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Girish D. Shelake[1]
Sarika K. Nilapgol[2]
and
Santosh B. Joshi[3]

BI-CONCAVE AND OZAKI-TYPE BI-CLOSE-TO-CONVEX FUNCTIONS ASSOCIATED WITH MILLER-ROSS TYPE POISSON DISTRIBUTION SUBORDINATE TO INVOLUTION NUMBERS

Abstract: The purpose of this article is to study new subclasses of bi-univalent functions related to the Miller-Ross type Poisson distribution, which is subordinate to the generalized telephone numbers. Here, we introduce two new subclasses of Ozaki-type bi-close-to-convex functions and bi-concave functions. For the functions, In these new classes, we estimate the first two Taylor-Maclaurin coefficients and Fekete-Szegö problem.

Keywords: Univalent Functions, Bi-Univalent Functions, Bi-Convex Function, Miller-Ross Type Poisson Distribution, Subordination, Fekete-Szegö Problem.

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1. Introduction

We begin by considering that $\mathfrak{B}$ represents the class of analytic functions defined as

$$\mathcal{T}(z) = z + \sum_{r=2}^{\infty} d_r z^r \qquad z \in \mathfrak{O}, \tag{1.1}$$

those are analytic in open unit disk $\mathfrak{O} = \{z : z \in \mathbb{C}, |z| < 1\}$. Let us denote the $\mathcal{G}$ as the family of all analytic and univalent functions in $\mathfrak{O}$.

Koebe one-quarter Theorem [11] states that, the image of $\mathfrak{D}$ under any univalent function $\tau \in \mathfrak{S}$ contains the disk of radius 1/4. As a result, every function τ has an inverse τ^{-1} given by

$$\tau^{-1}(z) = \mathcal{G}(z) = w - d_2 w^2 + (2d_2^2 - d_3)w^3 - (5d_2^3 - 5d_2 d_3 + d_4)w^4 + \cdots.$$

A function $\tau \in \mathfrak{S}$ is bi-univalent in $\mathfrak{D}$ if both τ and τ^{-1} are univalent in $\mathfrak{D}$. Let us denote Σ as the class of bi-univalent functions.

Assume that g_1 and g_2 are analytic functions that are defined in $\mathfrak{D}$. We say that g_1 is subordinate to g_2 i.e. $g_1(z) \prec g_2(z)$, when we can identify a function w with analytic properties in domain $\mathfrak{D}$, as follows:

$$w(0) = 0, |w(z)| < 1 \text{ and } g_1(z) = g_2(w(z)).$$

In particular, g_2 is univalent in $\mathfrak{D}$ then the below equivalence is obtained.

$$g_1 \prec g_2 \Leftrightarrow g_1(0) = g_2(0) \text{ and } g_1(\mathfrak{D}) \subset g_2(\mathfrak{D}).$$

A function $\tau : \mathfrak{S} \to \mathbb{C}$ belongs to the class of concave functions if τ satisfies conditions listed below:

- τ is analytic in $\mathfrak{D}$ and satisfying normalization conditions $\tau(0) = \tau'(0) - 1 = 0$.
- τ maps $\mathfrak{D}$ conformally onto a set whose complement with respect to $\mathbb{C}$ is convex.
- The opening angle $\tau(\mathfrak{D})$ at ∞ is less than or equal to $\pi\vartheta, \vartheta \in (1, 2]$.

The class $\mathcal{CV}(\vartheta)$ represents the class of concave analytic and univalent functions (for details, see [5; 3; 4; 25; 24]) and the functions of this class satisfy below inequality:

$$\Re\left(1+\frac{z\mathcal{T}''(z)}{\mathcal{T}'(z)}\right)<0 \qquad z\in\mathfrak{O}$$

Bhowmik B., Ponnusamy S., Wirths K. [7] established that a function maps $\mathfrak{O}$ onto an angled concave domain $\pi\vartheta$ if and only if

$$\Re\left\{\frac{1}{\vartheta-1}\frac{(\vartheta+1)(1+z)}{2(1-z)}-1-\frac{z\mathcal{T}''(z)}{\mathcal{T}'(z)}\right\}>0\,.$$

Numerous studies on bi-univalent function subclasses may be found in the varied publications [9; 8; 16; 21]. Motivated by works [27; 30; 28; 31; 20; 2], we analyze the novel subclasses of concave and bi-close-to-convex functions.

Let us consider the Miller-Ross function [17] (also see [15; 29]) and is denoted as

$$\mathcal{G}_{\upsilon,\eta}(z)=z^{\upsilon}\sum_{r=0}^{\infty}\frac{(\eta z)^r}{\Gamma(r+\upsilon+1)},\qquad \upsilon,\eta,z\in\mathbb{C}.$$

The two parameter Mittag-Leffler function [32], $\mathcal{E}_{\theta,\mu}(z)$ is given by

$$\mathcal{E}_{\theta,\mu}(z)=\sum_{r=0}^{\infty}\frac{z^r}{\Gamma(\theta r+\mu)},\qquad \theta,\mu,z\in\mathbb{C},\Re\{\theta,\mu\}>0.$$

For $\mu=1$, we have the Mittag-Leffler function [18],

$$\mathcal{E}_{\theta}(z)=\sum_{r=0}^{\infty}\frac{z^r}{\Gamma(\theta r+1)},\qquad \theta,z\in\mathbb{C},\Re\{\theta\}>0.$$

The Miller-Ross function [12] can be represented as

$$G_{\upsilon,\eta}(z)=z^{\upsilon}\mathcal{E}_{1,1+\upsilon}(\eta z).$$

Recently, Şeker *et al*. [26] represented a power series with the corresponding coefficients are the Miller-Ross type Poisson distribution, which is as shown below:

$$\mathcal{Q}_{\upsilon,\eta}^{\sigma}(z) = z\sum_{r=2}^{\infty}\frac{\sigma^{\upsilon}(\eta\sigma)^{r-1}}{\Gamma(r+\upsilon)\mathcal{G}_{\upsilon,\eta}(\sigma)}z^{r}, \qquad z \in \mathfrak{O}, \upsilon > -1, \eta > 0. \tag{1.2}$$

Now, consider the convolution of functions (1.1) and (1.2), an operator $\mathcal{M}_{\upsilon,\eta}^{\sigma} : \mathfrak{B} \to \mathfrak{B}$ written as:

$$\begin{aligned}\mathcal{M}_{\upsilon,\eta}^{\sigma}\mathcal{T}(z) &= \mathcal{Q}_{\upsilon,\eta}^{\sigma}(z) * \mathcal{T}(z)\\ &= z + \sum_{r=2}^{\infty}\frac{\sigma^{\upsilon}(\eta\sigma)^{r-1}}{\Gamma(r+\upsilon)\mathcal{G}_{\upsilon,\eta}(\sigma)}d_{r}z^{r}\\ &= z + \sum_{r=2}^{\infty}\varphi_{r}d_{r}z^{r}, \qquad z \in \mathfrak{O}, \upsilon > -1, \eta > 0.\end{aligned}$$

where, $\varphi_{r} = \dfrac{\sigma^{\upsilon}(\eta\sigma)^{r-1}}{\Gamma(r+\upsilon)\mathcal{G}_{\upsilon,\eta}(\sigma)}$.

In particular,

$$\varphi_{2} = \frac{\sigma^{\upsilon}(\eta\sigma)}{\Gamma(2+\upsilon)\mathcal{G}_{\upsilon,\eta}(\sigma)}, \text{ and } \varphi_{3} = \frac{\sigma^{\upsilon}(\eta\sigma)^{2}}{\Gamma(3+\upsilon)\mathcal{G}_{\upsilon,\eta}(\sigma)}. \tag{1.3}$$

1.1 Involution Numbers: Considering the involution numbers (that are also referred to as telephone numbers (TN)), the recurrence relation is

$$\mathcal{V}(r) = \mathcal{V}(r-1) + (r-1)\mathcal{V}(r-2), \quad r \geq 2,$$

with $\mathcal{V}(0) = \mathcal{V}(1) = 1$.

New generalized telephone numbers (GTNs) were recently identified in 2019 by Bednarz and Wolowiec-Musial [6] and are represented as

$$\mathcal{V}_{\epsilon}(r) = \mathcal{V}_{\epsilon}(r-1) + \epsilon(r-1)\mathcal{V}_{\epsilon}(r-2), \qquad r \geq 2, \epsilon \geq 1,$$

with $\mathcal{V}_{\epsilon}(0) = \mathcal{V}_{\epsilon}(1) = 1$.

GTNs are presented in exponential series form by

$$e^{x+\epsilon\frac{x^2}{2}} = \sum_{r=0}^{\infty} \mathcal{V}_{\epsilon}(r)\,\frac{x^r}{r!}, \qquad \epsilon \geq 1.$$

Thus, for $\epsilon = 1$, we have TNs $\mathcal{V}(r)$ and for specific values of r, $\mathcal{V}_{\epsilon}(r)$ is provided as

1. $\mathcal{V}_{\epsilon}(0) = \mathcal{V}_{\epsilon}(1) = 1,$
2. $\mathcal{V}_{\epsilon}(2) = 1 + \epsilon,$
3. $\mathcal{V}_{\epsilon}(3) = 1 + 3\epsilon,$
4. $\mathcal{V}_{\epsilon}(4) = 1 + 6\epsilon + 3\epsilon^2,$
5. $\mathcal{V}_{\epsilon}(5) = 1 + 10\epsilon + 15\epsilon^2.$

Let us consider the function

$$\begin{aligned}\Psi(x) &= e^{x+\epsilon\frac{x^2}{2}} \\ &= \sum_{r=0}^{\infty} \mathcal{V}_{\epsilon}(r)\,\frac{x^r}{r!} \\ &= 1 + x + (1+\epsilon)\,\frac{x^2}{2} + (1+3\epsilon)\,\frac{x^3}{6} + (1+6\epsilon+3\epsilon^2)\,\frac{x^4}{24} + \cdots.\end{aligned}$$

For $x \in \mathfrak{O}$. (see also [19; 10])

We introduce two novel subclasses of bi-univalent functions connected with the Poisson distribution of Miller-Ross type that are subordinate to GTNs in our current paper. In addition, we estimate the Fekete-Szego inequality and the Taylor-Maclaurin coefficients $|d2|$, $|d3|$, for the newly defined classes.

2. Ozaki-type Bi-Close-to-Convex Functions

In 1952, Kaplan [13] introduced the class $\mathfrak{K}$ of close-to-convex functions. In 1935, Ozaki [22] had already identified these functions, satisfying the following inequality:

$$\Re\left(1+\frac{z\mathcal{T}'(z)}{\mathcal{T}(z)}\right)>-\frac{1}{2}, \qquad z\in\mathfrak{D}. \tag{2.1}$$

Kaplan [8] states that the function which satisfy inequality (2.1) are close-to-convex functions and which are categorized under class $\mathfrak{S}$. The Ozaki inequality was further generalized by Kargar and Gebadian [14]. (For details see [22; 1]) A function $\mathcal{T}\in\mathfrak{B}$ is locally univalent and is said to be Ozaki close-to-convex function if it satisfy the condition:

$$\Re\left(1+\frac{z\mathcal{T}'(z)}{\mathcal{T}(z)}\right)>\frac{1}{2}-\vartheta, \qquad z\in\mathfrak{D},\vartheta\in(-\tfrac{1}{2},\tfrac{1}{2}]\,.$$

Definition 1: *The class* $OBCV_{\Sigma}(\vartheta,\sigma,\upsilon,\eta)$ *includes all functions* $\mathcal{T}\in\mathfrak{B}$ *if it satisfies the following subordination conditions*:

$$\frac{2\vartheta-1}{2\vartheta+1}+\frac{2}{2\vartheta+1}\left[\frac{((z\mathcal{M}^{\sigma}_{\upsilon,\eta}\mathcal{T}(z))')'}{(\mathcal{M}^{\sigma}_{\upsilon,\eta}\mathcal{T}(z))'}\right]\prec\Psi(z), \tag{2.2}$$

and

$$\frac{2\vartheta-1}{2\vartheta+1}+\frac{2}{2\vartheta+1}\left[\frac{((w\mathcal{M}^{\sigma}_{\upsilon,\eta}\mathcal{G}(w))')'}{(\mathcal{M}^{\sigma}_{\upsilon,\eta}\mathcal{G}(w))'}\right]\prec\Psi(z), \tag{2.3}$$

where $\mathcal{T}^{-1}(w)=\mathcal{G}(w)$ *and* $\frac{1}{2}\le\vartheta\le 1$.

Remark 1: *For* $\vartheta=\frac{1}{2}$, *the class* $OBCV_{\Sigma}(\vartheta,\sigma,\upsilon,\eta)=CV_{\Sigma}(\vartheta)$ *includes all functions* $\mathcal{T}\in\Sigma$ *if*

$$\frac{((z\mathcal{M}^{\sigma}_{\upsilon,\eta}\mathcal{T}(z))')'}{(\mathcal{M}^{\sigma}_{\upsilon,\eta}\mathcal{T}(z))'}\prec\Psi(z), \qquad \text{and} \qquad \frac{((w\mathcal{M}^{\sigma}_{\upsilon,\eta}\mathcal{G}(w))')'}{(\mathcal{M}^{\sigma}_{\upsilon,\eta}\mathcal{G}(w))'}\prec\Psi(z)$$

where $\mathcal{T}^{-1}(w)=\mathcal{G}(w)$.

The following lemma [23] will be necessary for proving the main findings.

Lemma 1: *If* $h \in \mathcal{P}$, *then* $|c_k| \leq 2$ *for each* k, *where* $\mathcal{P}$ *is the family of all functions* h, *analytic in* $\mathfrak{O}$, *for which* $\Re[h(z)] > 0$ $(z \in \mathfrak{O})$, *where*

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots (z \in \mathfrak{O}).$$

Theorem 1: *A function* $\tau \in \mathfrak{B}$ *form* (1.1) *is in class* $OBCV_{\Sigma}(\vartheta, \sigma, \upsilon, \eta)$, *then*

$$|d_2| \leq \min\left\{\frac{2\vartheta + 1}{4\varphi_2}, \frac{2\vartheta + 1}{2\sqrt{(2\vartheta + 1)(3\varphi_3 - 2\varphi_2^2) + 2(1 - \epsilon)\varphi_2^2}}\right\}, \tag{2.4}$$

and

$$|d_2| \leq \min\left\{\frac{2\vartheta + 1}{12\varphi_3} + \frac{(2\vartheta + 1)^2}{12\varphi_2^2}, \frac{2\vartheta + 1}{12\varphi_3} + \frac{(2\vartheta + 1)^2}{4[(2\vartheta + 1)(3\varphi_3 - 2\varphi_2^2) + 2(1 - \epsilon)\varphi_2^2}\right\} \tag{2.5}$$

where φ_2 *and* φ_3 *are as given in* (1.3).

Proof: Let us consider $s(z)$ and $t(z)$ as

$$s(z) = \frac{1 + l(z)}{1 - l(z)} = 1 + s_1 z + s_2 z^2 + \cdots, \tag{2.6}$$

$$t(w) = \frac{1 + m(w)}{1 - m(w)} = 1 + t_1 w + t_2 w^2 + \cdots, \tag{2.7}$$

or, equivalently,

$$l(z) = \frac{s(z) - 1}{s(z) + 1} = \frac{1}{2}\left[s_1 z + \left(s_2 - \frac{s_1^2}{2}\right) z^2 + \cdots\right], \tag{2.8}$$

and

$$m(w) = \frac{t(w) - 1}{t(w) + 1} = \frac{1}{2}\left[t_1 w + \left(t_2 - \frac{t_1^2}{2}\right) w^2 + \cdots\right], \tag{2.9}$$

Then $s(0) = t(0) = 1$, and $s(z)$ and $t(z)$ are analytic in $\mathfrak{O}$ with a positive real part in $\mathfrak{O}$.

Now consider,

$$\Psi(l(z)) = e^{l(z)+\epsilon\frac{[l(z)]^2}{2}}$$

$$\Psi(l(z)) = 1 + \frac{s_1}{2}z + \left(\frac{s_2}{2} + (1-\epsilon)\frac{s_1^2}{8}\right)z^2 + \left(\frac{s_3}{2} + (\epsilon-1)\frac{s_1 s_2}{4} + (1-3\epsilon)\frac{s_1^3}{48}\right)z^3 + \cdots. \tag{2.10}$$

Similarly,

$$\Psi(m(w)) = 1 + \frac{t_1}{2}w + \left(\frac{t_2}{2} + (1-\epsilon)\frac{t_1^2}{8}\right)w^2 + \left(\frac{t_3}{2} + (\epsilon-1)\frac{t_1 t_2}{4} + (1-3\epsilon)\frac{t_1^3}{48}\right)w^3 + \cdots. \tag{2.11}$$

From (2.2) and (2.3), we have

$$\frac{2\vartheta-1}{2\vartheta+1} + \frac{2}{2\vartheta+1}\left[\frac{((z\mathcal{M}_{v,\eta}^{\sigma}\mathcal{T}(z))')'}{(\mathcal{M}_{v,\eta}^{\sigma}\mathcal{T}(z))'}\right] = \Psi(l(z)), \tag{2.12}$$

and

$$\frac{2\vartheta-1}{2\vartheta+1} + \frac{2}{2\vartheta+1}\left[\frac{((w\mathcal{M}_{v,\eta}^{\sigma}\mathcal{G}(w))')'}{(\mathcal{M}_{v,\eta}^{\sigma}\mathcal{G}(w))'}\right] = \Psi(m(w)). \tag{2.13}$$

Using (2.10), (2.11) in (2.12), (2.13) and comparing the coefficients, we the following relations

$$\frac{4}{2\vartheta+1}d_2\varphi_2 = \frac{s_1}{2}, \tag{2.14}$$

$$\frac{4}{2\vartheta + 1}(3d_3\varphi_3 - 2d_2^2\varphi_2^2) = \frac{s_2}{2} + (\epsilon - 1)\frac{s_1^2}{8}, \tag{2.15}$$

$$-\left(\frac{4}{2\vartheta + 1}\right)d_2\varphi_2 = \frac{t_1}{2}, \tag{2.16}$$

$$\frac{4}{2\vartheta + 1}(3[2d_2^2 - d_3]\varphi_3 - 2d_2^2\varphi_2^2) = \frac{s_2}{2} + (\epsilon - 1)\frac{s_1^2}{8}. \tag{2.17}$$

From (2.14) and (2.16), it follows that

$$s_1 = -t_1. \tag{2.18}$$

Squaring and adding (2.14) and (2.16), we obtain that

$$\frac{128}{(2\vartheta + 1)^2}d_2^2\varphi_2^2 = s_1^2 + t_1^2. \tag{2.19}$$

Now, applying Lemma 1 to (2.19), we get

$$|d_2| \leq \frac{2\vartheta + 1}{4\varphi_2^2}. \tag{2.20}$$

Adding (2.15) and (2.17), we can find out that

$$\frac{4}{2\vartheta + 1}[6\varphi_3 d_2^2 - 4\varphi_2^2 d_2^2] = \frac{s_2 + t_2}{2} + (\epsilon - 1)\left(\frac{s_1^2 + t_1^2}{8}\right). \tag{2.21}$$

If we use (2.19) in (2.21), then we have

$$d_2^2 = \frac{(s_2 + t_2)(2\vartheta + 1)^2}{16[(2\vartheta + 1)(3\varphi_3 - 2\varphi_2^2) + 2(1 - \epsilon)\varphi_2^2]}. \tag{2.22}$$

Employing Lemma 1, we obtain

$$|d_2| \leq \frac{2\vartheta + 1}{2\sqrt{(2\vartheta + 1)(3\varphi_3 - 2\varphi_2^2) + 2(1-\epsilon)\varphi_2^2}}.$$

Subtracting (2.17) from (2.15) and using (2.18), it follows that

$$\frac{24}{2\vartheta + 1}(d_3 - d_2^2)\varphi_3 = \frac{s_2 - t_2}{2},$$

i.e.,

$$d_3 = \frac{(s_2 - t_2)(2\vartheta + 1)}{48\varphi_3} + d_2^2. \tag{2.23}$$

Substituting the value of d_2^2 from (2.19) in (2.23), we obtain

$$d_3 = \frac{(s_2 - t_2)(2\vartheta + 1)}{48\varphi_3} + \frac{(s_1^2 - t_1^2)(2\vartheta + 1)^2}{128\varphi_2^2}. \tag{2.24}$$

Applying Lemma 1, we obtain

$$|d_3| \leq \frac{2\vartheta + 1}{12\varphi_3} + \frac{(2\vartheta + 1)^2}{12\varphi_2^2}.$$

Using (2.22) in (2.23), we obtain

$$d_3 = \frac{(s_2 + t_2)(2\vartheta + 1)}{48\varphi_3} + \frac{(s_2 + t_2)(2\vartheta + 1)^2}{16[(2\vartheta + 1)(3\varphi_3 - 2\varphi_2^2) + 2(1-\epsilon)\varphi_2^2]}. \tag{2.25}$$

Using Lemma (1), it follows that

$$|d_3| \leq \frac{2\vartheta + 1}{12\varphi_3} + \frac{(2\vartheta + 1)^2}{4[(2\vartheta + 1)(3\varphi_3 - 2\varphi_2^2) + 2(1-\epsilon)\varphi_2^2]}. \qquad \square$$

Lemma 2 [13]: *Let* $b_1, b_2 \in \mathbb{R}$ *and* $c_1, c_2 \in \mathbb{C}$. *Suppose that* $|c_1|, |c_2| < \xi$, *then*

$$|(b_1 + b_2)c_1 + (b_1 - b_2)c_2| \leq \begin{cases} 2|b_1|\xi, & |b_1| \geq |b_2| \\ 2|b_2|\xi, & |b_1| \leq |b_2|. \end{cases}$$

Theorem 2: *A function* $\tau \in \mathfrak{B}$ *form* (1.1) *is in class* $OBCV_{\Sigma}(\vartheta, \sigma, \upsilon, \eta)$ *and* $\tau \in \mathbb{R}$, *then*

$$|d_3 - \tau d_2^2| \leq \begin{cases} \frac{2\vartheta+1}{12\varphi_3}, & 0 \leq \chi(\vartheta, \tau, \epsilon) \leq \frac{2\vartheta+1}{48\varphi_3} \\ 4|\chi(\vartheta, \tau, \epsilon)|, & \chi(\vartheta, \tau, \epsilon) \geq \frac{2\vartheta+1}{48\varphi_3}. \end{cases}$$

Proof: From (2.22) and (2.23), we have

$$d_3 - \tau d_2^2 = (1 - \tau)d_2^2 + \frac{(s_2 - t_2)(2\vartheta + 1)}{48\varphi_3}$$

$$= \frac{(s_2 + t_2)(1 - \tau)(2\vartheta + 1)^2}{16[(2\vartheta + 1)(3\varphi_3 - 2\varphi_2^2) + (1 - \epsilon)\varphi_2^2]} + \frac{(s_2 - t_2)(2\vartheta + 1)}{48\varphi_3}$$

$$= \left(\chi(\vartheta, \tau, \epsilon) + \frac{2\vartheta + 1}{48\varphi_3}\right)s_2 + \left(\chi(\vartheta, \tau, \epsilon) - \frac{2\vartheta + 1}{48\varphi_3}\right)t_2,$$

where

$$\chi(\vartheta, \tau, \epsilon) = \frac{(1 - \tau)(2\vartheta + 1)^2}{16[(2\vartheta + 1)(3\varphi_3 - 2\varphi_2^2) + (1 - \epsilon)\varphi_2^2]}$$

Applying Lemma 2, we deduce that

$$|d_3 - \tau d_2^2| \leq \begin{cases} \frac{2\vartheta+1}{12\varphi_3}, & 0 \leq \chi(\vartheta,\tau,\epsilon) \leq \frac{2\vartheta+1}{48\varphi_3} \\ 4|\chi(\vartheta,\tau,\epsilon)|, & \chi(\vartheta,\tau,\epsilon) \geq \frac{2\vartheta+1}{48\varphi_3}. \end{cases}$$ □

Corollary 1: *A function* $\tau \in \mathfrak{B}$ *form* (1.1) *is in class* $OBCV_\Sigma(\vartheta,\sigma,\upsilon,\eta)$, *then*

$$|d_3 - d_2^2| \leq \frac{2\vartheta+1}{12\varphi_3}.$$

3. Bi-concave Functions

Definition 2: *A function* $\tau \in \mathfrak{B}$ *is belongs to the class* $BCV_\Sigma(\vartheta,\sigma,\upsilon,\eta)$ *if it satisfies the following subordination conditions*:

$$\frac{2}{\varrho-1}\left[\frac{(1+\varrho)(1+z)}{2(1-z)} - 1 - \frac{z(\mathcal{M}_\sigma^{\upsilon,\eta}\tau(z))''}{(\mathcal{M}_\sigma^{\upsilon,\eta}\tau(z))'}\right] \prec \Psi(z), \tag{3.1}$$

and

$$\frac{2}{\varrho-1}\left[\frac{(1+\varrho)(1+w)}{2(1-w)} - 1 - \frac{w(\mathcal{M}_\sigma^{\upsilon,\eta}\tau(w))''}{(\mathcal{M}_\sigma^{\upsilon,\eta}\tau(w))'}\right] \prec \Psi(w) \tag{3.2}$$

where $\tau^{-1}(w) = \mathcal{G}(w)$ *and* $1 < \varrho \leq 2$.

Theorem 3: *A function* $\tau \in \mathfrak{B}$ *form* (1.1) *is in class* $BCV_\Sigma(\varrho,\sigma,\upsilon,\eta)$, *then*

$$|d_2| \leq \min\left\{\sqrt{\frac{9\varrho^2+6\varrho+1}{16\varphi_2^2}},\right.$$

$$\left.\left(\frac{1}{(\varrho-1)(2\varphi_2^2-3\varphi_3)+2(1-\epsilon)}\left[\frac{3\varrho^2-2\varrho-1}{4} + (\epsilon-1)\frac{\varrho 2+\varrho}{\varphi_2^2}\right]\right)^{\frac{1}{2}}\right\}, \tag{3.3}$$

and

$$|d_3| \leq \min\left\{\frac{9\varrho^2 + 6\varrho + 1}{16\varphi_2^2} + \frac{\varrho - 1}{12\varphi_3},\right.$$
$$\left.\frac{1}{(\varrho-1)(2\varphi_2^2 - 3\varphi_3) + 2(1-\epsilon)}\left[\frac{3\varrho^2 - 2\varrho - 1}{4} + (\epsilon - 1)\frac{\varrho 2 + \varrho}{\varphi_2^2}\right] + \frac{\varrho - 1}{12\varphi_3}\right\}, \quad (3.4)$$

where $1 < \varrho \leq 2$ *and* φ_2 *and* φ_3 *are as given in* (1.3).

Proof: From (3.1) and (3.2), we can write

$$\frac{2}{\varrho - 1}\left[\frac{(1+\varrho)(1+z)}{2(1-z)} - 1 - \frac{z(\mathcal{M}_\sigma^{\upsilon,\eta}\mathcal{T}(z))''}{(\mathcal{M}_\sigma^{\upsilon,\eta}\mathcal{T}(z))'}\right] \prec \Psi(l(z)), \quad (3.5)$$

and

$$\frac{2}{\varrho - 1}\left[\frac{(1+\varrho)(1+w)}{2(1-w)} - 1 - \frac{w(\mathcal{M}_\sigma^{\upsilon,\eta}\mathcal{G}(w))''}{(\mathcal{M}_\sigma^{\upsilon,\eta}\mathcal{G}(w))'}\right] \ll \Psi(m(w)). \quad (3.6)$$

Using (2.10), (2.11) in (3.5), (3.6) respectively and equating the coefficients, we obtain

$$\frac{2}{\varrho - 1}[(1+\varrho) - 2\varphi_2 d_2] = \frac{s_1}{2}, \quad (3.7)$$

$$\frac{2}{\varrho - 1}[(1+\varrho) + 4\varphi_2^2 d_2^2 - 6\varphi_3 d_3] = \frac{s_2}{2} + (\epsilon - 1)\frac{s_1^2}{8}, \quad (3.8)$$

$$-\frac{2}{\varrho - 1}[(1+\varrho) - 2\varphi_2 d_2] = \frac{t_1}{2}, \quad (3.9)$$

$$\frac{2}{\varrho - 1}[(1+\varrho) + 4\varphi_2^2 d_2^2 - 6\varphi_3(2d_2^2 - d_3)] = \frac{t_2}{2} + (\epsilon - 1)\frac{t_1^2}{8}. \quad (3.10)$$

From (3.7) and (3.9), it follows that

$$s_1 = t_1. \quad (3.11)$$

From (3.7) and (3.9), we can write

$$d_2 = \frac{1+\varrho}{2\varphi_2} - \frac{(\varrho-1)s_1}{8\varphi_2}, \tag{3.12}$$

$$d_2 = \frac{1+\varrho}{2\varphi_2} + \frac{(\varrho-1)t_1}{8\varphi_2}. \tag{3.13}$$

Squaring and adding (3.12) and (3.13), we obtain

$$d_2^2 = \frac{(1+\varrho)^2}{4\varphi_2^2} + \frac{(s_1^2+t_1^2)(\varrho-1)^2}{128\varphi_2^2} - \frac{(\varrho^2-1)(s_1-t_1)}{16\varphi_2^2}, \tag{3.14}$$

i.e.,

$$\frac{(s_1^2+t_1^2)(\varrho-1)^2}{64\varphi_2^2} = 2d_2^2 - \frac{(1+\varrho)^2}{2\varphi_2^2} + \frac{(\varrho^2-1)(s_1-t_1)}{8\varphi_2^2}. \tag{3.15}$$

Applying Lemma 1 to (3.14), we have

$$|d_2|^2 \leq \frac{9\varrho^2+6\varrho+1}{16\varphi_2^2}.$$

Adding (3.8) and (3.10), we obtain

$$\frac{2}{\varrho-1}\{2(1+\varrho)+8\varphi_2^2d_2^2-12\varphi_3d_2^2\} = \frac{(s_2+t_2)}{2} + (\epsilon-1)\left(\frac{s_1^2+t_1^2}{8}\right). \tag{3.16}$$

Implies that,

$$(2\varphi_2^2-3\varphi_3)d_2^2 = \frac{(\varrho-1)(s_2+t_2)}{16} + \frac{(\epsilon-1)(\varrho-1)(s_1^2+t_1^2)}{64} - \frac{(\varrho+1)}{2}.$$

Multiplying both sides by (ϱ 1), thus, we get

$$(\varrho-1)(2\varphi_2^2-3\varphi_3)d_2^2 = \frac{(\varrho-1)^2(s_2+t_2)}{16} + \frac{(\epsilon-1)(\varrho-1)^2(s_1^2+t_1^2)}{64} - \frac{(\varrho^2+1)}{2}. \tag{3.17}$$

Using (3.15) in (3.17), we obtain

$$(\varrho-1)(2\varphi_2^2-3\varphi_3)d_2^2+2(1-\epsilon)d_2^2=\frac{(\varrho-1)^2(s_2+t_2)}{16}-(\epsilon-1)\frac{(1+\varrho)^2}{2\varphi_2^2}$$
$$+(\epsilon-1)\frac{(\varrho^2-1)}{8\varphi_2^2}(s_1-t_1)-\left(\frac{(\varrho^2-1)}{2}\right). \quad (3.18)$$

Thus, we have

$$d_2^2=\frac{1}{[(\varrho-1)(2\varphi_2^2-3\varphi_3)+2(1-\epsilon)]}\left[\frac{(\varrho-1)^2(s_2+t_2)}{16}\right.$$
$$\left.-(\epsilon-1)\frac{(1+\varrho)^2}{2\varphi_2^2}+(\epsilon-1)\frac{(\varrho^2-1)}{8\varphi_2^2}(s_1-t_1)-\left(\frac{(\varrho^2-1)}{2}\right)\right]. \quad (3.19)$$

Applying Lemma 1 to (3.19), we obtain

$$|d_2^2|^2\leq\frac{1}{(\varrho-1)(2\varphi_2^2-3\varphi_3)+2(1-\epsilon)}\left[\frac{3\varrho^2-2\varrho-1}{4}+(\epsilon-1)\frac{\varrho^2+\varrho}{\varphi_2^2}\right].$$

Now, subtracting (3.10) from (3.8) and using (3.11), we get

$$d_2^2-d_3=\frac{(s_2-t_2)(\varrho-1)}{48\varphi_3}. \quad (3.20)$$

Using (3.14) in (3.20), we find that

$$d_3=\frac{(1+\varrho)^2}{4\varphi_2^2}+\frac{(s_1^2+t_1^2)(\varrho-1)^2}{128\varphi_2^2}-\frac{(\varrho^2-1)(s_1-t_1)}{16\varphi_2^2}-\frac{(s_2-t_2)(\varrho-1)}{48\varphi_3}. \quad (3.21)$$

According to Lemma 1, we deduce that

$$|d_3|\leq\frac{9\varrho^2+6\varrho+1}{16\varphi_2^2}+\frac{\varrho-1}{12\varphi_3}.$$

Next we use the value of d_2^2 form (3.19) in (3.20), we obtain

$$d_3 = \frac{1}{[(\varrho - 1)(2\varphi_2^2 - 3\varphi_3) + 2(1 - \epsilon)]}\left[\frac{(\varrho - 1)^2(s_2 + t_2)}{16}\right.$$

$$\left. - (\epsilon - 1)\frac{(1 + \varrho)^2}{2\varphi_2^2} + (\epsilon - 1)\frac{(\varrho^2 - 1)}{8\varphi_2^2}(s_1 - t_1) - \left(\frac{(\varrho^2 - 1)}{2}\right)\right] - \frac{(s_2 - t_2)(\varrho - 1)}{48\varphi_3}. \tag{3.22}$$

Using Lemma 1, we get

$$d_3 \leq \frac{1}{[(\varrho - 1)(2\varphi_2^2 - 3\varphi_3) + 2(1 - \epsilon)]}\left[\frac{3\varrho^2 - 2\varrho - 1}{4} + (\epsilon - 1)\frac{\varrho^2 + \varrho}{\varphi_2^2}\right] + \frac{\varrho - 1}{12\varphi_3}$$

□

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1. Department of Mathematics,
Willingdon College, Sangli,
E-mail: shelakegd@gmail.com

2. Department of Mathematics,
Shivaji University, Kolhapur,
E-mail: sarikanilapgol101@gmail.com

3. Department of Mathematics,
Walchand College of Engineering, Sangli,
E-mail: joshisb@hotmail.com

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