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Γ_2 GRAPH OVER THE RING OF GAUSSIAN INTEGERS MODULO n

Abstract: This work is an investigation on some properties of Γ_2 graph, first introduced by R. Sen Gupta. Sen Gupta primarily focused on the Γ_2 graph, with particular attention to Z_n. We have extended the same to the ring of Gaussian integers modulo *n* and generalised the structure of this graph for different prime power factorizations of *n*. Our work also determines the connectedness and various properties of this graphs including diameter, girth, independence number, domination number, completeness, eulerian characterstics and planarity.

Keywords: Gaussian Integers, Ideal, Bipartite Graph, Complete Graph.

Mathematics Subject Classification (2020) No.: 05C25, 13A99.

1. Introduction

Zero divisor graph holds an important place among the graphs associated to a ring because it unveiled several critical and intriguing aspects of the set of zero divisors. This inspired various scholars to define and investigate different types of graphs over rings altering the adjacency relations. The concept of zero divisor graph was first introduced by Beck [6], whose primary interest was coloring of a commutative ring. Later in [4], Anderson and Livingston defined the graph $\Gamma(R)$ taking the set of non-zero zero divisors as it's vertex set, as the undirected graph where, x and y are adjacent if and only if $xy = 0$. In [3], Anderson and Badawi

defined the total graph over a commutative ring, denoted by $T(\Gamma(R))$, where they used the addition operation involving the zero divisors. To have some more insights on zero divisor graphs, one can refer to [1, 2, 5, 10, 14].

In [13], Sengupta *et al*. presented a variant of the zero divisor graph, represented by Γ_1, that modifies the conventional zero divisor graph by adding multiplicative and additive operations on , as the undirected graph (V, E) in which $V = R \setminus \{0\}$ and for any $a, b \in V$, $ab \in E$ if and only if $a \neq b$ and either $a.b = 0$ or $b.a = 0$ or $a + b$ is a unit. Later in [12], Sengupta introduced another variant Γ_2, with the same set of vertices, altering the last segment of the adjacency from Γ_1.

The inspiration of this study comes from Sengupta's Γ_2 graph, which was first introduced and studied giving particular attention to Z_n in [12]. Our work is an extension to the ring of Gaussian integers modulo n. The ring of gaussian integers modulo n gives rise to distinctive graph representations based on varying adjacency criteria, providing valuable insights to symmetry, connectivity and other graph properties that reflect the behavior of modular arithmetic. First, we will consider the scenario in which n is a prime or a power of a prime. The general case is next examined, and characteristics such as diameter, girth, independence number, domination number, completeness, eulerian characteristics, and planarity are determined.

2. Preliminaries

Sengupta defined the graph Γ_2 in the following way:

Definition 2.1 (Sen Gupta, [12]): *For a ring R, Γ_2 is an undirected graph (V, E) in which $V = R \setminus \{0\}$ and for any $a, b \in V$, $ab \in E$ if and only if $a \neq b$ and either $a.b = 0$ or $b.a = 0$ or $a + b$ is a zero-divisor (including 0).*

Before delving into the main results, we first outline some essential notations, definitions and theorems that were instrumental in analyzing the graph Γ_2 for $\mathbb{Z}_n[i]$.

Let R be a ring with unity and Reg(R) denotes the set of units of the ring R, whereas $Z(R)$ denotes the set of zero divisors of R. Throughout this paper, we will

consider $\mathrm{Reg}(\Gamma_2(R))$ as the (induced) subgraph of $\Gamma_2(R)$ with the vertices Reg(R) and $Z(\Gamma_2(R))$ as the (induced) subgraph of $\Gamma_2(R)$ with the vertices $Z(R)$.

With the usual complex number operations and norm $N(a+ib) = a^2 + b^2$, the set of complex numbers $a+ib$, where a and b are integers, forms an Euclidean domain. This domain is called the ring of Gaussian integers and is denoted by $\mathbb{Z}[i]$. For a natural number n, let $< n >$ be the principal ideal generated by n in $\mathbb{Z}[i]$, and let $\mathbb{Z}_n$ be the ring of integers modulo n. In [8], Dresden and Dymacek obtained the following result:

Theorem 2.1 (Dresden and Dymacek, [8]):

$$\mathbb{Z}[i]/< n > \cong \mathbb{Z}_n[i] = \{\overline{a} + i\overline{b} : \overline{a}, \overline{b} \in \mathbb{Z}_n\}$$

The above result directly implies that $\mathbb{Z}_n[i]$ is a principal ideal ring. This ring is called the ring of Gaussian integers modulo n.

In this paper, p and q will denote prime integers such that $p \equiv 1 \pmod 4$ and $q \equiv 3 \pmod 4$. It is known that, $p \equiv 1 \pmod 4$ if and only if there are integers a, b such that $p = a^2 + b^2$. In addition to the results already available on $\mathbb{Z}_n[i]$, Osba *et al*. in [11] further added the following:

Lemma 2.1 (Osba *et al*., [11]): *$\overline{a} + i\overline{b}$ is a unit in $\mathbb{Z}_n[i]$ if and only if $a^2 + b^2$ is a unit of $\mathbb{Z}_n$.*

Corollary 2.1 (Osba *et al*., [11]): *If $n = \prod_{j=1}^{s} a_j^{k_j}$ is the prime power factorization of the positive integer n, then $\mathbb{Z}_n[i]$ is the direct product of the rings $\mathbb{Z}_{a_j^{k_j}}[i]$.*

While $\mathbb{Z}_n$ is a local ring if n is a power of a prime, this is not the case for $Z_n[i]$. The finding of Osba *et al*. regarding this is as follows:

Theorem 2.2 (Osba *et al.*, [11]): *If* $m = p^k$ *for some prime* p *and positive integer* k, *then* $\mathbb{Z}_m[i]$ *is a local ring if and only if* $p = 2$ *or* $p \equiv 3 \pmod 4$.

For usual graph-theoretic definitions and notations, one can look at [9]. Let nbd (v) be the set of vertices adjacent to v. A graph is called planar if it can be drawn in the plane so that its edges intersect only at their ends. A connected graph is called eulerian if there exists a closed trail containing every edge of G. A dominating set for a graph $G = (V, E)$ is a subset D of V such that every vertex not in D is adjacent to atleast one member of D. The domination number $\delta(G)$ is the minimum size of a dominating set in G. An independent vertex set of a graph G is a subset of the vertices of G such that no two vertices in the subset are adjacent to each other. The size of the maximum independent vertex set is called the independence number and is denoted by $\alpha(G)$.

3. Γ_2 Graph for $\mathbb{Z}_{2n}[i]$:

In this section, we consider the Γ_2 graph for $\mathbb{Z}_{2n}[i]$. As $2 = (1+i)(1-i) = -i(1+i)^2$, so $\mathbb{Z}_2[i]$ is isomorphic to $\mathbb{Z}[i]/<(1+i)^2>$, where $\{0, 1+i\}$ is its only maximal ideal. As both $1+i$ and $1-i$ will generate the same maximal ideal in $\mathbb{Z}_2[i]$, therefore $Z(\mathbb{Z}_2[i]) = <1+i>$, which implies that the subgraph induced by the zero divisors of $\mathbb{Z}_2[i]$ is the graph with one vertex only and no edge.

Now if n is an integer greater than 1, then $2^n = (-i)^n(1+i)^{2n}$ and $\mathbb{Z}_{2^n}[i] = \mathbb{Z}[i]/<2^n> = \mathbb{Z}[i]/<(1+i)^{2n}>$. Therefore $\mathbb{Z}_2[i]$ is local with its only maximal ideal $M = <1+i>$ and $Z(\mathbb{Z}_{2^n}[i]) = <1+i> = \{a+ib : a, b \in \mathbb{Z}_{2^n}$ and a, b are both even or both odd$\}$. Hence, set of zero divisors are basically all those elements $a+ib$ in $\mathbb{Z}_2[i]$ where a, b are both even or both odd. The number of units in $\mathbb{Z}_2[i]$ is 2^{2n-1}, see [7]. Therefore, number of zero divisors in $\Gamma_2(\mathbb{Z}_{2^n}[i])$ is $2^{2n} - 2^{2n-1} - 1 = 2^{2n-1} - 1$.

Theorem 3.1: *$\Gamma_2(\mathbb{Z}_{2^n}[i])$ is not connected and it has exactly two components consisting of the zero divisors and units of $\mathbb{Z}_{2n}[i]$ respectively. The first is $K_{2^{2n-1}-1}$, while the second is $K_{2^{2n-1}}$.*

Proof: If u is a unit and y is a zero divisor, $u+y$ is always of the form $a+ib$, where one is odd and the other is even, hence not a zero divisor. Therefore a unit and a zero divisor is never adjacent making the graph disconnected. Since, $Z(\mathbb{Z}_{2^n}[i])$ is closed under addition, thus $Z(\Gamma_2(\mathbb{Z}_{2^n}[i]))$ is the complete graph $K_{2^{2n-1}-1}$. For any $u, v \in \mathrm{Reg}(\mathbb{Z}_{2^n}[i])$, obviously $uv \neq 0$. But $u+v$ is always of the form $a+ib$ where a and b are both even or both odd, which implies that $u+v \in Z(\mathbb{Z}_{2^n}[i])$. Therefore $\mathrm{Reg}(\Gamma_2(\mathbb{Z}_{2^n}[i]))$ is the complete graph $K_{2^{2n-1}}$. □

Example 3.1: *Here two examples of Γ_2 graph of $\mathbb{Z}_{2^n}[i]$ are displayed in* Figure 1 *and* Figure 2.

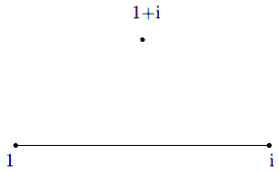


Figure 1: $\Gamma_2(\mathbb{Z}_{2^n}[i])$

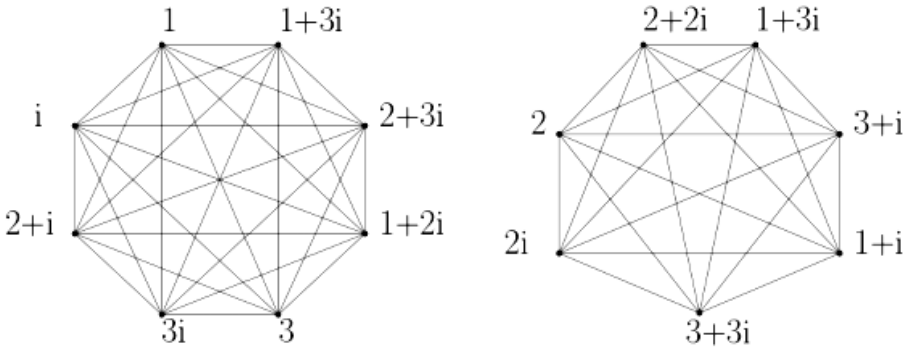


Figure 2: $\Gamma_2(\mathbb{Z}_4[i])$

Following the results from Theorem 3.1, we determine the diameter (diam) and girth (gr) of $\Gamma_2(\mathbb{Z}_{2^n}[i])$.

Corollary 3.1:

1. $\text{diam}(\Gamma_2(\mathbb{Z}_{2^n}[i])) = \infty$.

2. $\text{diam}(Z(\Gamma_2(\mathbb{Z}_{2^n}[i]))) = 1$.

3. $\text{diam}(\text{Reg}(\Gamma_2(\mathbb{Z}_{2^n}[i]))) = 1$.

Corollary 3.2: *If* $n \geq 2$, $\text{gr}(\Gamma_2(\mathbb{Z}_{2^n}[i]) = 3$.

4. Γ_2 Graph for $\mathbb{Z}_{q^n}[i], q \equiv 3 \pmod 4$

For a prime q congruent to 3 (mod 4), as $\mathbb{Z}_q[i]$ is isomorphic to the field $Z[i]/<q>$, therefore $\mathbb{Z}_q[i]$ has no nonzero zero divisors. If $n > 1$, then $Z_{q^n}[i] \cong \mathbb{Z}[i]/<q^n>$ is a local ring with the maximal ideal $<q>$. The number of units in $\mathbb{Z}_{q^n}[i]$ is $q^{2n}q^{2n-2}$, see [7]. Therefore the number of zero divisors in $\Gamma_2(\mathbb{Z}_{q^n}[i])$ is $q^{2n-2}-1$. We now look at the structure of Γ_2 by moving to the following results.

Theorem 4.1: $\Gamma_2(\mathbb{Z}_q[i])$ *is always disconnected having* $(q^2-1)/2$ *components, each being equal to* K_2.

Proof: Since, $\mathbb{Z}_q[i]$ has no nonzero zero divisors, no criterias of adjacency come into play except whenever the usual sum results to 0, which implies that every pair of additive inverses will form K_2. As the number of vertices is q^2-1, hence, there are $(q^2-1)/2$ components, each K_2 in $\Gamma_2(\mathbb{Z}_q[i])$. □

Example 4.1: *The graph* $\Gamma_2(\mathbb{Z}_3[i])$ *is given in* Figure 3.

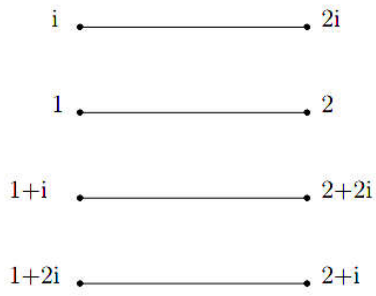


Figure 3: $\Gamma_2(\mathbb{Z}_3[i])$

Theorem 4.2: *For* $n \geq 2$, $\Gamma_2(\mathbb{Z}_{q^n}[i])$ *is disconnected with* $(q^2+1)/2$ *components. The component of the graph* $\Gamma_2(\mathbb{Z}_{q^n}[i])$; *having the zero divisors as the vertices is complete* $K_{q^{2n-2}-1}$. *Also the set of units form* $(q^2-1)/2$ *number of complete bipartite graphs each* $K_{q^{2n-2},\, q^{2n-2}}$.

Proof: Suppose there exists at least one zero divisor x which is adjacent to a unit v, then $x+v \in Z(\mathbb{Z}_{q^n}[i])$. Now the fact that $\mathbb{Z}_{q^n}[i]$ is a local ring with maximal ideal $< q >$, implies that the zero divisors are precisely the elements of the ideal $< q >$. So $x+v \in < q > \Rightarrow v = (x+v)+(-x) \in < q >$, hence this is a contradiction. Thus, $\Gamma_2(\mathbb{Z}_{q^n}[i])$ is never connected as zero divisors and units are not connected.

Any x, y belonging to the set of zero divisors will result in $x+y$ being a zero divisor. Therefore, the component of the graph $\Gamma_2(\mathbb{Z}_{q^n}[i])$ having the zero divisors as the vertices is complete and it is $K_{q^{2n-2}-1}$.

As $Z(\mathbb{Z}_{q^n}[i])$ is an ideal of $\mathbb{Z}_{q^n}[i]$, for two distinct elements in $u+Z(\mathbb{Z}_{q^n}[i])$, say $u+z_1$ and $u+z_2$, where $u \in \mathrm{Reg}(\mathbb{Z}_{q^n}[i])$; the adjacency of these two elements will imply; $(u+z_1)+(u+z_2) \in Z(\mathbb{Z}_{q^n}[i]) \Rightarrow 2u \in Z(\mathbb{Z}_{q^n}[i])$ $\Rightarrow 2 \in Z(\mathbb{Z}_{q^n}[i])$, hence this is a contradiction. So no two elements of

$u + Z(\mathbb{Z}_{q^n}[i])$ are adjacent to each other. Again, $u + Z(\mathbb{Z}_{q^n}[i])$ and $-u + Z(\mathbb{Z}_{q^n}[i])$ are disjoint. We prove this by method of contradiction. Suppose $u + Z(\mathbb{Z}_{q^n}[i])$ and $-u + Z(\mathbb{Z}_{q^n}[i])$ are not disjoint. Then for some $z_1,\ z_2 \in Z(\mathbb{Z}_{q^n}[i])$ and $v \in \mathrm{Reg}(\mathbb{Z}_{q^n}[i]), u + z_1 = v = -u + z_2 \Rightarrow u = v - z_1$. Thus, $-v + z_1 + z_2 = v$, $\Rightarrow z_1 + z_2 = 2v$ a zero divisor $\Rightarrow v$ is a zero divisor, which is a contradiction. Therefore $u + Z(\mathbb{Z}_{q^n}[i])$ and $-u + Z(\mathbb{Z}_{q^n}[i])$ are disjoint. But since each element of $u + Z(\mathbb{Z}_{q^n}[i])$ is adjacent to $-u + Z(\mathbb{Z}_{q^n}[i]),\ u + Z(\mathbb{Z}_{q^n}[i]) \cup -u + Z(\mathbb{Z}_{q^n}[i])$ is a complete bipartite subgraph of $\mathrm{Reg}(\mathbb{Z}_{q^n}[i])$. Also for $v + z_1$ being adjacent to $u + z_2$ where $u,\ v \in \mathrm{Reg}(\mathbb{Z}_{q^n}[i])$ and $z_1,\ z_2 \in Z(\mathbb{Z}_{q^n}[i]), u + v \in Z(\mathbb{Z}_{q^n}[i]) \Rightarrow v + Z(\mathbb{Z}_{q^n}[i])$ $= -u + Z(\mathbb{Z}_{q^n}[i])$; implying anything connected to $u + Z(\mathbb{Z}_{q^n}[i])$ is basically $-u + Z(\mathbb{Z}_{q^n}[i])$.

Now, $\left|u + Z(\mathbb{Z}_{q^n}[i])\right| = \left|Z(\mathbb{Z}_{q^n}[i])\right| = q^{2n-2}$. Hence regular elements form $K_{q^{2n-2},\, q^{2n-2}}$. Number of units being $q^{2n}q^{2n-2}$, there are number of complete bipartite graphs each $K_{q^{2n-2},\, q^{2n-2}}$. □

Following two results ensue directly from Theorem 4.1 and Theorem 4.2.

Corollary 4: 1. $\mathrm{diam}(\Gamma_2((\mathbb{Z}_{q^n}[i]))) = \infty$.

2. $\mathrm{diam}(Z(\Gamma_2((\mathbb{Z}_{q^n}[i]))) = 1,\ n \geq 2$.

3. $\mathrm{diam}(\mathrm{Reg}(\Gamma_2((\mathbb{Z}_{q^n}[i]))) = \infty$.

Corollary 4.2: $\mathrm{gr}(\Gamma_2(\mathbb{Z}_{q^n}[i]) = 3,\ n \geq 2$.

5. $\Gamma 2$ **Graph for** $\mathbb{Z}_{p^n}[i], p \equiv 1 \pmod 4$

If p is a prime congruent to 1 modulo 4, there exists $a,\ b \in Z$ such that $p = a^2 + b^2 = (a+ib)(aib)$. Since $\mathbb{Z}[i]$ is an unique factorization domain and $a+ib$ and $a-ib$ are gaussian primes in $\mathbb{Z}[i], < a+ib >$ and $< aib >$ are the only maximal ideals in $\mathbb{Z}[i]$ containing p. Thus, we have $\mathbb{Z}_p[i] \cong \mathbb{Z}[i]/< p > \cong (\mathbb{Z}[i]/< a+ib >) \times (\mathbb{Z}[i]/< a-ib >)$. It is trivial to see that in $\mathbb{Z}_p[i]$, the intersection of the maximal ideals is the singleton set $\{0\}$. Similarly, $p^n = (a^2+b^2)^n = (a+ib)^n(aib)^n$. Therefore p^n is contained in only two maximal ideals in $\mathbb{Z}[i]$, $< a+ib >$ and $< a-ib >$. Hence, $Z(\mathbb{Z}_{p^n}[i]) = < a+ib > \cup < a-ib >$ and $Z(\mathbb{Z}_{p^n}[i])$ is closed under multiplication by elements of the ring as $Z(\mathbb{Z}_{p^n}[i])$ is union of prime ideals. The number of units in $Z(\mathbb{Z}_{p^n}[i])$ is $(p^n p^{n-1})^2$, see [7]. Hence number of zero divisors in $\Gamma_2(\mathbb{Z}_{p^n}[i])$ is $2p^{2n-1}p^{2n-2} - 1$. In the ring $\mathbb{Z}_{p^n}[i]$, the cardinality of both maximal ideals is the same. Additionally, the set $W = \{a_1 p + b_1 pi : a_1,\ b_1 \in \mathbb{Z}_{p^n}\}$, which is the intersection of the ideals $< a+ib >$ and $< a-ib >$, forms an ideal itself, with a cardinality of p^{2n-2}. Furthermore, if two elements If $x \in < a+ib > \backslash W, y \in < a+ib > \backslash W$, then $x+y$ is a unit.

Theorem 5.1: *For every unit u, there exists a pair x, y, where $x \in < a+ib > \backslash W$ and $y \in < a-ib > \backslash W$ such that $u = x+y$. In $\mathbb{Z}_p[i]$, this pair $x, y \in Z(\mathbb{Z}_p[i])$ is unique.*

Proof: Existence: Suppose, $u = c+id \in \mathrm{Reg}(\mathbb{Z}_{p^n}[i])$. Then, $u = c+id = [d.(2b)^{-1} i.c.(2b)^{-1}](a+ib) + [-d.(2b)^{-1} + i.c.(2b)^{-1}](a-ib)$. Now $(2b)^{-1}$ exists as $b \in \mathbb{Z}_{p^n}$ and $b \neq 0$, $gcd(b,p) = 1 \Rightarrow gcd(2b,p) = 1 \Rightarrow gcd(2b, p^n) = 1$. Hence, there exists a pair $x, y \in Z(\mathbb{Z}_{p^n}[i])$ such that, $u = x+y$. Also it is clear that $x \in < a+ib > \backslash W$, $y \in < a+ib > \backslash W$.

Uniqueness: Suppose, there are two distinct pairs of zero divisors x, y and m, n such that $x+y = u = m+n$ where $x, m \in < a+ib > \backslash W$ and $y, n \in < a-ib > \backslash W$, also x, m and y, n are different.

Then for some $\alpha \in \mathbb{Z}_p[i], x = m + \alpha(a+ib) \Rightarrow m + \alpha(a+ib) + y = u$. Again, $m + n = u \Rightarrow \alpha(a+ib) + y = n \Rightarrow \alpha(a+ib) = n - y \in < a - ib >$, which is a contradiction as y and n are different, so $n - y \neq 0$ and the only possibility here is 0. If one of these two pairs is identical, suppose $n = y$, then for $x = m + \alpha(a+ib), \alpha(a+ib) = n - y = 0 \Rightarrow x = m$, which is a contradiction. Hence, for every unit u, there exists a unique pair $x,\ y \in Z(\mathbb{Z}_p[i])$ such that, $u = x + y$.

If a unit u can be expressed as sum of two zero divisors, we call them associated zero divisors of u. In $\mathbb{Z}_{p^n}[i]$, if x and y are associated zero divisors of a unit u, then the other associated pairs are $x + w_1$ and $y + w_2$ such that $w_1, w_2 \in W$ and $w_1 + w_2 = 0$. If for a unit u, the associated zero divisors are x and y, then u is adjacent to $-x$ and $-y$.

Theorem 5.2: *For a unit* $u, nbd(u) = \{-x + < a - ib >\} \cup$, $\{-y + < a + ib >\}$ *where* $x \in < a + ib > \backslash W$ *and* $y \in < a - ib > \backslash W$ *such that* $x + y = u$.

Proof: It is clear from Theorem 5.1 that for every unit u, there exists a pair $x,\ y \in Z(\mathbb{Z}_{p^n}[i])$ such that, $u = x + y$. Therefore every term in the coset $\{-x + < a - ib >\}$ is adjacent to u as for $z' \in < a - ib >, (-x + z') + u = (-x + z') + (x + y) = y + z' \in < a - ib >$. Similarly, every term in $\{-y + < a + ib >\}$ is adjacent to u and hence the result follows.

Corollary 5.1: *For any two distinct units* u_1 *and* u_2, *there exists atleast two units* u' *and* v' *such that* $u', v' \in \mathrm{nbd}(u_1) \cap \mathrm{nbd}(u_2)$.

Proof: Suppose the associated zero divisors of u_1 and u_2 are respectively x, y and x', y'. Then $\mathrm{nbd}(u_1) = \{-x + < a - ib >\} \cup \{-y + < a + ib >\}$ and similarly $\mathrm{nbd}(u_2) = \{-x' + < a - ib >\} \cup \{-y' + < a - ib >\}$. Hence, $-x - y'$ and $-x' - y$ are in the intersection of these two neighbourhoods. □

Theorem 5.3: $\Gamma_2(\mathbb{Z}_p[i])$ *is connected. Also the subgraph induced by the zero divisors is the complete graph* K_{2p-2}.

Proof: Assume that $x, y \in Z(\mathbb{Z}_p[i])$ and both belong to the same ideal. Then they are adjacent. But if x, y belong to different ideals, then $x = \alpha(a + ib)$ and $y = \beta(a - ib)$, where $\alpha, \beta \in \mathbb{Z}_p[i]$. Thus, $xy = \alpha(a + ib)\beta(a - ib) = \alpha\beta(a^2 + b^2) = \alpha\beta p = 0$, which implies that x and y are adjacent. Hence, the subgraph induced by the zero divisors is the complete graph K_{2p-2}.

Again for any unit u, there exists unique pairs of associated zero divisors $x, y \in Z(\mathbb{Z}_p[i])$ such that, $u = x + y$. Then for some $m \in Z(\mathbb{Z}_p[i])$ not adjacent to u, there will be a path $u - (x) - m$ or $u - (y) - m$.

Now from Corollary 5.1, for two non adjacent units u and v, there exists units u' and v' such that $u', v' \in nbd(u) \cap nbd(v)$. Thus, there will be paths $u - u' - v$ or $u - v' - v$. Hence, $\Gamma_2(\mathbb{Z}_p[i])$ is connected. □

Example 5.1: *The connectedness property of* $\Gamma_2(\mathbb{Z}_5[i])$ *can be seen in* Figure 4.

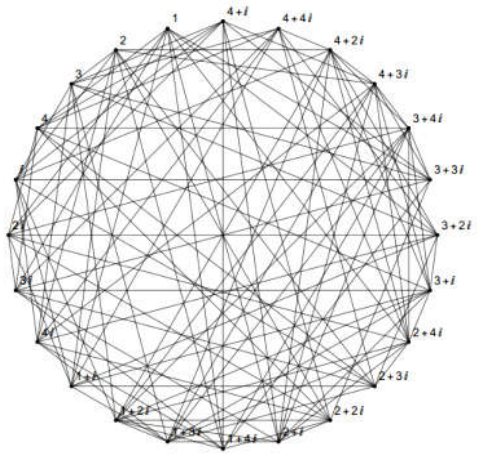


Figure 4: $\Gamma_2(\mathbb{Z}_5[i])$

Corollary 5.2: *In* $\Gamma_2(\mathbb{Z}_p[i])$, $\deg(u) = 2p-1$, *where* u *is a unit. On the other hand* $\deg(z) = 3p-4$, *where* z *is a zero divisor.*

Proof: Since $Z(\Gamma_2(\mathbb{Z}_p[i]))$ is complete with $2p-2$ vertices, number of adjacent zero divisors to a zero divisor is $2p-3$. Also a zero divisor is adjacent to $p-1$ units. Hence, degree of a zero divisor is $3p-4$. On the other hand, there are total $2p-1$ zero divisors including 0. As a unit is adjacent to a vertex if only their sum results in a zero divisor (including 0), hence degree of a unit is $2p-1$. □

Theorem 5.4: *For* $n \geq 2$, $Z(\Gamma_2(\mathbb{Z}_{p^n}[i]))$ *is* p^{2n-2}*-connected and* $\Gamma_2(\mathbb{Z}_{p^n}[i])$ *is connected.*

Proof: If two zero divisors x, y belong to the same ideal, then $x-y$. Again if x, y belong to different ideals and $m \in W$, then there is a path $x-m-y$. Hence, $Z(\Gamma_2(\mathbb{Z}_{p^n}[i]))$ is connected with $\operatorname{diam}(Z(\Gamma_2(\mathbb{Z}_p[i]))) = 2$. Note that deleting every vertex from the vertex set W will disconnect the subgraph induced by the zero divisors, $Z(\Gamma_2(\mathbb{Z}_{p^n}[i]))$ in two components. Hence, $Z(\Gamma_2(\mathbb{Z}_p[i]))$ is p^{2n-2} -connected.

Next assume that u is any unit and z_1 is a zero divisor such that they are not adjacent. Since there exists unique pairs of associated zero divisors $x, y \in Z(\mathbb{Z}_{p^n}[i])$ such that $u = x + y$, thus, there will be a path $u-(x)-z_1$ or $u-(y)-z_1$ depending on whether z_1 belongs to $< a+ib >$ or $< a-ib >$. Moreover from Corollary 5.1, for any two non adjacent units u and v, there exists units u' and v' such that $u', v' \in \operatorname{nbd}(u) \cap \operatorname{nbd}(v)$, thus having paths $u-u'-v$ or $u-v'-v$. Hence, $\Gamma_2(\mathbb{Z}_{p^n}[i])$ is connected.

We conclude this section with the following corollaries.

Corollary 5.3: $\operatorname{diam}(\Gamma_2(\mathbb{Z}_{p^n}[i])) = 2$.

Corollary 5.4: $\mathrm{gr}(\Gamma_2(\mathbb{Z}_{p^n}[i])) = 3$.

Corollary 5.5: *In* $\Gamma_2(\mathbb{Z}_{p^n}[i]), n \geq 2$; $\deg(u) = 2p^{2n-1} - p^{2n-2}$, *where* u *is a unit. On the other hand,* $\deg(z) = 2p^{2n-1} - p^{2n-2} - 2$, *where* z *is a zero divisor.*

Proof: A unit is adjacent to a vertex if only their sum is a zero divisor (including 0). Hence degree of a unit is $2p^{2n-1} - p^{2n-2}$. Since any $w \in W$ is adjacent to all the zero divisors in $\Gamma_2(\mathbb{Z}_{p^n}[i])$, therefore $\deg(w) = 2p^{2n-1} - p^{2n-2}$. Furthermore if $x \in < a + ib > \setminus W$, then x is adjacent to all the zero divisors of $< a + ib >$. Also, x is adjacent to $p^{2n-1} - p^{2n-2}$ units since $x + y$ is a unit for any $y \in < a - ib > \setminus W$. Therefore,

$$\deg(x) = [p^{2n-1} - 2] + [p^{2n-1} - p^{2n-2}] = 2p^{2n-1} - p^{2n-2} - 2.$$

6. $\Gamma 2$ Graph for $Z_n[i]$

In this section, we study the general case where n is any prime power factorization of distinct primes. Note that the set of zero divisors is nothing but the union of all the maximal ideals in $\mathbb{Z}_n[i]$. If $n = 2^m \prod_{j=1}^{s} {q_j}^{r_j} \prod_{i=1}^{t} {p_i}^{l_i}$, where $p_i \equiv 1(mod4)$ and $q_j \equiv 3(mod4)$, number of units in $\mathbb{Z}_n[i]$ is $2^{2m-1} \prod_{j=1}^{s} ({q_j}^{2r_j} - {q_j}^{2r_j-2}) \prod_{i=1}^{t} ({p_i}^{l_i} - {p_i}^{l_i-1})^2$. Thus number of zero divisors is $n^2 - 2^{2m-1} \prod_{j=1}^{s} ({q_j}^{2r_j} - {q_j}^{2r_j-2}) \prod_{i=1}^{t} ({p_i}^{l_i} - {p_i}^{l_i-1})^2$. The main goal of this section is to generalise the structure of $\Gamma_2(\mathbb{Z}_n[i])$ and to investigate the various properties of this graph like planarity, Eulerian characteristics, domination number and independence number.

Theorem 6.1: $\Gamma_2(\mathbb{Z}_n[i])$ *is not a complete graph for any* n.

Proof: $\Gamma_2(\mathbb{Z}_{2^r}[i])$ and $\Gamma_2(\mathbb{Z}_{q^r}[i])$ are always disconnected. For all other

n, $\mathbb{Z}_n[i]$ has atleast two units 1 and -1. Hence 1 and -2 are never adjacent. □

Theorem 6.2: $\Gamma_2(\mathbb{Z}_n[i])$ *is always connected if and only if* $n = p^r$ *or the prime power factorization of n has more than one prime factor and in the latter case* $\text{diam}(\Gamma_2(\mathbb{Z}_n[i])) = 2$ *and* $\text{gr}(\Gamma_2(\mathbb{Z}_n[i])) = 3$.

Proof: From the previous results, it is clear that $\Gamma_2(\mathbb{Z}_{2^r}[i])$ and $\Gamma_2(\mathbb{Z}_{q^r}[i])$ are always disconnected. Again $\Gamma_2(\mathbb{Z}_{p^n}[i])$ is connected with diameter 2 and girth 3. Now for the next part, from Theorem 2.8 of [12], we have, for $R = R_1 \times R_2 \times R_3 \ldots \times R_n$, where R_i's are rings with unity and $n > 1, n \in N$, $\Gamma_2(R)$ is connected with $\text{diam}(\Gamma_2(R)) \leq 2$ and $\text{gr}(\Gamma_2(R)) = 3$. If $n = \prod_{j=1}^{s} a_j^{k_j}$ is the prime power factorization of the positive integer n, then $\mathbb{Z}_n[i]$ is the direct product of the rings $\mathbb{Z}_{a_j}{}^{k_j}[i]$. But $\Gamma_2(\mathbb{Z}_n[i])$ is never a complete graph. Hence, $\Gamma_2(\mathbb{Z}_n[i])$ is always connected with $\text{diam}(\Gamma_2(\mathbb{Z}_n[i])) = 2$ and $\text{gr}(\Gamma_2(\mathbb{Z}_n[i])) = 3$. □

Corollary 6.1: $\Gamma_2(\mathbb{Z}_{q_1q_2}[i])$ *is connected with diameter 2 and girth 3. The subgraph induced by the zero divisors is* $K_{q_1^2+q_2^2-2}$.

Proof: The first part is trivial. Now for the second part, as the set of zero divisors is the union of the ideals $< q_1 >$ and $< q_2 >$, therefore zero divisors belonging to the same ideal will be adjacent to each other. If x and y belong to different ideals, then $xy = 0$. Hence, zero divisors belonging to different ideals will also be adjacent, thus, forming a complete graph. Since the number of non-zero zero divisors in $\mathbb{Z}_{q_1q_2}[i]$ is $(q_1q_2)^2 - [(q_1^2 - 1)(q_2^2 - 1)] - 1 = q_1^2 + q_2^2 - 2$, hence, the subgraph induced by the zero divisors is $K_{q_1^2+q_2^2-2}$. □

6.1 Planarity of $\Gamma_2(\mathbb{Z}_n[i])$

Theorem 6.3 (Kuratowski, [9]): *A graph is planar if and only if it contains no subgraph homeomorphic to* K_5 *or* $K_{3,3}$.

We also state the following result analogous to Kuratowski's theorem.

Theorem 6.4: $\Gamma_2(\mathbb{Z}_n[i])$ *is planar iff* $n = 2$ *or* n *is a prime* $q \equiv 3 \pmod 4$.

Proof: Since, $\Gamma_2(\mathbb{Z}_2[i])$ is disconnected with the two components K_1 and K_2, the graph is planar. When n is even, say $n = 2m$, the number of zero divisors in the ideal $< 1 + i >$ is $2m^2 \geq 8$. So, the subgraph of $\Gamma_2(\mathbb{Z}_n[i])$ induced by the elements of this ideal will be the complete graph K_{2m^2-1}. Hence, it will contain a subgraph homeomorphic to K_5.

When n is a prime $q \equiv 3 \pmod 4$, $\Gamma_2(\mathbb{Z}_n[i])$ is disconnected having $(q^2-1)/2$ components, each being equal to K_2, resulting the graph to be planar. But when $n = q^r, r > 1$, the component $Z(\Gamma_2(\mathbb{Z}_n[i]))$ is the complete graph $K_{q^{2r-2}-1}$, while the set of units form $(q^2-1)/2$ number of complete bipartite graphs each $K_{q^{2r-2},q^{2r-2}}$. Therefore it contains subgraph homeomorphic to both K_5 and $K_{3,3}$. When n is a prime $p \equiv 1 \pmod 4$, $Z(\Gamma_2(\mathbb{Z}_n[i]))$ is the complete graph K_{2p-2}. Again when $n = p^r, r > 1$, $\Gamma_2(\mathbb{Z}_n[i])$ has a complete subgraph $K_{q^{2r-2}-1}$ with the vertices of W. So $\Gamma_2(\mathbb{Z}_{p^r}[i])$ will always contain a subgraph homemorphic to K_5 for any r. Therefore whenever a prime $p \equiv 1 \pmod 4$ occurs in the prime power factorization of n, $\Gamma_2(\mathbb{Z}_n[i])$ will always contain a subgraph homemorphic to K_5. Hence, $\Gamma_2(\mathbb{Z}_n[i])$ is planar *iff* $n = 2$ or n is a prime $q \equiv 3 \pmod 4$ □

6.2 Non-eulerian characteristic of $\Gamma_2(\mathbb{Z}_n[i])$

Theorem 6.5 (Euler, [9]): *A connected graph G is eulerian if and only if the degree of each vertex of G is even.*

We state the following result now.

Theorem 6.6: *$\Gamma_2(\mathbb{Z}_n[i])$ is not eulerian for any n.*

Proof: Since, $\Gamma_2(\mathbb{Z}_{2^r}[i])$ and $\Gamma_2(\mathbb{Z}_{q^r}[i])$ are disconnected, they are not eulerian. Note that for any unit u in $\mathbb{Z}_n[i]$, u is adjacent to the vertices of the form $(z-u)$, where z is any zero divisor. But if n is even, then $2u \in Z(\mathbb{Z}_n[i])$ implying $u \in \{z-u : z \in Z(\mathbb{Z}_n[i])\}$. Thus, $\deg(u) = |\{z-u : z \in Z(\mathbb{Z}_n[i])\}| - 1$. Otherwise for the case where n is odd, $\deg(u) = |\{z-u : z \in Z(\mathbb{Z}_n[i])\}|$. When $n = 2^m \prod_{j=1}^{s} {q_j}^{r_j} \prod_{i=1}^{t} {p_i}^{l_i}, \deg(u) = n^2 - 2^{2m-1} \prod_{j=1}^{s} ({q_j}^{2r_j} - {q_j}^{2r_j-2}) \prod_{i=1}^{t} ({p_i}^{l_i} - {p_i}^{l_i-1})^2 - 1$ which is odd. But if $n = \prod_{j=1}^{s} {q_j}^{r_j} \prod_{i=1}^{t} {p_i}^{l_i}$, then $\deg(u) = n^2 - \prod_{j=1}^{s} ({q_j}^{2r_j} - {q_j}^{2r_j-2}) \prod_{i=1}^{t} ({p_i}^{l_i} - {p_i}^{l_i-1})^2$ which is again odd.

Hence, $\Gamma_2(\mathbb{Z}_n[i])$ is not eulerian for any n. □

6.3 The Domination Number for $\Gamma_2(\mathbb{Z}_n[i])$

Theorem 6.7: *The domination number $\delta(\Gamma_2(\mathbb{Z}_n[i])) = 2$ if $n = 2^r$.*

Proof: Since $\Gamma_2(\mathbb{Z}_{2^r}[i])$ has two components $K_{2^{2r-1}-1}$ and $K_{2^{2r-1}}$, we can include one vertex from each of these components in a set so that every vertex outside this set are adjacent to atleast one vertex in the set. □

Theorem 6.8: *The domination number*

$$\delta(\Gamma_2(\mathbb{Z}_n[i])) = \begin{cases} (q^2-1)/2, & n = q \\ q^2, & n = q^r, r > 1. \end{cases}$$

Proof: Since $\Gamma_2(\mathbb{Z}_q[i])$ is disconnected having $(q^2-1)/2$ components of K_2, therefore each component will contribute one vertex each to the dominating set. Thus, $\delta(\Gamma_2(\mathbb{Z}_q[i])) = (q^2-1)/2$.

For $r \geq 2$, $\Gamma_2(\mathbb{Z}_{q^r}[i])$ is disconnected with $(q^2+1)/2$ components. Since, the component of the graph $\Gamma_2(\mathbb{Z}_{q^r}[i])$ having the zero divisors as the vertices is complete $K_{q^{2r-2}-1}$, we can take only one vertex from this component. As the set of units form $(q^2-1)/2$ number of complete bipartite graphs each $K_{q^{2r-2},\ q^{2r-2}}$, we can include two vertices from each of these components, both from different partitions. Hence, $\delta(\Gamma_2(\mathbb{Z}_{q^r}[i])) = 1 + 2 \times [(q^2-1)/2] = 1 + q^2 - 1 = q^2$. □

Theorem 6.9: *The domination number*

$$\delta(\Gamma_2(\mathbb{Z}_n[i])) = \begin{cases} (p-1), & n = p \\ p, & n = p^r, r > 1. \end{cases}$$

Proof: In $\Gamma_2(\mathbb{Z}_p[i])$, zero divisors form a complete graph K_{2p-2}. If $x \in < a+ib > \backslash W$, then x is adjacent to $p-1$ number of unique units and the set of units adjacent to x is $-x + \{< a-ib > \backslash W\}$. Note that, inclusion of each element from $< a+ib > \backslash W$ in a set will result in all the outside vertices adjacent to atleast one vertex of this set. Hence, $\delta(\Gamma_2(\mathbb{Z}_p[i])$ = Number of zero divisors in $< a-ib > \backslash W = p-1$.

On the other hand, in $\Gamma_2(\mathbb{Z}_{p^r}[i]), r > 1$, the zero divisors from $\{< a+ib > \backslash W\}$ and $\{< a-ib > \backslash W\}$ are not adjacent. Vertices from W are adjacent to elements of both the ideals, but no units. So first we include one element (not 0) from W in the dominating set. For $x \in < a+ib > \backslash W$, each element of the set $\{x + w_i : w_i \in W\}$ has the same set of units adjacent to them. Thus, for any $x \in < a+ib > \backslash W$, there are p^{2n-2} zero divisors having the same set of adjacent units. If some $x' \notin \{x + w_i : w_i \in W\}$, then x' has unique set of units adjacent to it. Therefore total number of unique sets of units $= [p^{2n-1} - p^{2n-2}] / p^{2n-2} = p-1$. Adding only those zero divisors from $< a+ib > \backslash W$ which have different set of units adjacent to it, we get the the least dominating set. □

Hence, $\delta(\Gamma_2(\mathbb{Z}_n[i])) = 1 + (p-1) = p$. □

6.4 The Independence Number for $\Gamma_2(\mathbb{Z}_n[i])$

Theorem 6.10: *The independence number,* $\alpha(\Gamma_2(\mathbb{Z}_n[i])) = 2$ *if* $n = 2^r$.

Proof: Since in $\Gamma_2(\mathbb{Z}_{2^r}[i])$, both the components are complete, we can include only one vertex from each of these components in the independent set so that no two vertices inside this set are adjacent. □

Theorem 6.11: *The independence number*

$$\alpha(\Gamma_2(\mathbb{Z}_n[i])) = \begin{cases} \frac{q^2-1}{2} & n = q \\ 1 + \frac{q^{2r}-q^{2r-2}}{2}, & n = q^r, r > 1. \end{cases}$$

Proof: As $\Gamma_2(\mathbb{Z}_q[i])$ has $(q^2-1)/2$ components each being equal to K_2, so only one vertex from each component can be taken into the independent set. Hence, $\alpha(\Gamma_2(\mathbb{Z}_q[i])) = (q^2-1)/2$.

For $r \geq 2$, $\Gamma_2(\mathbb{Z}_{q^r}[i])$ has a component of the graph having the zero divisors, which is $K_{q^{2r-2}-1}$. Therefore we can only take one vertex from this component. Since the set of units form $(q^2-1)/2$ number of complete bipartite graphs each $K_{q^{2r-2},\, q^{2r-2}}$, we can take q^{2r-2} vertices from one partition of a component as they are not adjacent to one another. Hence, number of elements in the largest independent set that is $\alpha(\Gamma_2(\mathbb{Z}_{q^r}[i])) = 1 + [(q^2-1)/2] \times q^{2r-2} = 1 + [\frac{q^{2r}-q^{2r-2}}{2}]$. □

In $\mathbb{Z}_{p^r}[i]$, for any zero divisor $x \notin W$, we defne the elements of $-x + W$ as the W-inverses of x. Note that for $W = \{0\}$, they are simply the additive inverses of each other. It is trivial to see that, elements from $x + W$ has the same set of

W-inverses. For some $x' \notin x+W$, $x'+W$ will have di erent W-inverses than the elements from $x+W$. The set of W-inverses of the elements of $-x+W$ is nothing but the set $x+W$.

Theorem 6.12: *The independence number*

$$\alpha(\Gamma_2(\mathbb{Z}_n[i])) = \begin{cases} 1+(\frac{p-1}{2})^2, & n = p \\ 2+p^{2n-2}(\frac{p-1}{2})^2, & n = p^r \ for\ r > 1. \end{cases}$$

Proof: In $\Gamma_2(\mathbb{Z}_{p^r}[i])$, a unit is of the form $x+y$, where $x \in < a+ib > \backslash W$ and $y \in < a-ib > \backslash W$. Any two units $x+y$ and $m+n$ will be adjacent if either $x+m \in W$ or $y+n \in W$ or both, simply if $m \in -x+W$ or $n \in -y+W$. Therefore only those units of the form $\{x_i + y_j\}$ can be included in the independent set so that no two elements from $\{x_i\}$ or no two elements from $\{y_j\}$ are W-inverses of each other.

In $< a+ib > \backslash W$, number of cosets of the form $\{x_i + W\}$ such that no two elements from these cosets are W-inverses of each other is $\frac{p-1}{2}$. Similarly in $< a-ib > \backslash W$, number of cosets of the form $\{y_i + W\}$ such that no two elements from these cosets are W-inverses of each other is $\frac{p-1}{2}$. These two selected set of zero divisors after addition will result in mutually non-adjacent units. Note that for $m,\ n \in x+W$, the sets $m+y+W$ and $n+y+W$ are same. Therefore taking only one element each from the $\frac{p-1}{2}$ cosets of the form $\{x_i + W\}$ and all the elements from the selected $\frac{p-1}{2}$ cosets of the form $\{y_j + W\}$, we get the unique units that are mutualy non-adjacent. Thus, total units that can be included in the independent set $= \frac{p-1}{2}p^{2n-2} + \frac{p-1}{2}p^{2n-2} + \frac{p-1}{2}p^{2n-2} + \ldots$ upto $\frac{p-1}{2}$ terms $= (\frac{p-1}{2})^2 p^{2n-2}$.

In $\Gamma_2(\mathbb{Z}_p[i])$, since $W = \{0\}$, every unit of the form $\{x_i + y_j\}$ will be unique. Also only one zero divisor that too from the above selected sets $\{x_i\}$ or $\{y_j\}$

can be included in the independent set since the subgraph induced by the zero divisors is K_{2p-2}. Therefore, $\alpha(\Gamma_2(\mathbb{Z}_p[i])) = 1 + (\frac{p-1}{2})^2$.

In $\Gamma_2(\mathbb{Z}_{p^r}[i]), r \geq 2$; since elements from $< a + ib > \backslash W$ and $< a - ib > \backslash W$ are not adjacent, two zero divisors each from the above selected sets $\{x_i\}$ and $\{y_j\}$ can be included in the independent set. Hence, $\alpha(\Gamma_2(\mathbb{Z}_{p^r}[i])) = 2 + p^{2n-2}(\frac{p-1}{2})^2$.

We conclude this paper with the following example:

Example 6.1: *An independent set of* $\Gamma_2(\mathbb{Z}_5[i])$ *is* $\{1, 2, 1 + 2i, 4 + 4i, 4 + i\}$.

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Declarations

The authors declare that they have no conflict of interest.

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