; && \text{For } 1 \leqslant i \leqslant n-1,\\
f(x_ix_i') &= 3n+1-i; && \text{For } 1 \leqslant i \leqslant n,\\
f(x_ix_{i+1}) &= 3n+1+i; && \text{For } 1 \leqslant i \leqslant n-1,\\
f(x_1x_n) &= 3n+1.
\end{aligned}
$$

Above define edge labeling function will generate distinct vertex labels for all the vertices of $L(AC_n)$. Thus, f is an antimagic labeling.

Hence, the graph $L(AC_n)$ is an antimagic graph. □

Illustration 3.2: *The graph* $L(AC_6)$ *and its antimagic labeling is shown in* Figure 2.

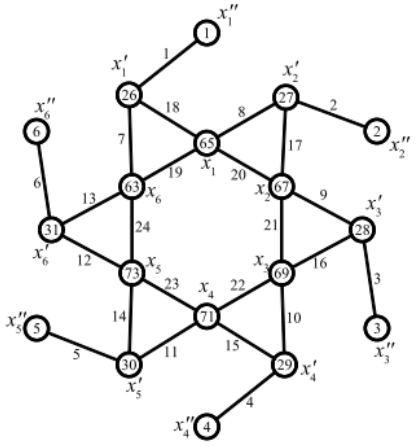


Figure 2: $L(AC_6)$ and its antimagic labeling.

Theorem 3.3: *The graph* $L(P_n \odot 2K_1)$ *is an antimagic graph.*

Proof: Let $P_n \odot 2K_1$ be a double comb graph with $V(P_n \odot 2K_1) = \{v_i, v_i', v_i'' / i = 1, 2, \ldots, n\}$ and $E(P_n \odot 2K_1) = \{e_i = v_i v_{i+1} / i = 1, 2, \ldots, n-1\} \cup \{e_i' = v_i v_i' / i = 1, 2, \ldots, n\} \cup \{e_i'' = v_i v_i'' / i = 1, 2, \ldots, n\}$. To construct $L(P_n \odot 2K_1)$, let the vertices corresponding to e_i be x_i, e_i' be x_i' and e_i'' be x_i'' for each i. Then $|V(L(P_n \odot 2K_1)| = 3n - 1$ and $|E(L(P_n \odot 2K_1)| = 6n - 6$.

We define $f : E(P_n \odot 2K1) \to \{1, 2, \ldots, 6n - 6\}$ as per following two cases.

Case 1: For $n = 2$.

The graph $L(P_2 \odot 2K_1)$ has 5 vertices and $\Delta(L(P_2 \odot 2K_1)) = 4$. Thus, by Preposition 2.1, $L(P_2 \odot 2K_1)$ is an antimagic graph.

Case 2: For $n \geqslant 3$.

$f(x_1'x_1'') = 1$,

$f(x_n'x_n'') = 2$,

$f(x_ix_i') = 2i + 1$; For $1 \leqslant i \leqslant n-1$,

$f(x_{i-1}x_i') = 2i$; For $2 \leqslant i \leqslant n$,

$f(x_ix_i'') = 4n - 2i$; For $1 \leqslant i \leqslant n-1$,

$f(x_ix_{i+1}'') = 4n - 1 - 2i$; For $1 \leqslant i \leqslant n-1$,

$f(x_ix_{i+1}) = 5n - 4 + i$; For $1 \leqslant i \leqslant n-2$,

$f(x_i'x_i'') = 4n - 3 + i$; For $2 \leqslant i \leqslant n-1$.

Above define edge labeling function will generate distinct vertex labels for

all the vertices of $L(P_n \odot 2K_1)$. Thus, f is an antimagic labeling.

Hence, the graph $L(P_n \odot 2K_1)$ is an antimagic graph. □

Illustration 3.3: *The graph* $L(P_7 \odot 2K_1)$ *and its antimagic labeling is shown in* Figure 3.

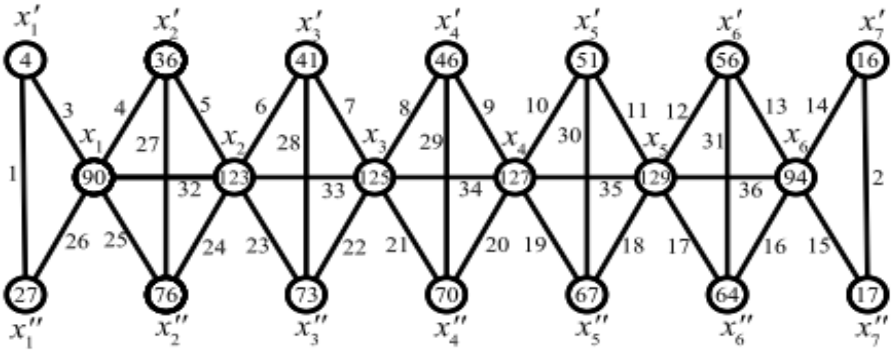


Figure 3: $L(P_7 \odot 2K_1)$ and its antimagic labeling.

Theorem 3.4: *The graph* $L(L_n)$ *is an antimagic graph.*

Proof: Let L_n be a ladder graph with $V(L_n) = \{v_i, v'_i / i = 1, 2, \ldots, n\}$ and $E(L_n) = \{e'_i = v_i v_{i+1} / i = 1, 2, \ldots, n-1\} \cup \{e_i = v_i v'_i / i = 1, 2, \ldots, n\}$ $\cup \{e''_i = v'_i v'_{i+1} / i = 1, 2, \ldots, n-1\}$. To construct $L(L_n)$, let the vertices corresponding to e_i be x_i, e'_i be x'_i and e''_i be x''_i for each i.

Then $|V(L(L_n)| = 3n - 2$ and $E(L(L_n)| = 6n - 8$.

We define $f : E(L(Ln)) \to \{1, 2, \ldots, 6n - 8\}$ as per following four cases.

Case 1: For $n = 2$.

The graph $L(L_2)$ is also known as cycle C_4. Thus, by Preposition 2.2, $L(L_2)$ is an antimagic graph.

Case 2: For $n = 4$.

The antimagic labeling of graph $L(L_4)$ is demonstrated in following Figure 4.

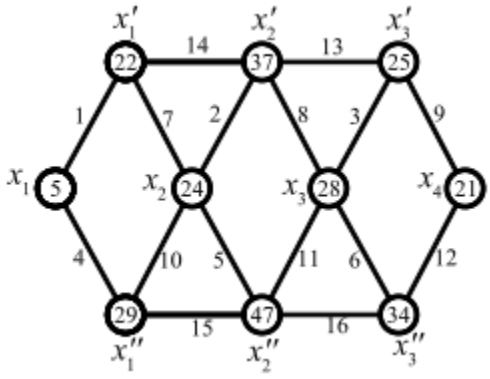


Figure 4: $L(L_4)$ and its antimagic labeling.

Case 3: For $n \equiv 0, 1, 3 \pmod 4$ and $n \neq 4$.

$$f(x_i x_i') = i; \qquad \text{For } 1 \leqslant i \leqslant n-1,$$

$$f(x_{i+1} x_i') = n - 1 + i; \qquad \text{For } 1 \leqslant i \leqslant n-1,$$

$$f(x_i x_i'') = 2(n-1) + i; \qquad \text{For } 1 \leqslant i \leqslant n-1,$$

$$f(x_{i+1} x_i'') = 3(n-1) + i; \qquad \text{For } 1 \leqslant i \leqslant n-1,$$

$$f(x_i' x_{i+1}') = 4(n-1) + i; \qquad \text{For } 1 \leqslant i \leqslant n-2,$$

$$f(x_i'' x_{i+1}'') = 5(n-1) - 1 + i; \qquad \text{For } 1 \leqslant i \leqslant n-2.$$

Case 4: For $n \equiv 2 \pmod 4$ and $n \neq 2$.

$$f(x_i x_i') = i; \qquad \text{For } 1 \leqslant i \leqslant n-1,$$

$$f(x_{i+1} x_i') = 2(n-1) + i; \qquad \text{For } 1 \leqslant i \leqslant n-1,$$

$$f(x_i x_i'') = n - 1 + i; \qquad \text{For } 1 \leqslant i \leqslant n-1,$$

$f(x_{i+1}x_i'') = 3(n-1) + i$; For $1 \leqslant i \leqslant n-1$,

$f(x_i'x_{i+1}') = 4(n-1) + i$; For $1 \leqslant i \leqslant n-2$,

$f(x_i''x_{i+1}'') = 5(n-1) - 1 + i$; For $1 \leqslant i \leqslant n-2$.

Above define edge labeling function will generate distinct vertex labels for all the vertices of $L(L_n)$. Thus, f is an antimagic labeling.

Hence, the graph $L(L_n)$ is an antimagic graph. □

Illustration 3.4: *The graph* $L(L_6)$ *and its antimagic labeling is shown in* Figure 5.

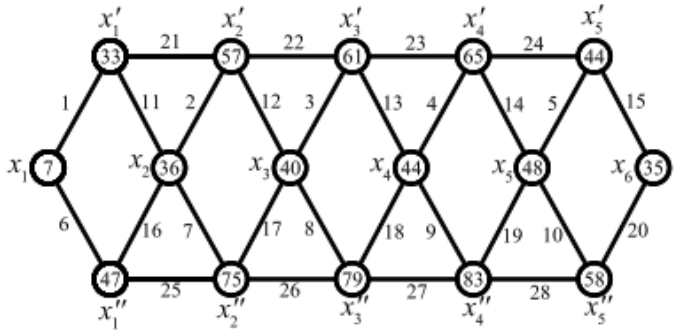


Figure 5: $L(L_6)$ and its antimagic labeling.

Theorem 3.5: *The graph* $L(W_n)$ *is an antimagic graph.*

Proof: Let W_n be a wheel graph with $V(W_n) = \{v, v_i / i = 1, 2, \ldots, n\}$ and $E(W_n) = \{e_i = v_iv_{i+1} / i = 1, 2, \ldots, n-1\} \cup \{e_n = v_1v_n\} \cup \{e_i' = vv_i / i = 1, 2, \ldots, n\}$.

To construct $L(W_n)$, let the vertices corresponding to e_i be x_i and e_i' be x_i' for each i.

Then $|V(L(W_n)| = 2n$ and $|E(L(W_n)| = \dfrac{n^2 + 5n}{2}$.

We define $f : E(L(W_n)) \to \left\{1, 2, \ldots, \dfrac{n^2 + 5n}{2}\right\}$ as follows.

$$f(x_i x_{i+1}) = i; \qquad \text{For } 1 \leqslant i \leqslant n-1,$$

$$f(x_n x_1) = n,$$

$$f(x_i x_i') = n + i; \qquad \text{For } 1 \leqslant i \leqslant n,$$

$$f(x_i' x_{i+1}) = 3n + 1 - i; \qquad \text{For } 1 \leqslant i \leqslant n-1,$$

$$f(x_n' x_1) = 2n + 1,$$

$$f(x_i' x_{i+j}') = i(n-1) + 2n + j + 1; \qquad \text{For } \begin{cases} 1 \leqslant i \leqslant 2, \\ 1 \leqslant j \leqslant n-i, \end{cases}$$

$$f(x_i' x_{i+j}') = i(n-1) + 2n + j + 1 - \frac{(i-1)(i-2)}{2}; \quad \text{For } \begin{cases} 3 \leqslant i \leqslant n-1, \\ 1 \leqslant j \leqslant n-i, \end{cases}$$

Above define edge labeling function will generate distinct vertex labels for all the vertices of $L(W_n)$.

Thus, f is an antimagic labeling.

Hence, the graph $L(W_n)$ is an antimagic graph. □

Illustration 3.5: *The graph* $L(W_5)$ *and its antimagic labeling is shown in* Figure 6.

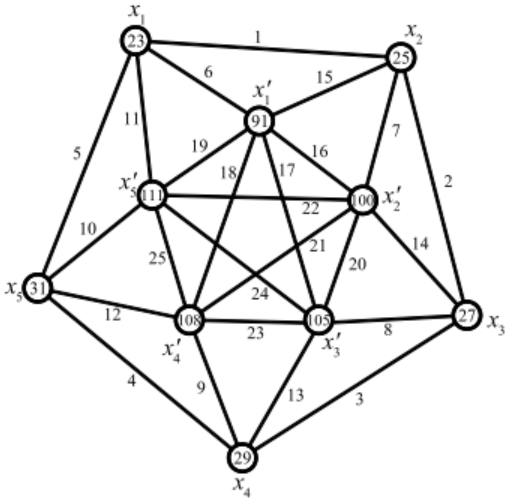


Figure 6: $L(W_5)$ and its antimagic labeling.

Theorem 3.6: *The graph* $L(T(n, m))$ *is an antimagic graph.*

Proof: Let $T(n, m)$ be a tadpole with $V(T(n, m)) = \{v_1, v_2, \dots, v_{m+n}\}$ and $E(T(n, m)) = \{e_i = v_i v_{i+1} / i=1, 2, \dots, n + m\} \cup \{e_{n+m} = v_{n+m} v_{m+1}\}$. To construct $L(T(n, m))$, let the vertices corresponding to e_i be x_i for each i.

Then $|V(L(T(n, m))| = n + m$ and $|E(L(T(n, m))| = n + m + 1$.

We define $f : E(L(T(n, m)) \to \{1, 2, \dots, n + m + 1\}$ as per following seven cases.

Case 1: For $n = 3$ and $m \geqslant n - 1$.

$f(x_i x_{i+1}) = i$; For $1 \leqslant i \leqslant n + m - 1$,

$f(x_{n+m} x_m) = n + m$,

$f(x_{m+1} x_{n+m}) = n + m + 1$.

Case 2: For $n \geqslant 4$ and $(m \geqslant n - 1$ or $m = n - 3)$.

$f(x_i x_{i+1}) = i$; For $1 \leqslant i \leqslant n + m - 1$,

$f(x_{n+m} x_m) = n + m$,

$f(x_{m+1} x_{n+m}) = n + m + 1$.

Case 3: For $(n = 3$ or $n = 4)$ and $m = n - 2$.

$f(x_i x_{i+1}) = i$; For $1 \leqslant i \leqslant n + m - 1$,

$f(x_{n+m} x_m) = n + m + 1$,

$f(x_{m+1} x_{n+m}) = n + m$.

Case 4: For odd $n \geqslant 5$ and $m = 1$.

$f(x_i x_{i+1}) = n - 2i + 2$; For $2 \leqslant i \leqslant \dfrac{n + m}{2}$

$f(x_i x_{i+1}) = 2i - (n + m)$; For $\dfrac{n + m}{2} + 1 \leqslant i \leqslant n + m - 1$

$$f(x_1x_2) = n + m,$$

$$f(x_1x_{n+m}) = n + m + 1,$$

$$f(x_2x_{n+m}) = n + m - 1.$$

Case 5: For even $n \geqslant 6$ and $m = 1$.

$$f(x_ix_{i+1}) = n - 2i + 2; \qquad \text{For } 2 \leqslant i \leqslant \frac{n + m - 1}{2}$$

$$f(x_ix_{i+1}) = 2i - (n + m); \qquad \text{For } \frac{n + m + 1}{2} \leqslant i \leqslant n + m - 1,$$

$$f(x_1x_2) = n + m,$$

$$f(x_1x_{n+m}) = n + m + 1,$$

$$f(x_2x_{n+m}) = n + m - 1.$$

Case 6: For $n \geqslant 5$ and $(1 < m \leqslant n - 4$ or $m = n - 2)$ and $n + m$ is even.

$$f(x_ix_{i+1}) = 2i; \qquad \text{For } 1 \leqslant i \leqslant \frac{n + m}{2},$$

$$f(x_ix_{i+1}) = 2i - (n + m) - 1; \qquad \text{For } \frac{n + m}{2} + 1 \leqslant i \leqslant n + m - 1$$

$$f(x_{n+m}x_m) = n + m - 1,$$

$$f(x_{m+1}x_{n+m}) = n + m + 1.$$

Case 7: For $n \geqslant 6$ and $(1 < m \leqslant n - 4$ or $m = n - 2)$ and $n + m$ is odd.

$$f(x_ix_{i+1}) = 2i; \qquad \text{For } 1 \leqslant i \leqslant \frac{n + m + 1}{2}$$

$$f(x_ix_{i+1}) = 2i - (n + m) - 2; \qquad \text{For } \frac{n + m + 1}{2} + 1 \leqslant i \leqslant n + m - 1$$

$$f(x_{n+m}x_m) = n + m - 2,$$

$$f(x_{m+1}x_{n+m}) = n + m.$$

Above define edge labeling function will generate distinct vertex labels for all the vertices of $L(T(n, m))$. Thus, f is an antimagic labeling.

Hence, the graph $L(T(n, m))$ is an antimagic graph. □

Illustration 3.6: *The graph* $L(T(5, 5))$ *and its antimagic labeling is shown in* Figure 7.

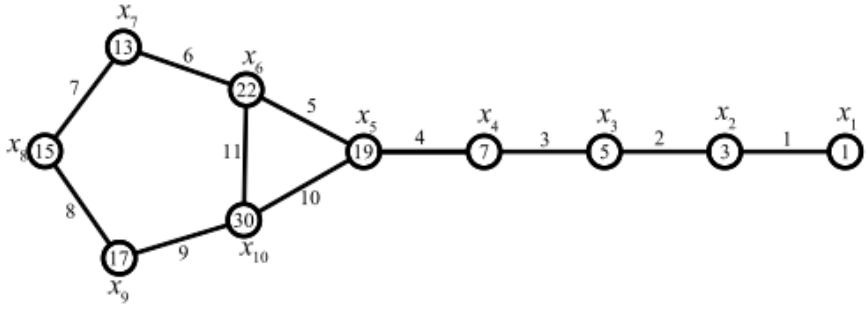


Figure 7: $L(T(5, 5))$ and its antimagic labeling.

4. Applications of Antimagic Labeling

Labeled graph has many applications in computer science, applied sciences, social sciences and cryptography. Development of encryption and decryption algorithm using antimagic labeling was studied by Krishnaa [16], Femina and Xavier [8] and Selvakumar and Gupta [20].

5. Conclusions

It is quite difficult to verify that the given connected graph admits antimagic labeling. Many authors [3, 4, 6, 15, 17, 24] have studied antimagic labeling for various graph operations.

While in this paper, antimagic labeling for the line graph of armed crown, double comb, ladder, wheel and tadpole is verified.

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1. Department of Mathematics,
Hemchandracharya North Gujarat University,
Patan - 384265, Gujarat, INDIA
E-mail: chirag.barasara@gmail.com

2. Department of Mathematics,
Hemchandracharya North Gujarat University,
Patan - 384265, Gujarat, INDIA
E-mail: palakprajapati733@gmail.com

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