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FRACTIONAL PRODUCT CORDIAL LABLEING OF WEB GRAPH, DUMBBELL GRAPH, JELLY FISH GRAPH, JEWEL GRAPH AND TADPOLE GRAPH

Abstract: Let $G = (V, E)$ be a (p, q) graph. Let

$$M = \begin{cases} 1, 2, \cdots, \frac{p}{2}, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{2}{p+1} & \text{if } p \text{ is even} \\ 1, 2, \cdots, \frac{p}{2}, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{2}{p+3} & \text{if } p \text{ is odd} \end{cases}$$

Let $\varphi: V(G) \to M$ be a bijection. For each edge xy assign the label $f(x)f(y)$.φ is called a fractional product cordial labeling (simply called FP-cordial labeling) if $\left| \prod_{\varphi}(0) \quad \prod_{\varphi}(1) \right| \leq 1$, where $\prod_{\varphi}(1)$ and $\prod_{\varphi}(0)$ respectively denotes the number of edges labelled with 1 and not labelled with 1. A graph with a fractional product cordial labeling is called a fractional product cordial graph (Simply FP-cordial graph). In this paper, we investigate the FP-cordial labeling of web, jewel, dumbbell, jelly fish and tadpole graph.

Keywords: Web Graph, Jewel Graph, Dumbbell Graph, Jelly Fish Graph and Tadpole Graph.

Mathematics Subject Classification No.: 05C78.

1. Introduction

We consider only finite, simple and undirected graphs. The number of vertices of a graph G is called the order of G and number of edges is called the size

of G. The cordial labeling concept was introduced by I. Cahit [1] and subsequently cordial related concept was studied in [2-8]. The notion of fractional product cordial labeling have been introduced in [9] and fractional product cordial labeling behavior of path, cycle, complete, star, wheel, book with triangle pages, ladder, comb, double comb, bistar, subdivision of the star and subdivision of the bistar have been investigated [9]. Fractional product cordial labeling behavior of subdivision of comb, subdivision of double comb, triangular snake, quadrilateral snake, slanting ladder, triangular ladder, fan graph, flower graph, sunflower graph, helm and closed helm have been investigated [10].

In this paper we investigate the fractional product cordial labeling of some special graphs like web graph, jewel graph, dumbbell graph, jelly fish graph and tadpole graph. x denotes the smallest integer $\geq x$.

2. Preliminaries

Definition 2.1 [1]: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The join $G_1 + G_2$ as $G_1 \cup G_2$ Together with all the edges joining vertices of V_1 to the vertices of V_2.

Definition 2.2 [1]: The graph $W_n = C_n + K_1$is called the wheel graph.

Definition 2.3 [2]: The jewel graph $J_n(n \geq 1)$, is the graph with vertex set $V(J_n) = \{u, v, x, y, u_i : 1 \leq i \leq n\}$ and edge set $E(J_n) = \{uv, xu, xv, yu, yv, xu_i, yu_i : 1 \leq i \leq n\}$. Then the jewel graph$n + 4$ vertices and $2n + 5$ edges.

Definition 2.4 [2]: The tadpole graph $T(m, n)(m \geq 3, n \geq 1)$, is the graph obtained by joining a cycle C_m to a path P_n with a bridge.

Definition 2.5 [2]: The graph obtained by joining two disjoint cycles $x_1x_2 \ldots x_nx_1$ and $y_1y_2 \ldots y_ny_1$ with an edge x_1y_1 is called dumbbell graph and it is denoted by $Db(m, n)(m, n \geq 3)$.

Definition 2.6 [2]: Jelly fish graphs $J(m, n)$ obtained from $u_1u_2u_3u_4u_1$ by joining u_1 and u_3 with an edge and appending m pendent edges to u_2 and n pendent edges to u_4.

Definition 2.7 [2]: The web graph $Wb_n(n \geq 3)$ is the graph obtained by joining the pendent vertices of a helm H_n to form a cycle and then adding a pendant edge to each vertex of outer cycle.

3. Fractional Product Cordial Labeling

Definition 3.1: Let $G = (V, E)$ be a (p, q) graph.

Let $$M = \begin{cases} 1, 2, \cdots, \frac{p}{2}, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{2}{p+1} & if\ p\ is\ even \\ 1, 2, \cdots, \frac{p-1}{2}, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{2}{p+3} & if\ p\ is\ odd \end{cases}$$

Let $\varphi: V(G) \to M$ be a bijection.For each edge xy assign the label $f(x)f(y)$. φ is called a fractional product cordial labeling (simply called FP-cordial labeling) if $\left|\prod_{\varphi}(0) \quad \prod_{\varphi}(1)\right| \leq 1$, where $\prod_{\varphi}(1)$ and $\prod_{\varphi}(0)$ respectively denotes the number of edges labelled with 1 and not labelled with 1. A graph with a fractional product cordial labeling is called a fractional product cordial graph (Simply FP-cordial graph).

Theorem 3.2: The web graph Wb_n is FP-cordial for all $n \geq 3$.

Proof: Let $V(Wb_n) = V(W_n) \cup \{z_i' : 1 \leq i \leq n\}$ where $W_n = C_n + K_1$, C_n be the cycle $y_1 y_2 \ldots y_n y_1$, $V(K_1) = \{y\}$ and $E(Wb_n) = E(W_n) \quad \{z_i z_i' : 1 \leq i \leq n\}$. Then there are $3n + 1$ vertices and $5n$ edges. This proof is divided into two cases:

Case (i): n is odd

Fix the central vertex y is assigned by 1. Assign the labels $2,3, \ldots, n+1$ to the vertices $y_1, y_3, \ldots, y_n$ respectively. We assign the label $n + 2$to z_n and assign the labels $n + 2, n + 3, \ldots, \frac{3n+1}{2}$ respectively to the vertices $z_n', z_1', \ldots, z_{\frac{n-5}{2}}'$. We now assign the labels $\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}$ to the vertices $z_1', z_2', \ldots, z_{n-1}'$ respectively. Also we assign the labels $\frac{1}{n+1}, \frac{1}{n+2}, \ldots, \frac{2}{3n+3}$ respectively to the vertices $z_{\frac{n-3}{2}}', z_{\frac{n-1}{2}}', \ldots, z_{n-1}'$.

Hence, $\prod_{\varphi}(0) = \frac{5n+1}{2}$ and $\prod_{\varphi}(1) = \frac{5n-1}{2}$.

Case (ii): n is even

As in Case (i), assign the label to the vertices y, z_n an d$y_i, 1 \leq i \leq n$. We assign the labels $n + 2, n + 3, \ldots, \frac{3n}{2}$ respectively to the vertices $z_n', z_1', \ldots, z_{\frac{n-6}{2}}'$. We now assign the labels $\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}$ to the vertices $z_1', z_2', \ldots, z_{n-1}'$ respectively. Also we assign the labels -$\frac{1}{n+1}, \frac{1}{n+2}, \ldots, \frac{2}{3n}$ respectively to the vertices $z_{\frac{n-4}{2}}', z_{\frac{n-2}{2}}', \ldots, z_{n-1}'$.

Hence, $\prod_{\varphi}(0) = \frac{5n}{2}$ and $\prod_{\varphi}(1) = \frac{5n}{2}$.

Theorem 3.3: The jewel graph J_n is FP-cordialif and only if n {1,3}.

Proof: Let $V(J_n) = \{u, v, x, y, u_i : 1 \leq i \leq n\}$ and $(J_n) = \{uv, xu, xv, yu, yv, xu_i, yu_i : 1 \leq i \leq n\}$. Then there are $n + 4$ vertices and $2n + 5$ edges. This proof is break up into three cases:

Case (i): n is odd and $n \geq 5$

Fix the labels 1,2,4,3 to the vertices $x, u, y \; and \; v$ respectively. Then we assign the labels $5,6, \ldots, \frac{n+3}{2}$ respectively to the vertices $u_1, u_2, \ldots, u_{\frac{n-5}{2}}$. We assign the labels $\frac{1}{2}, \frac{1}{3}, \ldots, \frac{2}{n+7}$ to the vertices $u_{\frac{n-3}{2}}, u_{\frac{n-1}{2}}, \ldots, u_n$ respectively.

Therefore, $\prod_{\varphi}(0) = \frac{2n+6}{2}$ and $\prod_{\varphi}(1) = \frac{2n+4}{2}$.

Case (ii): n is even and $n \geq 2$

Then the following subcases are arises.

Subcase (i): $n = 2$

A FP-cordial labeling J_2 is shown in given table.

n	x	u	y	v	u_1	u_2
2	1	2	$\frac{1}{2}$	3	$\frac{1}{3}$	$\frac{1}{4}$

Subcase (ii): $n \geq 4$

Fix the labels 1,2,4,3 to the vertices $x, u, y \; and \; v$ respectively. Then we assign the labels $5,6, \ldots, \frac{n+4}{2}$ respectively to the vertices $u_1, u_2, \ldots, u_{\frac{n-4}{2}}$. We assign the labels $\frac{1}{2}, \frac{1}{3}, \ldots, \frac{2}{n+6}$ to the vertices $u_{\frac{n-2}{2}}, u_{\frac{n}{2}}, \ldots, u_n$ respectively.

Therefore, $\prod_{\varphi}(0) = \frac{2n+6}{2}$and $\prod_{\varphi}(1) = \frac{2n+4}{2}$.

Subcase (iii): $n\epsilon\{1,3\}$

Suppose $J_n\epsilon\Omega_{fpc}$. When $n=1$, the vertex labels are $1,2,\frac{1}{2},\frac{1}{3},\frac{1}{4}$. Suppose 1 and 2 the vertex labels are non adjacent vertices then $\prod_\varphi(0)=0$ and $\prod_\varphi(1)=7$, a contradiction.

Suppose 1 and 2 the vertex labels are adjacent vertices then $\prod_\varphi(0)=1$ and $\prod_\varphi(1)=6$, not possible.

In the case $n=3$, the vertex labels are $1,2,3,\frac{1}{2},\frac{1}{3},\frac{1}{4}$. When 1,2 and 3 the vertex labels are non adjacent vertices then $\prod_\varphi(0)\leq 1$ a contradiction to the size of J_3 is 11.

When 1,2 and 3 the vertex labels are adjacent vertices then $\prod_\varphi(0)\leq 4$ again a contradiction to the size of J_3 is 11.

Theorem 3.4: The dumbbell graph $Db(n,n)$ is FP-cordial for all $n\geq 3$.

Proof: Let $V(Db(n,n))=\{x_i,y_i: 1\leq i\leq n\}$ and $E(Db(n,n))=\{x_ix_{i+1},y_iy_{i+1}: 1\leq i\leq n \quad 1\}\cup\{x_1x_n,y_1y_n,x_1y_1\}$. Then there are $2n$ vertices and $2n+1$ edges.

Assign the labels $1,2,\dots,n$ to the vertices $x_1,x_2,\dots,x_n$ respectively. We assign the labels $\frac{1}{2},\frac{1}{3},\dots,\frac{1}{n+1}$ respectively to the vertices $y_1,y_2,\dots,y_n$.

Therefore, $\prod_\varphi(0)=n$ and $\prod_\varphi(1)=n+1$

Theorem 3.5: The jellyfish graph $J(m,n)$ is FP-cordial for all n.

Proof: Let $V(J(m,n))=\{x,y,u,v,u_i,v_j: 1\leq i\leq m \text{ and } 1\leq j\leq n\}$ and $E(J(n,n))=\{xy,xu,xv,yu,yv,uu_i,vv_j: 1\leq i\leq m \text{ and } 1\leq j\leq n\}$. Then there are $m+n+4$ vertices and $m+n+5$ edges. Then this proof break up into two cases:

Case (i): $m+n+4$ **is even**

Fix the labels 1,2,3 to the vertices x,y and u respectively. Then the following subcases are arises:

Subcase (i): $\boldsymbol{m = n}$

We assign the labels $4,5,\ldots,n+2$ respectively to the vertices $u_1, u_2, \ldots, u_{n-1}$. We now assign the labels $\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n+3}$ to the vertices $u_n, v_1, v_2 \ldots, v_n$ respectively.

Therefore, $\prod_\varphi(0) = \frac{m+n+6}{2}$ and $\prod_\varphi(1) = \frac{m+n+4}{2}$.

Subcase (ii): $\boldsymbol{m > n}$

Assign the labels $4,5,\ldots,\frac{m+n+4}{2}$ respectively to the vertices $u_1, u_2, \ldots, u_{\frac{m+n-2}{2}}$ and assign the labels $\frac{1}{2}, \frac{1}{3}, \ldots, \frac{2}{m-n+4}$ to the vertice s$u_{\frac{m+n}{2}}, u_{\frac{m+n+2}{2}}, \ldots, u_m$ Now assign the labels $\frac{2}{m-n+6}, \frac{2}{m-n+8}, \ldots, \frac{2}{m+n+6}$to the vertices $v, v_1, v_2, \ldots, v_n$ respectively.

Hence, $\prod_\varphi(0) = \frac{m+n+6}{2}$ and $\prod_\varphi(1) = \frac{m+n+4}{2}$.

Subcase (iii): $\boldsymbol{m < n}$

Assign the labels $4,5,\ldots,\frac{m+n+4}{2}$ respectively to the vertices $v_1, v_2, \ldots, v_{\frac{m+n-2}{2}}$ and assign the labels $\frac{1}{2}, \frac{1}{3}, \ldots, \frac{2}{n-m+4}$ to the vertices $v_{\frac{m+n}{2}}, v_{\frac{m+n+2}{2}}, \ldots, v_n$ Now assign the labels $\frac{2}{n-m+6}, \frac{2}{n-m+8}, \ldots, \frac{2}{m+n+2}$to the vertices $u, u_1, u_2, \ldots, u_m$ respectively.

Hence, $\prod_\varphi(0) = \frac{m+n+6}{2}$ and $\prod_\varphi(1) = \frac{m+n+4}{2}$.

Case (ii): $\boldsymbol{m + n + 4}$ **is odd**

Fix the labels 1,2,3 to the vertices x, y an du respectively. Then the following subcases are arises:

Subcase (i): $\boldsymbol{m > n}$

Assign the labels $4,5,\ldots,\frac{m+n+3}{2}$ respectively to the vertices $u_1, u_2, \ldots, u_{\frac{m+n-3}{2}}$ and assign the labels $\frac{1}{2}, \frac{1}{3}, \ldots, \frac{2}{m-n+5}$ to the vertices $u_{\frac{m+n-1}{2}}, u_{\frac{m+n+1}{2}}, \ldots, u_m$. Now

assign the labels $\frac{2}{m-n+7}, \frac{2}{m-n+9}, \ldots, \frac{2}{m+n+7}$ to the vertices $v, v_1, v_2, \ldots, v_n$ respectively.

Hence, $\prod_{\varphi}(0) = \frac{m+n+5}{2}$ and $\prod_{\varphi}(1) = \frac{m+n+5}{2}$.

Subcase (ii): $\boldsymbol{m < n}$

Assign the labels $4,5, \ldots, \frac{m+n+3}{2}$ respectively to the vertices $v_1, v_2, \ldots, v_{\frac{m+n-3}{2}}$ and assign the labels $\frac{1}{2}, \frac{1}{3}, \ldots, \frac{2}{n-m+5}$ to the vertices $v_{\frac{m+n-1}{2}}, v_{\frac{m+n+1}{2}}, \ldots, v_n$ Now assign the labels $\frac{2}{n-m+7}, \frac{2}{n-m+9}, \ldots, \frac{2}{m+n+7}$ to the vertices $u, u_1, u_2, \ldots, u_m$ respectively.

Hence, $\prod_{\varphi}(0) = \frac{m+n+5}{2}$ and $\prod_{\varphi}(1) = \frac{m+n+5}{2}$.

Theorem 3.6: The tadpole graph $T(m,n)$ is FP-cordial for all $m \geq 3$ an dfor all n, $T(3,1)$ an d$T(3,2)$ are not FP-cordial.

Proof: Let $V(T(m,n)) = \{u_i, v_j : 1 \leq i \leq m$ an d$1 \leq j \leq n\}$ and $E(T(m,n)) = \{u_i u_{i+1}, v_j v_{j+1} : 1 \leq i \leq m \quad 1 \; and \; 1 \leq j \leq n \quad 1\} \cup \{u_m u_1, u_1 v_1\}$. Then there are $m+n$ vertices and $m+n$ edges. This proof divided into three cases:

Case (i): $\boldsymbol{m+n}$ **is even**

Then the following subcases are arises:

Subcase (i): $\boldsymbol{m = n}$

Assign the labels $1,2, \ldots, \frac{m+n}{2}$ respectively to the vertices $y_n, y_{n-1}, \ldots, y_1$ andassign the labels $\frac{1}{2}, \frac{1}{3}, \ldots, \frac{2}{m+n+2}$ to the vertices $x_1, x_2, \ldots, x_m$ respectively.

Therefore, $\prod_{\varphi}(0) = \frac{m+n}{2}$ and $\prod_{\varphi}(1) = \frac{m+n}{2}$.

Subcase (ii): $\boldsymbol{m < n}$

Assign the labels $1,2,\ldots,\frac{m+n}{2}$ respectively to the vertices $y_n, y_{n-1}, \ldots, y_{\frac{n-m}{2}+1}$ and assign the labels $\frac{1}{2},\frac{1}{3},\ldots,\frac{2}{n-m+2}$ to the vertices $y_{\frac{n-m}{2}}, y_{\frac{n-m}{2}-1}, \ldots, y_1$. Now assign the labels $\frac{2}{n-m+4},\frac{2}{n-m+6},\ldots,\frac{2}{m+n+2}$to the vertices $x_1, x_2, \ldots, x_m$ respectively.

Hence, $\prod_\varphi(0) = \frac{m+n}{2}$ and $\prod_\varphi(1) = \frac{m+n}{2}$.

Subcase (iii): $\boldsymbol{m > n}$

Assign the labels $1,2,\ldots,n$ respectively to the vertices $y_n, y_{n-1}, \ldots, y_1$ and assign the labels $n+1, n+2, \ldots, \frac{m+n}{2} x_1, x_2, \ldots, x_{\frac{m-n}{2}}$ and assign the labels $\frac{1}{2},\frac{1}{3},\ldots,\frac{2}{m+n+2}$ to the vertices $x_{\frac{m-n}{2}+1}, x_{\frac{m-n}{2}+2}, \ldots, x_m$ respectively.

Hence, $\prod_\varphi(0) = \frac{m+n}{2}$ and $\prod_\varphi(1) = \frac{m+n}{2}$.

Case (ii): $\boldsymbol{m+n}$ **is odd**

Then the following subcases are arises:

Subcase (i): $\boldsymbol{m < n}$

Assign the labels $1,2,\ldots,\frac{m+n-1}{2}$ respectively to the vertices $y_n, y_{n-1}, \ldots, y_{n-m}$ and assign the labels $\frac{1}{2},\frac{1}{3},\ldots,\frac{2}{n-m+3}$ to the vertices$y_{n-m-1}, y_{n-m-2}, \ldots, y_1$. We now assign the labels $\frac{2}{n-m+5},\frac{2}{n-m+7},\ldots,\frac{2}{m+n+3}$ to the vertices $x_1, x_2, \ldots, x_m$ respectively.

Hence, $\prod_\varphi(0) = \frac{m+n-1}{2}$ and $\prod_\varphi(1) = \frac{m+n+1}{2}$.

Subcase (ii): $\boldsymbol{m > n}$

Assign the labels $1,2,\ldots,n$ respectively to the vertices $y_n, y_{n-1}, \ldots, y_1$ and assign the labels $n+1, n+2, \ldots, \frac{m+n-1}{2}$ to the vertices $x_1, x_2, \ldots, x_{\frac{m-n}{2}}$. Now assign the labels $\frac{1}{2},\frac{1}{3},\ldots,\frac{2}{m+n+3}$ to the vertices $x_{\frac{m-n+1}{2}}, x_{\frac{m-n+3}{2}}, \ldots, x_m$ respectively.

Hence, $\prod_{\varphi}(0) = \frac{m+n+1}{2}$ and $\prod_{\varphi}(1) = \frac{m+n-1}{2}$.

Case (iii): $\boldsymbol{m = 3}$ and $\boldsymbol{n \epsilon \{1, 2\}}$

Suppose $T(m, n) \epsilon \Omega_{fpc}$. When $n = 1$, the vertex labels are $1,2,\frac{1}{2},\frac{1}{3}$. Suppose 1 and 2 the vertex labels are non adjacent vertices then $\prod_{\varphi}(0) = 0$ and $\prod_{\varphi}(1) = 4$, a contradiction.

Suppose 1 and 2 the vertex labels are adjacent vertices then $\prod_{\varphi}(0) = 1$ and $\prod_{\varphi}(1) = 3$, which is not possible.

When $n = 2$, the vertex labels are $1,2,\frac{1}{2},\frac{1}{3},\frac{1}{4}$. Suppose 1 and 2 the vertex labels are non adjacent vertices then $\prod_{\varphi}(0) = 0$ and $\prod_{\varphi}(1) = 5$, again a contradiction.

Suppose 1 and 2 the vertex labels are adjacent vertices then $\prod_{\varphi}(0) = 1$ and $\prod_{\varphi}(1) = 4$, not possible.

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