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Jeyanthi Venkatapathy[1] *and Madhan Velayuthan*[2]

AXION FIXED POINT THEOREM: A NEW FRAMEWORK BRIDGING HILBERT MANIFOLDS AND HILBERT SPACES

Abstract: This paper presents a novel fixed-point framework on Hilbert manifolds, called Axion. The local and global structure of manifolds can be better understood by using contraction mappings to define axion points. By using an Axion structure (a, Ψ, Γ), where Ψ is a diffeomorphism and Γ is its inverse meeting a contraction condition, the Axion Fixed Point Theorem extends conventional fixed-point findings to infinite-dimensional spaces. By establishing the existence and uniqueness of axion points, this approach advances our knowledge of fixed points in functional spaces.

Key Words and Phrases: Axion Set, Embedding, Hilbert Manifold, Hilbert Space.

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1. Introduction and Preliminaries

A Hilbert space is an infinite-dimensional generalization of Euclidean space, equipped with an inner product that induces a norm and a complete metric topology. Fixed point theorems are essential in analysis, topology, and geometry, providing fundamental results in nonlinear functional analysis, differential equations, and dynamical systems. The Banach Fixed Point Theorem, one of the most well-known results, guarantees the existence and uniqueness of fixed points under contraction

mappings in complete metric spaces. In 1956, Nash established the fundamental theory for embedding abstract Riemannian manifolds into Euclidean spaces, which remains a cornerstone in differential geometry [10]. A few years later, Hamilton (1982) contributed critical insights into curvature evolution, significantly influencing modern perspectives in differential geometry and general relativity [6]. In 2006, Chavel provided an extensive treatment of modern Riemannian geometry, focusing on embedding theorems and geometric flows [5]. Lee (2013) presented a contemporary perspective on smooth manifolds and Lie groups, which has been instrumental in advancing research in differential structures [8]. Between 2020 and 2024, significant progress was made in isometric embeddings and Hilbert manifold structures. Chattopadhyay *et al*. (2020) investigated the isometric embeddability of S_q^m into S_p^n, contributing to a deeper understanding of embeddings between finite-dimensional spaces [4]. In 2024, Capdeville examined the isometric embeddings of n-point spaces for $n \leq 4$, laying the groundwork for further studies in discrete metric spaces [3]. Looking ahead to 2025, Madhan Velayuthan and Jeyanthi Venkatapathy have extended embedding theories by addressing diffeomorphic embeddings of higher-dimensional Hilbert manifolds into Hilbert spaces. Their work introduces innovative techniques for handling infinite-dimensional structures and preserving geometric and topological properties [9].

Definition 1.1 ([8]): *A topological space $\mathcal{M}$ is called an n-**dimensional manifold** if*:

1. ***Local Euclidean Property***: $\forall p \in \mathcal{M}, \exists$ *a neighborhood* $U \subset \mathcal{M}$ *and a homeomorphism* $\sigma : U \to V \subset \mathbb{R}^n$, *such that* σ *and* $\sigma - 1$ *are continuous*.

2. ***Hausdorff Property***: $\mathcal{M}$ *is Hausdorff, i.e.,* $\forall p,\ q \in \mathcal{M},\ p \neq q, \exists$ *disjoint open sets* $U_p,\ U_q$ *such that* $p \in U_p$ *and* $q \in U_q$.

3. ***Second-Countability***: *The topology of* $\mathcal{M}$ *has a countable basis.*

If, in addition, $\mathcal{M}$ *is equipped with an atlas* $\{(U_j, \sigma_j)\}_{j \in \mathcal{J}}$ *such that for any two overlapping charts* (U_j, σ_j) *and* (U_k, σ_k), *the transition maps*

$\sigma_k \circ \sigma_j^{-1} : \sigma_j(U_j \cap U_k) \to \sigma_k(U_j \cap U_k)$ *are infinitely differentiable* (C^∞), *then* $\mathcal{M}$ *is called a* ***smooth manifold****.*

Definition 1.2 ([5]): *A* ***Hilbert manifold*** $\mathbb{H}_\mathcal{M}$ *is a smooth manifold modeled on an Hilbert space* $\mathcal{H}$. *Specifically,* $\mathbb{H}_\mathcal{M}$ *satisfies*:

1. $\exists$ *an atlas* $\{(U_\alpha, \sigma_\alpha)\}_{\alpha \in \mathcal{A}}$ *such that each chart* $\sigma_\alpha : U_\alpha \to \sigma_\alpha(U_\alpha) \subset \mathcal{H}$ *is a bijective homeomorphism mapping onto an open subset of* $\mathcal{H}$.

2. *Transition maps between overlapping charts,* $\sigma_\beta \circ \sigma_\alpha^{-1} : \sigma_\alpha(U_\alpha \cup U_\beta) \to \sigma_\beta(U_\alpha \cap U_\beta)$, *are infinitely differentiable* (C^∞).

3. *The topology of* $\mathbb{H}_\mathcal{M}$ *is induced by* $\mathcal{H}$, *i.e.,* $A \subset \mathbb{H}_\mathcal{M}$ *is open if and only if* $\sigma_\alpha(A)$ *is open in* $\mathcal{H}$ *for each chart* σ_α.

Definition 1.3 [9]: *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be Hilbert manifolds. A mapping* $\Psi : \mathcal{M} \to \mathcal{N}$ *is called a* ***diffeomorphism*** *if it satisfies the following conditions*:

1. ***Bijectivity***: *The map* Ψ *is a bijection, meaning it is both injective and surjective.*

2. ***Smoothness***: *The map* Ψ *is infinitely differentiable, i.e.,* $\Psi \in C^\infty(\mathcal{M}, \mathcal{N})$.

3. ***Smooth Inverse***: *The inverse mapping* $\Psi^{-1} : \mathcal{N} \to \mathcal{M}$ *exists and is also smooth, ensuring that* Ψ *establishes a smooth one-to-one correspondence between* $\mathcal{M}$ *and* $\mathcal{N}$.

If such a map exists, we say that $\mathcal{M}$ *and* $\mathcal{N}$ *are* ***diffeomorphic****, denoted as* $\mathcal{M} \cong \mathcal{N}$.

Definition 1.4 [8]: *Given a local chart* $\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathcal{H}$, *the induced metric on* U_α *is defined by* $d_{\mathcal{M}}(x,y) = \| \varphi_\alpha(x) - \varphi_\alpha(y) \|_{\mathcal{H}}$, *where* $\|\cdot\|_{\mathcal{H}}$ *denotes the norm in the Hilbert space* $\mathcal{H}$.

Theorem 1.5 ([8]): *Let* $\mathbb{S}_d$ *be a complete metric space and let* $C : \mathbb{S}_d \to \mathbb{S}_d$ *be a contraction mapping, i.e.,* $\exists c \in [0,1]$ *such that* $d(\mathcal{C}(h), \mathcal{C}(t)) \le c \cdot d(h,t), \forall h,\ t \in \mathbb{S}_d$. *Then,* $\mathcal{C}$ *has a unique fixed point* $h^* \in \mathbb{S}_d$.

2. Axion Fixed Point Theorem

This section presents new definition Axion and Axion fixed point theorem.

Definition 2.1: *Let* $\mathcal{M}$ *be a Hilbert manifold modeled on a Hilbert space* $\mathcal{H}$, *with an atlas* $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$. *An* ***Axion*** *is an ordered triplet* $(a,\ \Psi, \Gamma)$ *satisfying*:

1. ***Accumulation***: $a \in \mathcal{M}$ *is an accumulation point of* $S \subset M$, *i.e.,* $\forall\, U' \subset \mathcal{M}, a \in U' \Rightarrow U' \cap S \ = \varnothing$.

2. ***Smooth Chart***: $\exists$ *a chart* $(U_\alpha, \varphi_\alpha)$ *with* $a \in U_\alpha$ *and a smooth diffeomorphism* $\Psi : U_\alpha \to \Psi(U_\alpha) \subset \mathcal{H}$.

3. ***Inverse Mapping***: $\Gamma = \Psi^{-1} : \Psi(U_\alpha) \to U_\alpha$ *is smooth.*

The set of all Axions $\mathcal{X} = \{(a, \Psi, \Gamma) \mid \Psi : U\alpha \to \Psi(U_\alpha)$ *is a diffeomorphism,* $\Gamma = \Psi^{-1}\}$.

Theorem 2.2 (Axion Fixed Point Theorem): *Let* (a, Ψ, Γ) *be an Axion in a Hilbert manifold* $\mathcal{M}$ *with respect to a chart* $(U_\alpha, \varphi_\alpha)$, *where*:

1. $\Psi : U_\alpha \to \Psi(U_\alpha) \subset \mathcal{H}$ *is a smooth diffeomorphism.*

2. $\Gamma : \Psi(U_\alpha) \to U_\alpha$ *is the inverse of* Ψ, *i.e.*, $\Gamma = \Psi^{-1}$.

3. Γ *satisfies the contraction condition*: $\exists$ *a constant* $c \in [0,1)$ *such that* $d_{\mathcal{M}}(\Gamma(x), \Gamma(y)) \le c \cdot d_{\mathcal{M}}(x, y), \ \forall x, \ y \in U_\alpha$.

Then, $\exists$ *a unique fixed point* $a^* \in U_\alpha$ *such that* $\Gamma(a^*) = a^*$.

Proof: The local chart $\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathcal{H}$ induces a metric on U_α defined by: $d_{\mathcal{M}}(x, y) = \| \varphi_\alpha(x) - \varphi_\alpha(y) \|_{\mathcal{H}}$, where $\|\cdot\|_{\mathcal{H}}$ is the norm in $\mathcal{H}$. This metric provides a distance measure for elements of U_α. To apply the Banach Fixed Point Theorem, we must show that $(U_\alpha, d_{\mathcal{M}})$ is a complete metric space. Let (x_n) be a Cauchy sequence in U_α with respect to $d_{\mathcal{M}}$. By definition, $d_{\mathcal{M}}(x_n, x_m) = \| \varphi_\alpha(x_n) - \varphi_\alpha(x_m) \|_{\mathcal{H}}$.

Since, (x_n) is Cauchy in U_α, the sequence $\varphi_\alpha(x_n)$ is Cauchy in $\mathcal{H}$. Since, $\mathcal{H}$ is a Hilbert space, it is complete, and thus, $\exists$ a limit point $y \in \mathcal{H}$ such that: $\varphi_\alpha(x_n) \to y$ as $n \to \infty$. Since, $\Psi = \varphi_\alpha^{-1}$ is a diffeomorphism. By the continuity of Ψ, $x_n = \Psi(\varphi_\alpha(x_n)) \to \Psi(y)$ as $n \to \infty$. Since $y \in \Psi(U_\alpha)$, we have $\Psi(y) \in U_\alpha$, proving that U_α is complete. By assumption, $\exists\, c \in [0,1)$ such that: $d_{\mathcal{M}}(\Gamma(x), \Gamma(y)) \le c \cdot d_{\mathcal{M}}(x, y), \ \forall x, \ y \in U_\alpha$. This confirms that Γ is a strict contraction mapping.

Since $(U_\alpha, d_{\mathcal{M}})$ is a complete metric space and Γ is a contraction, the Banach Contraction Theorem guarantees the existence of a unique fixed point

$a^* \in U_\alpha$ such that: $\Gamma(a^*) = a^*$. Suppose there exist two fixed points $a_1,\ a_2 \in U_\alpha$ such that $\Gamma(a_1) = a_1$ and $\Gamma(a_2) = a_2$.

Then, $d_{\mathcal{M}}(a_1, a_2) = d_{\mathcal{M}}(\Gamma(a_1), \Gamma(a_2)) \leq c \cdot d_{\mathcal{M}}(a_1, a_2)$.

Since, $c \in [0,1)$, it follows that $d_{\mathcal{M}}(a_1, a_2) = 0$, implying $a_1 = a_2$.

Thus, the fixed point is unique. □

3. Conclusion

The Axion triplet (a, Ψ, Γ) is introduced in the Axion fixed point Theorem, which extends the standard Banach fixed point theorem to Hilbert manifolds. This framework preserves the underlying geometric structure of the manifold while enabling the analysis of contraction mappings inside local charts. In order to provide stability under smooth transformations, the theorem ensures that fixed points for such mappings exist and are unique. The Banach contraction theorem is used in the proof to demonstrate convergence, taking use of the contraction quality of Γ and the completeness of the induced metric space.

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1, 2 Department of Mathematics,
Sri Krishna Arts and Science
College, Coimbatore, 641008 - India

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1. E-mail: jeyanthiv@skasc.ac.in
2. E-mail: madhanvmaths@gmail.com