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SOLITONS ON VARIOUS GEOMETRIC STRUCTURES OF PARA-SASAKIAN MANIFOLDS ADMITTING SCHOUTEN-VAN KEMPEN CONNECTION

Abstract. In this paper we investigate properties of para-Sasakian manifold by the help of Schouten-van Kampen connection. We also study para-Sasakian manifolds of various equivalent structures admitting conformal Ricci soliton and conformal η-Ricci soliton with respect to Schouten-van Kampen connection.

Key words and phrases: Para-Sasakian Manifold, Schouten-van Kampen Connection, Conformal Ricci Soliton, Conformal η-Ricci Soliton.

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1. Introduction

In 1979, the notion of para-Sasakian (briefly, P-Sasakian) and special para-Sasakian (briefly, SP-Sasakian) manifolds were introduced by Sato and Matsumoto [28]. Later, Adati and Matsumoto investigate some interesting results on P-Sasakian manifolds and SP-Sasakian manifolds in [1]. The properties of para-Sasakian manifold have been studied by many authors. For instance, we see [2, 16, 17, 19, 27, 30] and their references.

The notion of Schouten-van Kampen connection (shortly, SVK-connection) was introduced in the third decade of last century for a study of non-holomorphic manifolds [29, 37]. In 2006, Bejancu [3] studied Schouten-van Kampen connection on Foliated manifolds. Recently, Biswas and Baisya [4, 5] investigated some

properties of pesudo symmetric Sasakian manifolds with respect to SVK-connectiopn. Most recently, this connection has been introduced on para-Sasakian manifold by Sundriyal and Upreti [31]. They studied projective curvature tensor, concircular curvature tensor and Nijenhuis tensor for the para-sasakian manifold with respect to this connection. SVK-connection $(\bar{\nabla})$ for an *n*-dimensional almost contact metric manifold M equipped with an almost contact metric structure (ϕ, ξ, η, g) consisting of a (1, 1) tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g, is defined by

$$\bar{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi, \tag{1.1}$$

for all $X, Y \in \chi(M)$, where $\chi(M)$ is the set of all vector fields on M and ∇ being the Levi-Civita connection on M.

The concept of Ricci flow was first introduced by R. S. Hamilton in the early 1980s. Hamilton [13] observed that the Ricci flow is an excellent tool for simplifying the structure of a manifold. It is the process which deforms the metric of a Riemannian manifold by smoothing out the irregularities. The Ricci flow equation is given by

$$\frac{\partial g}{\partial t} = -2S, \tag{1.2}$$

where g is a Riemannian metric, S is Ricci tensor and t is time. The solitons for the Ricci flow is the solutions of the above equation, where the metrices at different times differ by a diffeomorphism of the manifold. A Ricci soliton is represented by a triple (g, V, λ), where V is a vector field and λ is a scalar, which satisfies the equation

$$L_V g + 2S + 2\lambda g = 0, \tag{1.3}$$

where S is Ricci curvature tensor and $L_V g$ denotes the Lie derivative of g along the vector field V. A Ricci soliton is said to be shrinking, steady, expanding according as $\lambda < 0$, $\lambda = 0$, $\lambda > 0$, respectively. The vector field V is called potential vector field and if it is a gradient of a smooth function, then the Ricci soliton (g, V, λ) is called a gradient Ricci soliton and the associated function is called the potential function. Ricci soliton was further studied by many researchers. For instance, we see [18, 25, 35, 36] and their references.

In 2005, Fischer [12] introduced conformal Ricci flow which is a generalisation of the Ricci flow equation that modifies the unit volume constraint to a scalar curvature constraint. The conformal Ricci flow equation is given by

$$\frac{\partial g}{\partial t} + 2\left(S + \frac{g}{n}\right) = -pg \tag{1.4}$$

$$r(g) = -1 \tag{1.5}$$

where $r(g)$ is the scalar curvature of the manifold, p is a non-dynamical scalar field and n is the dimension of the manifold. In 2015, corresponding to the conformal Ricci flow equation, Basu and Bhattacharyya [7] introduced the notion of conformal Ricci soliton as a generalisation of Ricci soliton and it is given by

$$L_V g + 2S + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g = 0, \tag{1.6}$$

where λ is a constant.

As a generalization of Ricci soliton, the η-Ricci soliton was introduced by Cho and Kimura [9]. This notion has also been studied by Cälin and Crasmarearu [10]. Later, remarkable studies on η-Ricci soliton have been made by Blaga [6] and Prakasha [24]. Let M be a Riemannian manifold with structure (ϕ, ξ, η, g). Consider the equation

$$L_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0\ , \tag{1.7}$$

where S is Ricci curvature tensor, $L_V g$ denotes the Lie derivative of g along the vector field V, λ and μ are real constants. The data (g, V, λ, μ) which satisfies the equation (1.7) is called an η-Ricci soliton on M. In particular, when $\mu = 0$, the notion of η-Ricci soliton simply reduces to the notion of Ricci soliton. And when $\mu \neq 0$, (g, V, λ, μ) is called proper η-Ricci soliton on M.

In 2018, Siddiqi [34] introduced the notion of conformal η-Ricci soliton as

$$L_V g + 2S + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g + 2\beta\eta \otimes \eta = 0, \tag{1.8}$$

where $L_V g$ denotes the Lie derivative of g along the vector field V, λ and β are real

constants and p is a non-dynamical scalar field.

Definition 1.1: *Let* F *and* Ω *be two tensors of rank* 4. *A Riemanian manifold* (*or, pseudo Riemannian manifold*) M *is said to be* Ω*-semisymmetric type if* $F(X,Y).\Omega = 0$ *for all smooth vector fields X, Y on M, where* F *acts on* Ω *as derivation of tensor algebra.*

In the above definition if we consider $F = \Omega = R$, then the manifold M is called semi-symmetric [32]. Semi-symmetry and other conditions of semi-symmetry type are studied in detail in [8, 15, 20, 33]. In 2013, Kundu and Shaikh [26] investigated the equivalency of the various geometric structures depending on conditions of semi-symmetry. They have established the following conditions

(i) $E.R = 0$, $E.P = 0$, $E.E = 0$, $E.P = 0$, $E.\mathcal{M} = 0$, $E.W_i = 0$ and $E.\mathcal{W}_i^* = 0$ (for all $i = 1,2,...9$) are equivalent and named such a class by C_1;

(ii) $R.R = 0$, $R.P = 0$, $R.E = 0$, $R.P = 0$, $R.M = 0$, $R.W_i = 0$ and $R.W_i^* = 0$ (for all $i = 1,2,$ .....9) are equivalent and named such a class by C_2;

(iii) $R.K = 0$ and $R.C = 0$ are equivalent and named such a class by C_3;

(iv) $E.C = 0$ and $E.K = 0$ are equivalent and named such a class by C_4;

where the symbols C, E, P, K, M and Wi stand for conformal curvature tensor [11], concircular curvature tensor [38], projective curvature tensor [38], conharmonic curvature tensor [14], M-projective curvature tensor [22], Wi-curvature tensor [21, 22, 23] and W_i^*-curvature tensor [22], respectively.

$$C(X,Y) = R(X,Y) - \frac{1}{n-2}\left[(X \wedge_g QY) + (QX \wedge_g Y) + \frac{r}{n-1}(X \wedge_g Y)\right], \tag{1.9}$$

$$E(X,Y) = R(X,Y) - \frac{r}{n(n-1)}(X \wedge_g Y), \tag{1.10}$$

$$P(X,Y) = R(X,Y) - \frac{1}{n-1}(X \wedge_g Y), \tag{1.11}$$

$$K(X,Y) = R(X,Y) - \frac{1}{n-2}[(X \wedge_g QY) + (QX \wedge_g Y)], \tag{1.12}$$

$$\mathcal{M}(X,Y) = R(X,Y) - \frac{1}{2(n-1)}[(X \wedge_g QY) + (QX \wedge_g Y)], \tag{1.13}$$

$$\mathcal{W}_0(X,Y) = R(X,Y) - \frac{1}{n-1}(X \wedge_g QY), \tag{1.14}$$

$$\mathcal{W}_0^*(X,Y) = R(X,Y) + \frac{1}{n-1}(X \wedge_g QY), \tag{1.15}$$

$$\mathcal{W}_1(X,Y) = R(X,Y) - \frac{1}{n-1}(X \wedge_S Y), \tag{1.16}$$

$$\mathcal{W}_1^*(X,Y) = R(X,Y) + \frac{1}{n-1}(X \wedge_S Y), \tag{1.17}$$

$$\mathcal{W}_2(X,Y) = R(X,Y) - \frac{1}{n-2}\big[(QX \wedge_g Y) + (X \wedge_g QY) - (X \wedge_S Y)\big], \tag{1.18}$$

$$\mathcal{W}_2^*(X,Y) = R(X,Y) + \frac{1}{n-2}\big[(QX \wedge_g Y) + (X \wedge_g QY) - (X \wedge_S Y)\big], \tag{1.19}$$

$$\mathcal{W}_3(X,Y) = R(X,Y) - \frac{1}{n-1}(Y \wedge_g QX), \tag{1.20}$$

$$\mathcal{W}_3^*(X,Y) = R(X,Y) + \frac{1}{n-1}(Y \wedge_g QX), \tag{1.21}$$

$$\mathcal{W}_5(X,Y) = R(X,Y) - \frac{1}{n-1}[(X \wedge_g QY) - (X \wedge_S Y)], \tag{1.22}$$

$$\mathcal{W}_5^*(X,Y) = R(X,Y) + \frac{1}{n-1}[(X \wedge_g QY) - (X \wedge_S Y)], \tag{1.23}$$

$$\mathcal{W}_7(X,Y) = R(X,Y) + \frac{1}{n-1}[(QX \wedge_g Y) - (X \wedge_S Y)], \tag{1.24}$$

$$\mathcal{W}_7^*(X,Y) = R(X,Y) - \frac{1}{n-1}[(QX \wedge_g Y) - (X \wedge_S Y)], \tag{1.25}$$

$$\mathcal{W}_4(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[g(X,Z)QY - g(X,Y)QZ], \tag{1.26}$$

$$\mathcal{W}_4^*(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(X,Z)QY - g(X,Y)QZ], \tag{1.27}$$

$$\mathcal{W}_6(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(X,Z)QY - g(X,Y)QZ], \tag{1.28}$$

$$\mathcal{W}_6^*(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[S(Y,Z)X - g(X,Y)QZ], \tag{1.29}$$

$$\mathcal{W}_8(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - g(X,Y)Z], \tag{1.30}$$

$$\mathcal{W}_8^*(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[S(Y,Z)X - g(X,Y)Z], \tag{1.31}$$

$$\mathcal{W}_9(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(X,Y)Z - g(Y,Z)QX], \tag{1.32}$$

$$\mathcal{W}_9^*(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[S(X,Y)Z - g(Y,Z)QX], \tag{1.33}$$

where

$$(X \wedge_D Y)Z = D(Y,Z)X - D(X,Z)Y .$$

for all $X, Y, Z \in \chi(M)$, where R is the Riemannian curvature tensor of type (1, 3) and r is the scalar curvature.

Definition 1.2: *A para-Sasakian manifold M is called an Einstein manifold if its Ricci tensor is of the form*

$$S(Y,Z) = kg(Y,Z),$$

for all $Y, Z \in \chi(M)$, where k being a scalar.

Definition 1.3: *A para-Sasakian manifold M is called an η-Einstein manifold if its Ricci tensor is of the form*

$$S(Y,Z) = l_1 g(Y,Z) + l_2 \eta(Y)\eta(Z),$$

for all $Y, Z \in \chi(M)$, *where* l_1, l_2 *are scalars.*

Definition 1.4: *A para-Sasakian manifold* M *is called a generalized* η*-Einstein manifold if its Ricci tensor is of the form*

$$S(Y,Z) = k_1 g(Y,Z) + k_2 \eta(Y)\eta(Z) + k_3 g(Y,\phi Z),$$

for all $Y, Z \in \chi(M)$, *where* k_1, k_2 *and* k_3 *are scalars.*

This paper is structured as follows:

First two sections of the paper has been kept for introduction and preliminaries. In Section-3, we study properties of para-Sasakian manifold with respect to SVK-connection. In Section-4, we introduce conformal Ricci soliton on para-Sasakian manifold with respect to SVK-connection. In Section-5, we study conformal η-Ricci soliton on para-Sasakian manifold with respect to SVK-connection. Section-6 concerns with conformal η-Ricci soliton with respect to SVK-connection on para-Sasakian manifolds of class C_1, C_2, C_3 and C_4.

2. Preliminaries

Let M be an n-dimensional differentiable manifold with structure (ϕ, ξ, η), where η is a 1-form, ξ is the structure vector field, ϕ is a (1, 1)-tensor field satisfying [28]

$$\phi^2(X) = X - \eta(X)\xi, \quad \eta(\xi) = 1 \tag{2.1}$$

$$\phi(\xi) = 0, \eta \circ \phi = 0, \tag{2.2}$$

for all vector field X on M is called almost paracontact manifold. If an almost paracontact manifold M with structure (ϕ, ξ, η) admits a pseudo-Riemannian metric g such that [39]

$$g(\phi X, \phi Y) = -g(X,Y) + \eta(X)\eta(Y), \tag{2.3}$$

then we say that M is an almost paracontact metric manifold with an almost paracontact metric structure (ϕ, ξ, η, g). From (2.3) one can deduce that

$$g(X,\phi Y) = -g(\phi X,Y), \tag{2.4}$$

$$g(X,\xi) = \eta(\xi). \tag{2.5}$$

An almost paracontact metric structure of M becomes a paracontact metric structure [39] if

$$g(X,\phi Y) = d\eta(X,Y),$$

for all vector fields X, Y on M, where

$$d\eta(X,Y) = \frac{1}{2}\{X\eta(Y) - Y\eta(X) - \eta([X,Y])\}.$$

The manifold M is called a para-Sasakian manifold if

$$(\nabla_X \varphi)Y = -g(X,Y)\xi + \eta(Y)X, \tag{2.6}$$

for any smooth vector fields X, Y on M.

In a para-Sasakian manifold the following relations also hold [39]

$$(\nabla_X \eta)Y = g(X,\phi Y), \nabla_X \xi = -\phi X, \tag{2.7}$$

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{2.8}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.9}$$

$$R(\xi,X)Y = -g(X,Y)\xi + \eta(Y)X, \tag{2.10}$$

$$R(X,\xi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.11}$$

$$R(\xi,X)\xi = X - \eta(X)\xi, \tag{2.12}$$

$$S(X,\xi) = -(n-1)\eta(X), \tag{2.13}$$

$$S(\xi,\xi) = -(n-1), \quad Q\xi = -(n-1)\xi, \tag{2.14}$$

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y). \tag{2.15}$$

for any smooth vector fields X, Y, Z on M.

3. Schouten-Van Kampen Connection on Para-Sasakian Manifolds

In this section we get the relation between SVK-connection and Levi-Civita connection on para-Sasakian manifold M. Then we obtain Rie-mannian curvature tensor, Ricci curvature tensor, Ricci operator and scalar curvature of M with respect to the SVK-connection. We also establish here the first Bianchi identity with respect to SVK-connection on M.

In view of (1.1), (2.7) and (2.5), we get the expression for SVK-connection in a para-Sasakian manifold M as

$$\bar{\nabla}_X Y = \nabla_X Y + g(X, \phi Y)\xi + \eta(Y)\phi X \ , \tag{3.1}$$

with torsion tensor

$$\bar{T}(X, Y) = 2g(X, \phi Y)\xi + \eta(Y)\phi X - \eta(X)\phi Y \ .$$

On para-Sasakian manifold the connection $\bar{\nabla}$ has the following properties

$$\bar{\nabla}_X \xi = 0, (\bar{\nabla}_X \eta) Y = g(\phi X, Y), \tag{3.2}$$

$$(\bar{\nabla}_X g)(Y, Z) = g(\phi X, Y)\eta(Z) + g(\phi Y, Z)\eta(X). \tag{3.3}$$

for all $X, Y \in \chi(M)$.

Proposition 3.1: *The SVK-connection on a para-Sasakian manifold is non metric compatible connection.*

Proposition 3.2: *The SVK-connection on a para-Sasakian manifold is non symmetric connection.*

Proposition 3.3: *The structure vector field of a para-Sasakian manifold is parallel with respect to SVK-connection.*

Let $\bar{R}$ be the Riemannian curvature tensor with respect to SVK-connection on a para-Sasakian manifold defined as

$$\overline{R}(X,Y)Z = \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z\,. \tag{3.4}$$

Then using (2.6), (2.7) and (3.1) in (3.4) we get

$$\begin{aligned}\overline{R}(X,Y)Z &= R(X,Y)Z + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &\quad + g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X \\ &\quad + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\,. \end{aligned}\tag{3.5}$$

Writing the equation (3.5) by cyclic permutations of X,Y and Z and using the fact that $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$, we have

$$\overline{R}(X,Y)Z + \overline{R}(Y,Z)X + \overline{R}(Z,X)Y = 0\ ,$$

for all $X,Y,Z \in \chi(M)$.

Taking inner product of (3.5) with a vector field U and contracting over X and U we get

$$\overline{S}(Y,Z) = S(Y,Z) + (n-1)\eta(Y)\eta(Z) - \psi g(Y,\phi Z)\,, \tag{3.6}$$

where $\overline{S}$ denotes Ricci curvature tensor with respect to $\overline{\nabla}$ and $\psi = trace(\phi)$.

Proposition 3.4: *The SVK-connection on para-Sasakian manifold satisfies the first Bianchi identity.*

Lemma 3.5: *Let M be an n-dimensional para-Sasakian manifold admitting SVK-connection, then*

$$\overline{R}(X,Y)\xi = 0,\ \ \overline{R}(\xi,Y)Z = 2\,[g(Y,Z)\xi - \eta(Z)Y]\,, \tag{3.7}$$

$$\overline{R}(X,\xi)Z = -2[g(X,Z)\xi + \eta(Z)X]\,, \tag{3.8}$$

$$\overline{S}(X,\xi) = 0 = \overline{S}(\xi,Y)\,, \tag{3.9}$$

$$\bar{Q}X = QX + (n-1)\eta(X)\xi + \phi X\psi,\ \ \bar{Q}\xi = 0\,, \tag{3.10}$$

$$\bar{r} = r + (n-1) - \psi^2\,, \tag{3.11}$$

for all $X,Y,Z \in \chi(M)$, *where* $\bar{R}$, $\bar{Q}$ *and* $\bar{r}$ *denote Riemannian curvature tensor, Ricci operator and scalar curvature with respect to* $\bar{\nabla}$, *respectively.*

Remark 3.6: *Eigen value of Ricci operator with respect to SVK-connection corresponding to the eigen vector* ξ *is zero.*

4. Conformal Ricci Soliton on Para-Sasakian Manifold with Respect to SVK-Connection

In this section we find a para-Sasakian manifold M admitting conformal Ricci soliton with respect to SVK-connection in which the potential vector field being pointwise collinear with the structure vector field of M .

Let $V = \alpha\xi$, where α is some non-zero smooth function. Taking covariant derivative of V in the direction of X and using (2.7) we get

$$\nabla_X V = X(\alpha)\xi - \alpha\phi X\,. \tag{4.1}$$

In view of (3.1) and (4.1) we have

$$\bar{\nabla}_X V = X(\alpha)\xi - \alpha\phi X + g(X,\phi V)\xi + \eta(V)\phi X\,. \tag{4.2}$$

Writing equation (1.6) with respect to SVK-connection we have

$$\begin{aligned} 0 &= (\bar{L}_V g)(X,Y) + 2\bar{S}(X,Y) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X,Y) \\ &= g(\bar{\nabla}_X V, Y) + g(X, \bar{\nabla}_Y V) \\ &\quad + 2\bar{S}(X,Y) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X,Y)\,. \end{aligned} \tag{4.3}$$

Using (4.2) in (4.3) we get

$$0 = X(\alpha)\eta(Y) + Y(\alpha)\eta(X) + 2\overline{S}(X,Y) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X,Y). \tag{4.4}$$

Setting $X = \xi$ and using (3.9) in (4.4) we get

$$0 = \xi(\alpha)\eta(Y) + Y(\alpha) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right]\eta(Y). \tag{4.5}$$

Replacing Y by ξ in (4.5) we obtain

$$\xi(\alpha) = \left[\left(\frac{p}{2} + \frac{1}{n}\right) - \lambda\right]. \tag{4.6}$$

Using (4.6) in (4.5) we get

$$Y(\alpha) = \left[\left(\frac{p}{2} + \frac{1}{n}\right) - \lambda\right]. \tag{4.7}$$

If we consider $Y(\alpha) = 0$, then equation (4.7) yields

$$\lambda = \frac{p}{2} + \frac{1}{n}.$$

Therefore we have the following theorem

Theorem 4.1: *Let $M(\phi, \xi, \eta, g)$ be a para-Sasakian manifold admitting conformal Ricci soliton (g, V, λ) with respect to SVK-connection. If V is pointwise collinear with ξ, then V is a constant multiple of ξ provided $\lambda = \frac{p}{2} + \frac{1}{n}$.*

Now setting $V = \xi$ in (4.3) we have

$$0 = 2\overline{S}(X,Y) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X,Y). \tag{4.8}$$

Using (3.6) in (4.8) we get

$$S(X,Y)=\left[\left(\frac{p}{2}+\frac{1}{n}\right)-\lambda\right]g(X,Y) -(n-1)\eta(X)\eta(Y)+\psi g(X,\phi Y). \tag{4.9}$$

Corollary 4.2: *If a para-Sasakian manifold M admits conformal Ricci soliton (g,ξ,λ) with respect to SVK-connection, then M is generalized η-Einstein.*

5. Conformal η-Ricci Soliton on Para-Sasakian Manifold with Respect to SVK-Connection

Writing equation (1.6) with respect to SVK-connection we have

$$0=(\bar{L}_{\xi}g)(X,Y)+2\bar{S}(X,Y) +\left[2\lambda-\left(p+\frac{2}{n}\right)\right]g(X,Y)+2\beta\eta(X)\eta(Y). \tag{5.1}$$

Expanding (5.1) we get

$$0=g(\bar{\nabla}_X\xi,Y)+g(X,\bar{\nabla}_Y\xi)+2\bar{S}(X,Y) +\left[2\lambda-\left(p+\frac{2}{n}\right)\right]g(X,Y)+2\beta\eta(X)\eta(Y). \tag{5.2}$$

Using (3.2) in (5.2) we obtain

$$0=2\bar{S}(X,Y)+\left[2\lambda-\left(p+\frac{2}{n}\right)\right]g(X,Y)+2\beta\eta(X)\eta(Y). \tag{5.3}$$

Setting $X=\xi$ in (5.3) we have

$$\lambda=\left(\frac{p}{2}+\frac{1}{n}\right)-\beta. \tag{5.4}$$

Hence, we have the following theorem

Theorem 5.1: *If an n-dimensional para-Sasakian manifold admits a conformal η-Ricci soliton (g,ξ,λ,β) with respect to SVK-connection, then the relation between the soliton scalars are given by*

$$\lambda = \left(\frac{p}{2} + \frac{1}{n}\right) - \beta .$$

Using (3.6) in (5.3) we obtain

$$S(X,Y) = \left[\left(\frac{p}{2} + \frac{1}{n}\right) - \lambda\right] g(X,Y) - (n + \beta - 1)\eta(X)\eta(Y) + \psi g(X,\phi Y), \qquad (5.5)$$

which shows that M is generalized η-Einstein manifold.

Corollary 5.2: *If an n-dimensional para-Sasakian manifold M contains a conformal η-Ricci soliton with respect to SVK-connection, then M is generalized η-Einstein manifold.*

Contracting (5.5) over X and Y we get

$$r = \frac{n}{2}(p-2) - \lambda n + \beta + \psi^2 + 2 . \qquad (5.6)$$

Corollary 5.3: *If an n-dimensional para-Sasakian manifold M contains a conformal η-Ricci soliton with respect to SVK-connection, then the scalar curvature of M is given by equation* (5.6).

6. Conformal η-Ricci Soliton with Respect to SVK-Connection on Equivalence Classes C_1, C_2, C_3 and C_4

In this section we consider η-Ricci soliton (g,ξ,λ,β) with respect to SVK-connection on the manifolds belong to the equivalence classes C_1, C_2, C_3 and C_4 and obtain the relation between the soliton constants.

Conformal η-Ricci soliton with respect to $\bar{\nabla}$ on class C_1 : The condition that must be satisfied by the Riemannian curvature tensor (R) is

$$(E(\xi,X).R)(Y,Z)V = 0, \tag{6.1}$$

for all $X,Y,Z,V \in \chi(M)$.

Equation (6.1) gives

$$\begin{aligned} E(\xi,X).R(Y,Z)V = {} & R(E(\xi,X)Y,Z)V \\ & + R(Y,E(\xi,X)Z)V + R(Y,Z)E(\xi,X)V. \end{aligned} \tag{6.2}$$

Setting $V = \xi$ and using (1.10), (2.9)-(2.11) in (6.2) we get

$$0 = [r + n(n-1)][g(X,Y)Z - g(X,Z)Y] - [r + n(n-1)]R(Y,Z)X. \tag{6.3}$$

Taking an inner product of (6.3) with a vector field U we get

$$\begin{aligned} 0 = {} & [r + n(n-1)]\ [g(X,Y)g(Z,U) - g(X,Z)g(Y,U)] \\ & - [r + n(n-1)]g(R(Y,Z)X,U). \end{aligned} \tag{6.4}$$

Contracting (6.4) over Z and U we have

$$S(X,Y) = -(n-1)g(X,Y), \tag{6.5}$$

if $r \neq -n(n-1)$.

In view of (5.5) and (6.5) we obtain

$$\begin{aligned} 0 = {} & \left[\left(\frac{p}{2} + \frac{1}{n}\right) - \lambda + n - 1\right] g(X,Y) \\ & - (n + \beta - 1)\eta(X)\eta(Y) + \psi g(X,\phi Y). \end{aligned} \tag{6.6}$$

Setting $Y = \xi$ in (6.6) we have

$$\lambda = \left(\frac{p}{2} + \frac{1}{n}\right) - \beta. \tag{6.7}$$

Thus, we have the following theorem:

Theorem 6.1: *Let $M(\phi,\xi,\eta,g)$ be an η-dimensional para-Sasakian manifold of class C_1. If M admits a conformal η-Ricci soliton with respect to SVK-connection, then the soliton constants are given by*

$$\lambda = \left(\frac{p}{2} + \frac{1}{n}\right) - \beta,$$

provided $r \neq -n(n-1)$.

Corollary 6.2: *A para-Sasakian manifold of class C_1 is Einstein manifold if* $r \neq -n(n-1)$.

Corollary 6.3: *If a para-Sasakian manifold of class C_1 contains conformal η-Ricci soliton with respect to SVK-connection, then the manifold is generalized η-Einstein, provided* $r \neq -n(n-1)$.

Conformal η-Ricci Soliton with Respect to $\overline{\nabla}$ on Class C_2: The condition that must be satisfied by the Riemannian curvature tensor (*R*) is

$$(E(\xi,X).R)(Y,Z)V = 0,$$

for all $X,Y,Z,V \in \chi(M)$

$$R(\xi,X).R(Y,Z)V = R(R(\xi,X)Y,Z)V + R(Y,R(\xi,X)Z)V + R(Y,Z)R(\xi,X)V. \quad (6.8)$$

Setting $V = \xi$ and using (2.8)-(2.11) in (6.8) we get

$$0 = [g(X,Y)Z - g(X,Z)Y] - R(Y,Z)X. \quad (6.9)$$

Taking an inner product of (6.9) with a vector field W we get

$$0 = [g(X,Y)g(Z,W) - g(X,Z)g(Y,W)] - g(R(Y,Z)X,W). \quad (6.10)$$

Contracting (6.10) over Z and W we have

$$S(X,Y) = -(n-1)\,g(X,Y), \tag{6.11}$$

In view of (5.5) and (6.11) we obtain

$$\begin{aligned} 0 = &\left[\left(\frac{p}{2}+\frac{1}{n}\right)-\lambda+n-1\right]g(X,Y) \\ &-(n+\beta-1)\,\eta(X)\,\eta(Y)+\psi g(X,\phi Y), \end{aligned} \tag{6.12}$$

Setting $Y = \xi$ in (6.12) we have

$$\lambda = \left(\frac{p}{2}+\frac{1}{n}\right)-\beta.$$

This leads to the following theorem:

Theorem 6.4: *Let $M(\phi,\xi,\eta,g)$ be an n-dimensional para-Sasakian manifold of class C_2. If M admits a conformal η-Ricci soliton with respect to SVK-connection, then the soliton constants are given by*

$$\lambda = \left(\frac{p}{2}+\frac{1}{n}\right)-\beta.$$

Corollary 6.5: *A para-Sasakian manifold of class C_2 is always Einstein manifold.*

Conformal η-Ricci Soliton with Respect to $\overline{\nabla}$ on Class C_3: The condition that must be satisfied by conformal curvature tensor (C) is

$$(R(\xi,X).C)(Y,Z)V = 0, \tag{6.13}$$

for all $X,Y,Z,V \in \chi(M)$.

Equation (6.13) gives

$$R(\xi,X).C(Y,Z)V = C(R(\xi,X)Y,Z)V + C(Y,R(\xi,X)Z)V + C(Y,Z)R(\xi,X)V. \quad (6.14)$$

Setting $V = \xi$ in (6.14) we have

$$R(\xi,X).C(Y,Z)\xi = C(R(\xi,X)Y,Z)\xi + C(Y,R(\xi,X)Z)\xi + C(Y,Z)R(\xi,X)\xi. \quad (6.15)$$

Using (1.9), (2.9)-(2.11) in (6.14) and taking inner product of (6.15) with a vector field U and then contracting over Z, U we get

$$S(X,Y) = \left[\frac{n+r-1}{n-1}\right]g(X,Y) - \left[\frac{n^2-n+r}{n-1}\right]\eta(X)\eta(Y). \quad (6.16)$$

In consequence of (5.5) and (6.16) we obtain

$$0 = \left[\left(\frac{p}{2}+\frac{1}{n}\right) - \lambda - \frac{n+r-1}{n-1}\right]g(X,Y) + \left[\frac{n^2-n+r}{n-1} - n - \beta + 1\right]\eta(X)\eta(Y) + \psi g(X,\phi Y), \quad (6.17)$$

Setting $Y = \xi$ in (6.17) we have

$$\lambda = \left(\frac{p}{2}+\frac{1}{n}\right) - \beta,$$

which gives the following theorem:

Theorem 6.6: *Let $M(\phi,\xi,\eta,g)$ be an n-dimensional para-Sasakian manifold of class C_3. If M admits a conformal η-Ricci soliton with respect to SVK-connection, then the soliton constants are given by*

$$\lambda = \left(\frac{p}{2}+\frac{1}{n}\right) - \beta.$$

Corollary 6.7: *a para-Sasakian manifold of class* C_3 *is always an* η *-Einstein manifold.*

Conformal η-Ricci soliton with respect to $\bar{\nabla}$ on class C_4 : The condition that must be satisfied by conformal curvature tensor (C) is

$$(E(\xi,X).C)(Y,Z)V=0, \tag{6.18}$$

for all $X,Y,Z,V\in\chi(M)$.

Equation (6.13) gives

$$E(\xi,X).C(Y,Z)V = C(E(\xi,X)Y,Z)V + C(Y,E(\xi,X)Z)V + C(Y,Z)E(\xi,X)V. \tag{6.19}$$

Setting $V=\xi$ in (6.14) we have

$$E(\xi,X).C(Y,Z)\xi = C(E(\xi,X)Y,Z)\xi + C(Y,E(\xi,X)Z)\xi + C(Y,Z)E(\xi,X)\xi. \tag{6.20}$$

Using (1.9), (1.10), (2.9)-(2.11) in (6.14) and taking inner product of (6.15) with a vector field U and then contracting over Z, U we get

$$S(X,Y)=\left[\frac{n+r-1}{n-1}\right]g(X,Y)-\left[\frac{n^2-n+r}{n-1}\right]\eta(X)\eta(Y),$$

if $r\neq -n(n-1)$.

In view of (5.5) and (6.16) we obtain

$$0=\left[\left(\frac{p}{2}+\frac{1}{n}\right)-\lambda-\frac{n+r-1}{n-1}\right]g(X,Y) + \left[\frac{n^2-n+r}{n-1}-n-\beta+1\right]\eta(X)\eta(Y)+\psi g(X,\phi Y), \tag{6.22}$$

Setting $Y = \xi$ in (6.22) we have

$$\lambda = \left(\frac{p}{2} + \frac{1}{n}\right) - \beta,$$

which gives the following theorem:

Theorem 6.8: *Let $M(\phi, \xi, \eta, g)$ be an n-dimensional para-Sasakian manifold of class C_4. If M admits a conformal η-Ricci soliton with respect to SVK-connection, then the soliton constants are given by*

$$\lambda = \left(\frac{p}{2} + \frac{1}{n}\right) - \beta.$$

provided $r \neq -n(n-1)$.

Corollary 6.9: *A para-Sasakian manifold of class C_4 is an η-Einstein manifold if $r \neq -n(n-1)$.*

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