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EDGE ODD GRACEFUL LABELING OF SOME SNAKE GRAPHS

Abstract: An edge odd graceful labeling of graph G is a bijection f from the edges of the graph to $\{1, 3, \ldots, 2q-1\}$ such that, when each vertex is assigned the sum of all the edges incident to it $\bmod 2q$ the resulting vertex labels are distinct. A graph is called an edge odd graceful graph as it admits an edge odd graceful labeling. It was intoduced by Solairaju and Chithra in 2008. In this research paper, Edge odd graceful labeling of some snake graphs such as double alternate triangular snake $DA(T_n)$, double alternate quadrilateral snake $DA(Q_n)$ and alternate pentagonal snake $A(PS_n)$ have been discussed.

Keywords: Edge Odd Graceful Labeling, Edge Odd Graceful Graph, Snake Graphs.

Mathematics Subject Classification (2000) No.: 05C78.

1. Introduction

In this research article, all graphs $G = (V(G), E(G))$ are finite, simple, connected and undirected. Here $V(G)$ be the vertex set and $E(G)$ be the edge set of a graph. Graph labeling is an assignment of integers to edges or vertices or both, subject to certain conditions. For an extensive survey on graph labeling and

bibliographic references, we refer to Gallian [2]. A graceful labeling of a graph G, which was introduced by Rosa [6] in 1967, is a injection f from the vertices of the graph to the set $\{1, 2, \ldots, q\}$ such that the induced function f^* from the set of edges to the set $\{0, 1, 2, \ldots, q\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$, is bijective. Soleha *et al*. [10] have proved that the alternate triangular snake and alternate quadrilateral snake graphs are edge odd graceful.

Definition 1.1 [9]: *A function f is called an edge odd graceful labeling of a graph G if $f : E(G) \to \{1, 3, \ldots, 2q-1\}$ is bijective and the induced function $f^* : V(G) \to \{0, 1, 2, \ldots, 2q-1\}$, defined as $f^*(u) = \sum_{uv \in E(G)} f(uv) \pmod{2q}$ is injective.*

A graph which admits an edge odd graceful labeling is called an edge odd graceful graph.

Definition 1.2 [1]: *An alternate triangular snake $A(T_n)$ is obtained from a path P_n with vertices $u_1, u_2, \ldots, u_n$ by joining u_i and $u_i + 1$ (alternatively) to new vertex v_i, where $1 \leqslant i \leqslant n-1$ for even n and for $1 \leqslant i \leqslant n-2$ for odd n.*

That is every alternate edge of a path P_n is replaced by C_3.

Definition 1.3 [1]: *A double alternate triangular snake $DA(T_n)$ is obtained from a path P_n with vertices $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1} (alternatively) to two new vertices v_i and w_i, where $1 \leqslant i \leqslant n-1$ for even n and for $1 \leqslant i \leqslant n-2$ for odd n.*

In other words, the double alternate triangular snake $DA(T_n)$ consists of two alternate triangular snakes that have a common path.

Definition 1.4 [1]: *An alternate quadrilateral snake $A(Q_n)$ is obtained from a path P_n with vertices $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1}*

(alternatively) to two new vertices v_i *and* w_i, *respectively and then joining* v_i, *and* w_i *where* $1 \leqslant i \leqslant n-1$ *for even* n *and for* $1 \leqslant i \leqslant n-2$ *for odd* n.

That is every alternate edge of a path P_n *is replaced by* C_4.

Definition 1.5 [1]: *The double alternate quadrilateral snake* $DA(Q_n)$ *obtained from a path* P_n *with vertices* $u_1, u_2, ..., u_n$ *by joining* u_i *and* $u_i + 1$ *(alternatively) to two new vertices* v_i, w_i *and* v_{i+1}, w_{i+1} *respectively and then joining* v_i, v_{i+1} *and* w_i, w_{i+1}, *where* $1 \leqslant i \leqslant n-1$ *for even* n *and for* $1 \leqslant i \leqslant n-2$ *for odd* n .

In other words, the double alternate quadrilateral snake graph $DA(Q_n)$ *consists of two alternate quadrilateral snakes that have a common path.*

Definition 1.6 [8]: *An alternate pentagonal snake* $A(PS_n)$ *is obtained from a path* P_n *with vertices* $u_1, u_2, ..., u_n$ *by joining* u_i *and* $u_i + 1$ *(alternatively) to new vertices* v_i *and* w_i *respectively and then joining* v_i *and* w_i *to the new vertex* x_i, *where* $1 \leqslant i \leqslant n-1$ *for even* n *and for* $1 \leqslant i \leqslant n-2$ *for odd* n.

That is, every alternate edge of path P_n *is replaced by a cycle* C_5.

2. Main Results

Theorem 2.1: *The double alternate triangular snake* $DA(T_n)$ *is an edge odd graceful graph for all* $n \geqslant 2$.

Proof: Let G be a double alternate triangular snake $DA(T_n)$ which is obtained from a path P_n with vertices $u_1, u_2, ..., u_n$ by joining u_i and $u_i + 1$ (alternatively) to two new vertices v_j and w_j, where $1 \leqslant i \leqslant n-1$ for even n, $1 \leqslant i \leqslant n-2$ for odd n and $1 \leqslant j \leqslant \lfloor \frac{n}{2} \rfloor$.

Therefore

$V(G) = \{u_i, v_j, w_j \ / \ 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant \lfloor \frac{n}{2} \rfloor\}$

$E(G) = \{u_i u_{i+1} / 1 \leqslant i \leqslant n-1\} \cup \{u_{2i-1} v_i / 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor\} \cup \{u_{2i} v_i / 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor\}$
$\cup \{u_{2i-1} w_i / 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor\} \cup \{u_{2i} w_i / 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor\}.$

Here note that

$$|V(G)| = \begin{cases} 2n, & \text{if } n \equiv 0 \pmod 2 \\ 2n-1, & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

$$|E(G)| = \begin{cases} 3n-1, & \text{if } n \equiv 0 \pmod 2 \\ 3n-3, & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

Case 1: $n \equiv 0, 2 \pmod 4$

Subcase 1: $n = 2$

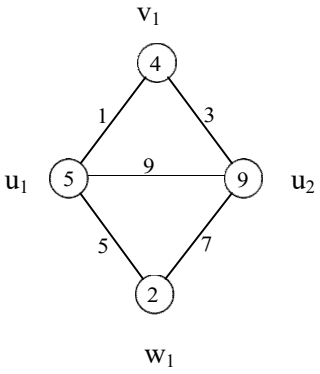


Figure 1: Edge odd graceful labeling of double alternate triangular snake $DA(T_2)$

Here Figure 1 shows that the double alternate triangular snake $DA(T_2)$ is an edge odd graceful graph.

Subcase 2: $n \equiv 0, 2 \pmod 4$ and $n \neq 2$

Define edge labeling $f : E(G) \to \{1, 3, 5, \ldots, 6n-3\}$ is as follows:

$f(u_iu_{i+1}) = 4n - 2i - 1;\ 1 \leqslant i \leqslant n - 1$

$f(u_{2i-1}v_i) = 4i - 3;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f(u_{2i}v_i) = 4i - 1;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f(u_{2i-1}w_i) = 6n - 4i + 1;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f(u_{2i}w_i) = 6n - 4i - 1;\ 1 \leqslant i \leqslant \frac{n}{2}$

The corresponding labels of vertices u_i and $v_i, i = 1, 2, 3\ldots \mod (6n - 2)$ are

$f^*(u_1) = f(u_1u_2) + f(u_1v_1) + f(u_1w_1) = 4n - 3\,;$

$f^*(u_i) = f(u_{i-1}u_i) + f(u_iu_{i+1}) + f(u_iv_{\lceil \frac{i}{2} \rceil}) + f(u_iw_{\lceil \frac{i}{2} \rceil})$

$= 2n - 4i + 2;\ 2 \leqslant i \leqslant n - 1$

$f^*(u_n) = f(u_{n-1}u_n) + f(u_nv_{\frac{n}{2}}) + f(u_nw_{\frac{n}{2}}) = 2n + 1\,;$

$f^*(v_i) = f(u_{2i-1}v_i) + f(u_{2i}v_i) = 8i - 4;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f^*(w_i) = f(u_{2i-1}w_i) + f(u_{2i}w_i) = 6n - 8i + 2;\ 1 \leqslant i \leqslant \frac{n}{2}$

The labels of edges are in the set $\{1, 3, 5, \ldots, 6n - 3\}$. Then the labels of vertices are in the set

$\{4, 12, \ldots, 4n - 4\} \cup \{4n - 3\} \cup \{2n - 6, 2n - 10, \ldots, 4n + 4\} \cup \{2n + 1\}$
$\cup\ \{6n - 6, 6n - 14, \ldots, 2n + 2\}.$

Here $\forall i \neq j,\ f(v_i) \neq f(v_j)$.

Therefore, the induced function $f^* : V(G) \to \{0, 1, 2, \ldots, 2q - 1\}$, defined as $f^*(u) = \sum_{uv \in E(G)} f(uv) \mod (6n - 2)$ is injective.

Case 2: $n \equiv 1 \pmod 4$

Define edge labeling $f : E(G) \to \{1, 3, 5, \ldots, 6n - 7\}$ is as follows:

$f(u_iu_{i+1}) = 4n + 2i - 3;\ 1 \leqslant i \leqslant n - 2$

$f(u_{n-1}u_n) = 2n - 1\,;$

$f(u_{2i-1}v_i) = 4i - 3;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f(u_{2i}v_i) = 4i - 1;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f(u_{2i-1}w_i) = 4n - 4i + 1;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f(u_{2i}w_i) = 4n - 4i - 1;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

The corresponding labels of vertices u_i and $v_i, i = 1, 2, 3 \ldots \bmod (6n - 6)$ are

$f^*(u_1) = f(u_1u_2) + f(u_1v_1) + f(u_1w_1) = 8n - 3\,;$

$f^*(u_i) = f(u_{i-1}u_i) + f(u_iu_{i+1}) + f(u_iv_{\lceil \frac{i}{2} \rceil}) + f(u_iw_{\lceil \frac{i}{2} \rceil})$

$= 6n + 4i - 4;\ 2 \leqslant i \leqslant n - 2$

$f^*(u_{n-1}) = f(u_{n-2}u_{n-1}) + f(u_{n-1}u_n) + f(u_{n-1}v_{\lfloor \frac{n}{2} \rfloor}) + f(u_nw_{\lfloor \frac{n}{2} \rfloor}) = 6n - 4$

$f^*(u_n) = f(u_{n-1}u_n) = 2n - 1\,;$

$f^*(v_i) = f(u_{2i-1}v_i) + f(u_{2i}v_i) = 8i - 4;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f^*(w_i) = f(u_{2i-1}wi) + f(u_{2i}w_i) = 2n - 8i + 6;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

The labels of edges are in the set $\{1, 3, 5, \ldots, 6n - 7\}$. Then the labels of vertices are in the set $\{4, 12, \ldots, 4n - 8\} \cup \{2n + 3\} \cup \{6n + 4, 6n + 8, \ldots, 4n - 6\}$ $\cup\ \{2n - 1\} \cup \{2n - 2, 2n - 10, \ldots, 4n + 4\}$.

Here $\forall i \neq j,\ f(v_i) \neq f(v_j)$.

Therefore, the induced function $f^* : V(G) \to \{0, 1, 2, \ldots, 2q-1\}$, defined as $f^*(u) = \sum_{uv \in E}(G) f(uv) \bmod (6n-6)$ is injective.

Case 3: $n \equiv 3 \pmod 4$

Define edge labeling $f : E(G) \to \{1, 3, 5, \ldots, 6n-7\}$ is as follows:

$f(u_i u_{i+1}) = 2n + 2i - 3;\ 1 \leqslant i \leqslant n-1$

$f(u_{2i-1} v_i) = 4i - 3;\ ;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f(u_{2i} v_i) = 4i - 1;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f(u_{2i-1} w_i) = 4n + 4i - 7;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f(u_{2i} w_i) = 4n + 4i - 5;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

The corresponding labels of vertices u_i and $v_i, i = 1, 2, 3\ldots \bmod (6n-6)$ are

$f^*(u_1) = f(u_1 u_2) + f(u_1 v_1) + f(u_1 w_1) = 6n - 3\,;$

$f^*(u_i) = f(u_{i-1} u_i) + f(u_i u_{i+1}) + f(u_i v_{\lceil \frac{i}{2} \rceil}) + f(u_i w_{\lceil \frac{i}{2} \rceil})$

$= 2n + 8i - 8;\ 2 \leqslant i \leqslant n-1$

$f^*(u_n) = f(u_{n-1} u_n) = 4n - 5\,;$

$f^*(v_i) = f(u_{2i-1} v_i) + f(u_{2i} v_i) = 8i - 4;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f^*(w_i) = f(u_{2i-1} w_i) + f(u_{2i}\, w_i) = ;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

The labels of edges are in the set $\{1, 3, 5, \ldots, 6n-7\}$. Then the labels of vertices are in the set $\{4, 12, \ldots, 4n-4\} \cup \{6n-3\} \cup \{2n+8, 2n+16, \ldots, 4n-10\} \cup \{4n-5\} \cup \{2n+2, 2n+10, \ldots, 6n-10\}$.

Here $\forall i \neq j,\ f(v_i) \neq f(v_j)$.

Therefore, the induced function $f^* : V(G) \to \{0, 1, 2, \ldots, 2q-1\}$, defined as $f^*(u) = \sum_{uv \in E(G)} f(uv) \bmod (6n-6)$ is injective.

Example 2.2: *The edge odd graceful labeling of double alternate triangular snake* $DA(T_8)$, $DA(T_9)$ *and* $DA(T_{11})$ *is shown in* Figure 2, 3, *and* 4.

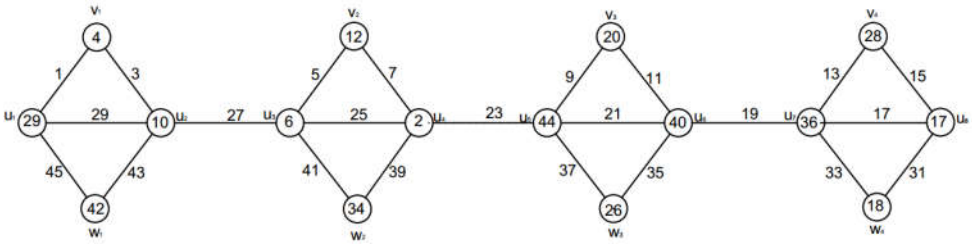


Figure 2: The edge odd graceful labeling of double alternate triangular snake $DA(T_8)$

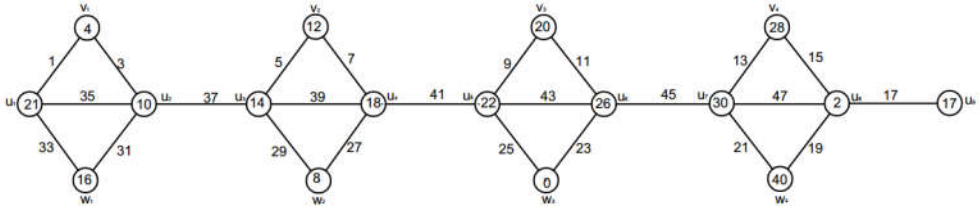


Figure 3: The edge odd graceful labeling of double alternate triangular snake $DA(T_9)$

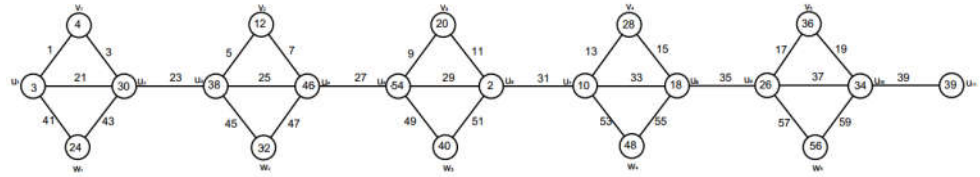


Figure 4: The edge odd graceful labeling of double alternate triangular snake $DA(T_{11})$.

In each possibility the graph under consideration satisfies the vertex conditions and edge conditions for an edge odd graceful labeling. Hence, the double alternate triangular snake $DA(T_n)$ is an edge odd graceful graph for all $n \geqslant 2$. □

Theorem 2.3: *The double alternate quadrilateral snake $DA(Q_n)$ is an edge odd graceful graph for all $n \geqslant 2$.*

Proof: Let G be a double alternate quadrilateral snake $DA(Q_n)$ which is obtained from a path P_n with vertices $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1} (alternatively) to two new vertices v_i, w_i and v_{i+1}, w_{i+1} respectively and then joining v_i, v_{i+1} and w_i, w_{i+1}, where $1 \leqslant i \leqslant n-1$ for even n and for $1 \leqslant i \leqslant n-2$ for odd n.

Therefore,

$$V(G) = \{u_i, v_i, w_i / 1 \leqslant i \leqslant n\}$$

$$E(G) = \{u_i u_{i+1} / 1 \leqslant i \leqslant n-1\} \cup \{v_{2i-1}v_{2i} / 1 \leqslant i \leqslant \lfloor \tfrac{n}{2} \rfloor\}$$
$$\cup \{w_{2i-1}w_{2i} / 1 \leqslant i \leqslant \lfloor \tfrac{n}{2} \rfloor\} \cup \{u_i v_i / 1 \leqslant i \leqslant n\} \cup \{v_i w_i / 1 \leqslant i \leqslant n\}.$$

Here note that

$$|V(G)| = \begin{cases} 3n, & \text{if } n \equiv 0 \pmod 2 \\ 3n-2, & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

$$|E(G)| = \begin{cases} 4n-1, & \text{if } n \equiv 0 \pmod 2 \\ 4n-4, & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

Case 1: $n \equiv 0 \pmod 2$

Subcase 1: $n = 2$

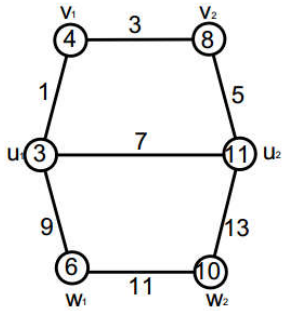


Figure 5: Edge odd graceful labeling of double alternate quadrilateral snake graph $DA(Q_2)$

Here Figure 5 shows that double alternate quadrilateral snake $DA(Q_2)$ is an edge odd graceful graph.

Subcase 2: $n \equiv 0 \pmod 2$ and $n \neq 2$

Define edge labeling $f : E(G) \rightarrow \{1, 3, 5, \ldots 8n - 3\}$ is as follows:

$f(u_i u_{i+1}) = 3n + 2i - 1;\ 1 \leqslant i \leqslant n - 1$

$f(u_{2i-1} v_{2i-1}) = 6i - 5;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f(u_{2i} v_{2i}) = 6i - 1;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f(v_i v_{i+1}) = 6i - 3;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f(u_{2i-1} w_{2i-1}) = 8n - 6i + 3;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f(u_{2i} w_{2i}) = 8n - 6i - 1;\ 2 \leqslant i \leqslant \frac{n}{2}$

$f(w_i w_{i+1}) = 8n - 6i + 1;\ 1 \leqslant i \leqslant \frac{n}{2}$

The corresponding labels of vertices u_i and v_i, $i = 1, 2, 3 \ldots \bmod (8n - 2)$ are

$f^*(u_1) = f(u_1 u_2) + f(u_1 v_1) + f(u_1 w_1) = 3n + 1;$

$f^*(u_i) = f(u_{i-1}u_i) + f(u_iu_{i+1}) + f(u_iv_i) + f(u_iw_i) = 6n + 4i - 4;\ 2 \leqslant i \leqslant n - 1$

$f^*(u_n) = f(u_{n-1}u_n) + f(u_nv_n) + f(u_nw_n) = 5n - 3$

$f^*(v_{2i-1}) = f(u_{2i-1}v_{2i-1}) + f(v_{2i-1}v_{2i}) = 12i - 8;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f^*(v_{2i}) = f(v_{2i-1}v_{2i}) + f(v_{2i}u_{2i}) = 12i - 4;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f^*(w_{2i-1}) = f(u_{2i-1}w_{2i-1}) + f(w_{2i-1}w_{2i}) = 8n - 12i + 6;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f^*(w_{2i}) = f(w_{2i-1}w_{2i}) + f(w_{2i}u_{2i}) = 8n - 12i + 2;\ 1 \leqslant i \leqslant \frac{n}{2}$

The labels of edges are in the set $\{1, 3, 5, \ldots, 8n - 3\}$. Then the labels of vertices are in the set $\{4, 16, \ldots, 6n - 8\} \cup \{8, 20, \ldots, 6n - 4\} \cup \{3n + 1\} \cup \{6n + 4, 6n + 8, \ldots, 2n - 6\} \cup \{5n - 3\} \cup \{8n - 6, 8n - 18, \ldots, 2n + 6\} \cup \{8n - 10, 8n - 22, \ldots, 2n + 2\}$. Here $\forall i \neq j,\ f(v_i) \neq f(v_j)$.

Therefore, the induced function $f^* : V(G) \to \{0, 1, 2, \ldots, 2q - 1\}$, defined as $f^*(u) = \sum_{uv \in E(G)} f(uv) \bmod (8n - 2)$ is injective.

Case 2: $n \equiv 1 \pmod 2$

Define edge labeling $f : E(G) \to \{1, 3, 5, \ldots 8n - 9\}$ is as follows:

$f(u_iu_{i+1}) = 6n + 2i - 7;\ 1 \leqslant i \leqslant n - 1$

$f(u_{2i-1}v_{2i-1}) = 4i - 3;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f(u_{2i}v_{2i}) = 4n + 4i - 7;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f(v_iv_{i+1}) = 2n + 4i - 5;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f(u_{2i-1}w_{2i-1}) = 4n + 4i - 5;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f(u_{2i}w_{2i}) = 4i - 1;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f(w_iw_{i+1}) = 2n + 4i - 3;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

The corresponding labels of vertices u_i and v_i, $i = 1, 2, 3\ldots \mod (8n - 8)$ are

$f^*(u_1) = f(u_1u_2) + f(u_1v_1) + f(u_1w_1) = 2n + 3$;

$f^*(u_{2i-1}) = f(u_{2i-2}u_{2i-1}) + f(u_{2i-1}u_{2i}) + f(u_{2i-1}v_{2i-1}) + f(u_{2i-1}w_{2i-1})$,
$= 8n + 16i - 20;\ 2 \leqslant i \leqslant \lfloor \frac{n-2}{2} \rfloor$

$f^*(u_{2i}) = f(u_{2i-1}u_{2i}) + f(u_{2i}u_{2i+1}) + f(u_{2i}v_{2i}) + f(u_{2i}w_{2i})$
$= 8n + 16i - 16;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f^*(un) = f(u_{n-1}u_n) = 8n - 9$

$f^*(v_{2i-1}) = f(u_{2i-1}v_{2i-1}) + f(v_{2i-1}v_{2i}) = 2n + 8i - 8;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f^*(v_{2i}) = f(u_{2i}v_{2i}) + f(v_{2i-\ 1}v_{2i}) = 6n + 8i - 12;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f^*(w_{2i-1}) = f(u_{2i-1}w_{2i-1}) + f(w_{2i-1}w_{2i}) = 6n + 8i - 8;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f^*(w_{2i}) = f(u_{2i}w_{2i}) + f(w_{2i-1}w_{2i}) = 2n + 8i - 4;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

The labels of edges are in the set $\{1, 3, 5, \ldots, 8n - 9\}$. Then the labels of vertices are in the set $\{2n,\ 2n + 8, \ldots, 6n - 12\} \cup \{6n - 4, 6n + 4, \ldots, 2n - 8\} \cup \{2n + 3\} \cup \{8, 24, \ldots, 8n - 16\} \cup \{2n + 3\} \cup \{20, 36, , \ldots, 8n - 20\} \cup \{8n - 9\} \cup \{6n, 6n + 8, \ldots, 2n - 4\} \cup \{2n + 4, 2n + 12, \ldots, 6n - 8\}$.

Here $\forall i \neq j,\ f(v_i) \neq f(v_j)$.

Therefore, the induced function $f^* : V(G) \to \{0, 1, 2, \ldots, 2q - 1\}$, defined $f^*(u) = \sum_{uv \in E(G)} f(uv) \mod (8n - 8)$ is injective.

Example 2.4: *The edge odd graceful labeling of the double alternate quadrilateral snake* $DA(Q_6)$ *and* $DA(Q_7)$ *is shown in the following* Figure 6 *and* 7.

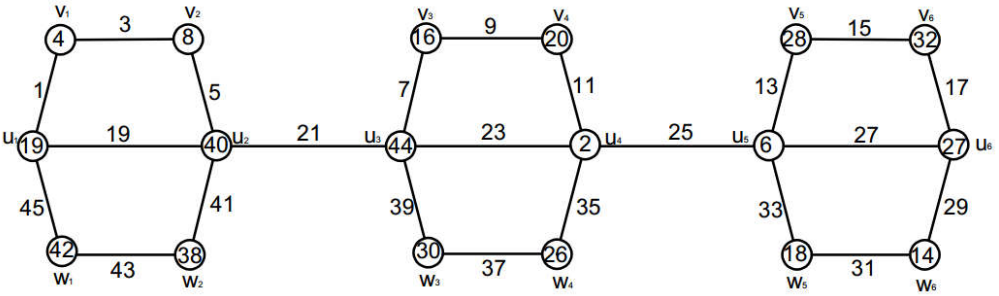


Figure 6: The edge odd graceful labeling of double alternate quadrilateral snake $DA(Q6)$

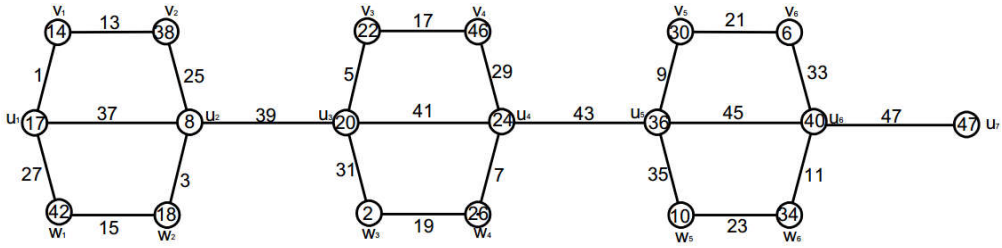


Figure 7: The edge odd graceful labeling of double alternate quadrilateral snake $DA(Q_7)$.

In each possibility the graph under consideration satisfies the vertex conditions and edge conditions for an edge odd graceful labeling. Hence, the double alternate quadrilateral snake $DA(Q_n)$ is an edge odd graceful graph for all $n \geqslant 2$. □

Theorem 2.5: *The alternate pentagonal snake* $A(PS_n)$ *is an edge odd graceful graph for all* $n \geqslant 2$.

Proof: Let G be a alternate pentagonal snake $A(PS_n)$ which is obtained from a path P_n with vertices $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1} (alternatively) to new vertices v_j and w_j respectively and then joining v_j and w_j to the new

vertex x_j, where $1 \leqslant i \leqslant n-1$ for even $n, 1 \leqslant i \leqslant n-2$ for odd n and $1 \leqslant j \leqslant \lfloor \frac{n}{2} \rfloor$.

Therefore,

$$V(G) = \{u_i, v_j, w_j, x_j / 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant \lfloor \tfrac{n}{2} \rfloor\}$$

$$E(G) = \{u_i u_{i+1} / 1 \leqslant i \leqslant n-1\} \cup \{u_{2i-1} v_i / 1 \leqslant i \leqslant \lfloor \tfrac{n}{2} \rfloor\} \cup \{v_i x_i / 1 \leqslant i \leqslant \lfloor \tfrac{n}{2} \rfloor\}$$
$$\cup \{x_i w_i / 1 \leqslant i \leqslant \lfloor \tfrac{n}{2} \rfloor\} \cup \{w_i u_{2i} / 1 \leqslant i \leqslant \lfloor \tfrac{n}{2} \rfloor\}.$$

Here note that

$$|V(G)| = \begin{cases} \frac{5n}{2}, & \text{if } n \equiv 0 \pmod 2 \\ \lfloor \frac{5n}{2} \rfloor + 1, & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

$$|E(G)| = \begin{cases} 3n-1, & \text{if } n \equiv 0 \pmod 2 \\ 3n-3, & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

Case 1: $n \equiv 0 \pmod 2$

Subcase 1: $n \equiv 0 \pmod 6$

Define edge labeling $f : E(G) \to \{1, 3, 5, \ldots, 6n-2\}$ is as follows:

$f(u_i u_{i+1}) = 4n + 2i - 1;\ 1 \leqslant i \leqslant n-1$

$f(u_{2i-1} v_i) = 2n + 4i - 3;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f(v_i x_i) = 4i - 3;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f(w_i x_i) = 2n + 4i - 1;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f(u_{2i} w_i) = 4i - 1;\ 1 \leqslant i \leqslant \frac{n}{2}$

The corresponding labels of vertices u_i and $v_i, i = 1, 2, 3\ldots \mod(6n-2)$ are

$f^*(u_1) = f(u_1u_2) + f(u_1v_1) = 6n + 2$

$f^*(u_{2i}) = f(u_{2i-1}u_{2i}) + f(u_{2i}u_{2i+1}) + f(u_{2i}w_i) = 2n + 12i - 3;\ 1 \leqslant i \leqslant \frac{n-2}{2}$

$f^*(u_{2i-1}) = f(u_{2i-1}u_{2i}) + f(u_{2i}u_{2i+1}) + f(u_{2i-1}v_i) = 4n + 12i - 9;\ 2 \leqslant i \leqslant \frac{n}{2}$

$f^*(u_n) = f(u_{n-1}u_n) + f(w_{n/2}u_n) = 8n - 4$;

$f^*(v_i) = f(u_{2i-1}v_i) + f(v_ix_i) = 2n + 8i - 6;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f^*(x_i) = f(v_ix_i) + f(x_iw_i) = 2n + 8i - 4;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f^*(w_i) = f(x_iw_i) + f(w_iu_{2i}) = 2n + 8i - 2;\ 1 \leqslant i \leqslant \frac{n}{2}$

The labels of edges are in the set $\{1, 3, 5, \ldots, 6n-3\}$ .Then the labels of vertices are in the set $\{6n+2\} \cup \{2n+9, 2n+21, \ldots, 2n-13\}$ $\cup \{4n+15, 4n+ \quad 27, \ldots, 4n-7\} \cup \{2n-2\} \cup \{2n+2, 2n+10, \ldots, 6n-6\}$ $\cup \{2n+4, 2n+12, \ldots, 6n-4\} \cup \{2n+6, 2n+14, \ldots, 6n-2\}$.

Here $\forall i \neq j,\ f(v_i) \neq f(v_j)$.

Therefore, the induced function $f^* : V(G) \to \{0, 1, 2, \ldots, 2q-1\}$, defined as $f^*(u) = \sum_{uv \in E(G)} f(uv) \mod (6n-2)$ is injective.

Subcase 2: $n \equiv 2, 4 \pmod 6$

Define edge labeling $f : E(G) \to \{1, 3, 5, \ldots, 6n-2\}$ is as follows:

$f(u_iu_{i+1}) = 4n + 2i - 1;\ 1 \leqslant i \leqslant n-1$

$f(u_{2i-1}v_i) = 8i - 7;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f(v_ix_i) = 8i - 5;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f(w_i x_i) = 8i - 3;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f(u_{2i} w_i) = 8i - 1;\ 1 \leqslant i \leqslant \frac{n}{2}$

The corresponding labels of vertices v_i and $v_i, i = 1, 2, 3\ldots \mod (6n - 2)$ are

$f^*(u_1) = f(u_1 u_2) + f(u_1 v_1) = 4n + 2\,;$

$f^*(u_{2i}) = f(u_{2i} u_{2i+1}) + f(u_{2i-1} u_{2i}) + f(u_{2i} w_i) = 2n + 16i - 3;\ 1 \leqslant i \leqslant \frac{n-2}{2}$

$f^*(u_{2i-1}) = f(u_{2i-1} u_{2i}) + f(u_{2i} u_{2i+1}) + f(u_{2i-1} v_i) = 2n + 16i - 13;\ 2 \leqslant i \leqslant \frac{n}{2}$

$f^*(u_n) = f(u_{n-1} u_n) + f(w_{n/2} u_n) = 4n - 2\,;$

$f^*(v_i) = f(u_{2i-1} v_i) + f(v_i x_i) = 16i - 12;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f^*(x_i) = f(v_i x_i) + f(x_i w_i) = 16i - 8;\ 1 \leqslant i \leqslant \frac{n}{2}$

$f^*(w_i) = f(x_i w_i) + f(w_i u_{2i}) = 16i - 4;\ 1 \leqslant i \leqslant \frac{n}{2}$

The labels of edges are in the set $\{1, 3, 5, \ldots, 6n - 3\}$. Then the labels of vertices are in the set $\{4n + 2\} \cup \{2n + 13, 2n + 29, \ldots, 4n - 17\} \cup \{2n + 19, 2n + 35, \ldots, 4n - 11\} \cup \{8, 24, \ldots, 2n - 6\} \cup \{12, 28, \ldots, 2n - 2\}$.

Here $\forall i \neq j,\ f(v_i) \neq f(v_j)$.

Therefore, the induced function $f^* : V(G) \to \{0, 1, 2, \ldots, 2q - 1\}$, defined as $f^*(u) = \sum_{uv \in E(G)} f(uv) \mod (6n - 2)$ is injective.

Case 2: $n \equiv 1 \pmod 2$

Here note that

$$|V(G)| = 5\lfloor \tfrac{n}{2} \rfloor + 1$$

$$|E(G)| = 3n - 3$$

Define edge labeling $f : E(G) \to \{1, 3, 5, \ldots 6n - 7\}$ is as follows:

$f(u_i u_{i+1}) = 6n - 2i - 5;\ 1 \leqslant i \leqslant n - 1$

$f(u_{2i-1} v_i) = 4i - 3;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f(v_i x_i) = 2n + 4i - 5;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f(w_i x_i) = 4i - 1;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f(u_{2i} w_i) = 2n + 4i - 3;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

The corresponding labels of vertices u_i and v_i, $i = 1, 2, 3 \ldots \mod (6n - 6)$ are

$f^*(u_1) = f(u_1 u_2) + f(u_1 v_1) = 6n - 6\,;$

$f^*(u_{2i}) = f(u_{2i} u_{2i+1}) + f(u_{2i-1} u_{2i}) + f(u_{2i}\, w_i) = 2n - 4i + 1;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f^*(u_{2i-1}) = f(u_{2i-1} u_{2i}) + f(u_{2i} u_{2i+1}) + f(u_{2i-1} v_i) = 6n - 4i - 1;\ 2 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f^*(u_n) = f(u_{n-1} u_n) = 4n - 3\,;$

$f^*(v_i) = f(u_{2i-1} v_i) + f(v_i x_i) = 2n + 8i - 8;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f^*(x_i) = f(v_i x_i) + f(x_i w_i) = 2n + 8i - 6;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

$f^*(w_i) = f(x_i w_i) + f(w_i u_{2i}) = 2n + 8i - 4;\ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$

The labels of edges are in the set $\{1, 3, 5, \ldots, 6n - 7\}$.

Then the labels of vertices are in the set $\{6n - 6\} \cup \{2n - 3, 2n - 7, \ldots, 6n - 3\}$ $\cup \{6n - 9, 6n - 13, \ldots, 4n + 1\} \cup \{4n - 3\} \cup \{2n, 2n + 8, \ldots, 6n - 12\}$ $\cup \{2n + 2, 2n + 10, \ldots, 6n - 10\} \cup \{2n + 4, 2n + 12, \ldots, 6n - 8\}$.

Here $\forall i \neq j,\ f(v_i) \neq f(v_j)$.

Therefore, the induced function $f^* : V(G) \to \{0, 1, 2, \ldots, 2q - 1\}$, defined as $f^*(u) = \sum_{uv \in E(G)} f(uv) \bmod (6n - 6)$ is injective.

Example 2.6: *The edge odd graceful labeling of the alternate pentagonal snake* $A(PS_6)$, $A(PS_{10})$ *and* $A(PS_9)$ *is shown in* Figure 8, 9 *and* 10.

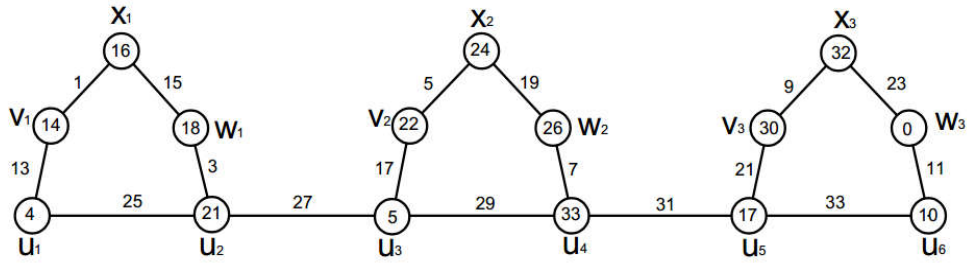


Figure 8: The edge odd graceful labeling of alternate pentagonal snake $A(PS_6)$.

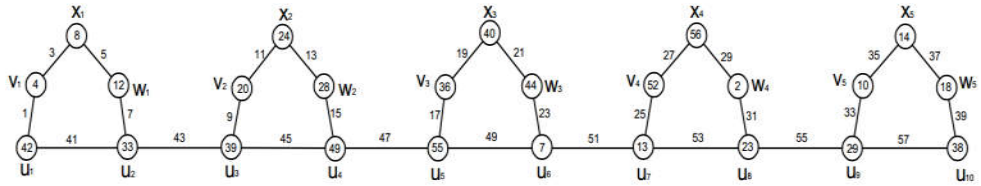


Figure 9: The edge odd graceful labeling of alternate pentagonal snake $A(PS_{10})$.

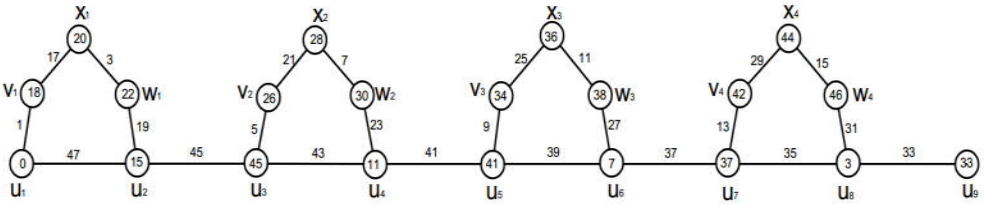


Figure 10: The edge odd graceful labeling of alternate pentagonal snake graph $A(PS_9)$

In each possibility the graph under consideration satisfies the vertex conditions and edge conditions for an edge odd graceful labeling. Hence, the alternate pentagonal snake $A(PS_n)$ is an edge odd graceful graph for all $n \geqslant 2$. □

3 Conclusion

In this paper, it is proved that double alternate triangular snake $DA(T_n)$, double alternate quadrilateral snake $DA(Q_n)$ and alternate pentagonal snake $A(PS_n)$ are edge odd graceful graphs. To derive new families of graphs that admit edge odd graceful labeling is an open area of research.

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