Journal of Indian Acad. Math. ISSN: 0970-5120
Vol. 46, No. 2 (2024) pp. 257-264

Amitabh Kumar

SOLUTION OF KAPLER'S EQUATION

Abstract: The profound contributions of Johannes Kepler to the field of astronomy and mathematics continue to shape our understanding of the universe. The amalgamation of these derivations establishes *Kapler's* planetary motion *Equations* collectively known as his "Revelations on Planetary Motion," stand as a testament to his meticulous observations, innovative thinking, and unwavering dedication to unraveling the mysteries of the cosmos. *In this paper we solve the Kapler's* planetary motion *Equations.*

Key words: Kepler Equation, Celestial Mechanics, Iteration Method.

Mathematics Subject Classification (2010) No.: 83C40, 83C45.

1. Introduction: Unraveling Heavens

Throughout history, humanity's fascination with the night sky has driven the pursuit of understanding the motions of celestial bodies. In the early 17th century, Johannes Kepler emerged as a key figure in this quest, overturning traditional cosmological beliefs and introducing a new paradigm for comprehending planetary orbits. Kepler's groundbreaking laws provided an illustrious confirmation of his revolutionary ideas.

2. Kepler's Laws: The Cornerstones of Celestial Mechanics

2.1 The Law of Ellipses: A Paradigm Shift: Kepler's first law, the Law of Ellipses, shattered the prevailing notion of circular planetary orbits by revealing that

planets trace elliptical paths around the Sun. This insight not only corrected inaccuracies in earlier models, but also laid the foundation for a more accurate representation of planetary motion. The law's formulation established the Sun as a focal point within these elliptical paths, transforming the understanding of the solar system's geometry.

2.2 The Law of Equal Areas: Dynamic Orbits: Kepler's second law, the Law of Equal Areas, introduced the concept of equal areas being swept out in equal times by a line joining a planet and the Sun. This law highlighted the non-uniform nature of planetary motion, dispelling the notion of constant planet speed along its orbit. Instead, Kepler's insight revealed that planets experience acceleration and deceleration in response to their varying distances from the Sun.

2.3 The Harmonic Law: A Cosmic Symphony: Kepler's third law, The Harmonic Law, often referred to as the Harmonic Law, presented a mathematical expression between a planet's orbital period and its distance from the Sun. This elegant formula, which expressed the square of a planet's orbital period as proportional to the cube of its semi-major axis, illuminated the underlying harmony in the solar system. This law not only solidified Kepler's findings but also laid the groundwork for Newton's later work on universal gravitation.

Newton's Proclamation of Motion: A Trilogy of Fundamental Laws

In an epoch-defining moment, Sir Isaac Newton unveiled a triumvirate of elegantly straightforward laws of motion, whose harmonious interplay, when coupled with his revolutionary Law of Universal Gravitation, constructs the very edifice of theoretical mechanics.

The First Law: The Pinnacle of Inertia

Newton's inaugural decree posits that an object, unswervingly immersed in a state of tranquility or ceaseless uniform motion along a linear trajectory, remains ensconced in that very state unless coerced into metamorphosis by the imposing hands of external forces.

The Second Law: The Dance of Momentum and Force

Casting his gaze upon momentum's metamorphosis, Newton's second tenet divulges that the metamorphosis rate of momentum, gauged relative to an inertial vantage, stands in perfect proportion to the vigour of the force that imposes itself, unfurling as if in choreographed synchrony, along the identical compass of that force.

The Third Law: The Balancing Act of Action and Reaction

Emanating wisdom akin to a cosmic riddle, Newton's third dictum entwines every action with an equal and opposite reaction. A symphony of harmonious counterbalance governs the interactions of the universe, elegantly ensuring equilibrium in every altercation.

Mathematically embodied, the second law assumes the following form:

$$F = m\frac{d^2\mathbf{r}}{dt^2}, \tag{1}$$

Where, F = Vector sum of all forces,
m = mass,

Newton's Universal Gravitation law

In the illustrious tapestry of scientific revelations, Newton's indelible contribution shines brightly through his formulation of the Law of Universal Gravitation. An exquisite cosmic symphony unfolds as this law proclaims that any two celestial bodies share an intimate dance of attraction, a force that waltzes in direct proportion to the product of their masses while gracefully waltzing inversely to the square of the distance that separates them.

Mathematically embodied, this celestial pact is elegantly inscribed as:

$$F = \frac{Gm_1m_2}{r^3}\mathbf{r} \tag{2}$$

Here:

F = gravitational force between the two objects,
G = gravitational constant (approximately 6.67430 x 10^ 11 N(m/kg)^2),
m1 and m2 = masses of the two objects,
r = distance between the centers of the two objects.

Solution of Kapler's Equations

Newton stood as the vanguard in articulating and resolving the intricate enigma of the two-body problem. This celestial quandary boasts significance twofold: Primarily, it addresses the gravitational interactions within spherical entities whose masses nestle within their spherical shells-an exclusive predicament in

gravitational studies amenable to rigorous, and somewhat uncomplicated, resolutions. Secondly, it serves as the bedrock for practical quandaries concerning orbital movements, allowing them to be approximated as two-body problems. In essence, the two-body solution emerges as the cardinal cornerstone, from which the edifice of more precise calculations unfurls.

Equations (1) and (2) combine well to provide the two-body problem's mathematical representation. The radial vector r embraces a positive connotation as it sweeps away from the point of origin since the earth's core serves as the origin of our coordinate system. Equation (1) and (2) undergo a transmutation, orchestrating them to articulate the force that caresses the satellite endowed with mass m—a force that serenades in response to the gravitational allure cast by the colossal mass M of the earth

$$F_m = m\frac{d^2\mathbf{r}}{dt^2}$$

$$F_m = -GmM\frac{\mathbf{r}}{r^3} \tag{1}$$

$$\frac{d^2\mathbf{r}}{dt^2} + GM\frac{\mathbf{r}}{r^3} = \frac{d^2}{dt^2}\mathbf{r} + \mu\frac{\mathbf{r}}{r^3} = 0 \tag{3}$$

The process of entering the basic equation, known as the "process of integration of two equations," is simpler for individuals with a basic understanding of vector analysis. This process is completed by determining the validity of Kepler's law.

Verification of Invariable Vector

Performing the operation of cross-multiplication on equation (3) with respect to the factor r:

$$\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2} + \frac{\mu}{r^3}\mathbf{r}\times\mathbf{r} = 0$$

The second term evaluates to zero due to the property that the cross product of a vector with itself, i.e., r × r, results in zero. Consequently,

$$\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2} = 0 \tag{4}$$

Calculate the derivative of the angular momentum L:

$$\frac{dL}{dt^2} = \frac{d}{dt^2}\left(\mathbf{r} \times \frac{d\mathbf{r}}{dt^2}\right) = \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} + \frac{d\mathbf{r}}{dt} \times \frac{d\mathbf{r}}{dt} = 0 \tag{5}$$

The initial term on the right-hand side becomes zero as a result of equation (4). Similarly, the second term is effective because the cross product of two parallel vectors is zero. Thus, $\frac{dL}{dt} = 0,$ we arrive at the conclusion: or, in alternative words, the vector L remains constant.

Solution of Kepler's 1st Law equation

Cross-multiplying equation (3) with the angular momentum L:

$$\frac{d^2\mathbf{r}}{dt^2} \times L = -\frac{\mu}{r^3}\mathbf{r} \times L = -\frac{\mu}{r^3}\mathbf{r} \times \left(\mathbf{r} \times \frac{d\mathbf{r}}{dt^2}\right)$$

By triple vector product, $x \times (y \times z) = (x \cdot z)y - (x \cdot y)z$, the resulting expression takes the form:

$$-\frac{\mu}{r^3}\left[\left(\mathbf{r} \cdot \frac{d\mathbf{r}}{dt}\right)\mathbf{r} - (\mathbf{r} \cdot \mathbf{r})\frac{d\mathbf{r}}{dt}\right] = -\frac{\mu}{r^3}\left(r\frac{d\mathbf{r}}{dt}\mathbf{r} - r^2\frac{d\mathbf{r}}{dt}\right) = \mu\left(\frac{1}{r}\frac{d\mathbf{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\mathbf{r}\right) = \mu\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right),$$

Hence, $$\frac{d^2\mathbf{r}}{dt^2} \times L = \mu\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right)$$

This equation is amenable to direct integration due to the constancy of L:

$$\frac{d\mathbf{r}}{dt} \times L = \mu\left(\frac{\mathbf{r}}{r}\right) + c,$$

where c is a constant vector of integration. To clarify matters that will be elucidated subsequently, we represent c as μe. Consequently, the last expression transforms into:

$$\frac{d\mathbf{r}}{dt} \times L = \frac{\mu}{r}(\mathbf{r} + er) \tag{6}$$

Finally, compute the dot product of equation (6) with vector r, utilizing the property:

$$x \,.\, y \times z = z \,.\, x \times y$$

$$\mathbf{r}\cdot\left(\frac{d\mathbf{r}}{dt}\times L\right)=\frac{\mu}{r}\left(\mathbf{r}\cdot\mathbf{r}+e\cdot\mathbf{r}r\right),$$

$$\text{or, } L\cdot\left(\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right)=L\cdot L=L^2=\frac{\mu}{r}\left(r^2+er^2\cos v\right),$$

where v denotes the angle between vectors e and r-

Therefore ,

$$r=\frac{L^2/\mu}{1+e\cos v} \tag{7}$$

Equation (7) serves as the all-encompassing polar coordinate representation describing a conic section, where the origin of coordinates is located at one of the focal points.

If $0 \le e \le 1$, the orbit coincides with an ellipse. Thus, we get Kepler's 1st equation of planetary motion.

Solution of Kepler's 2nd Law equation

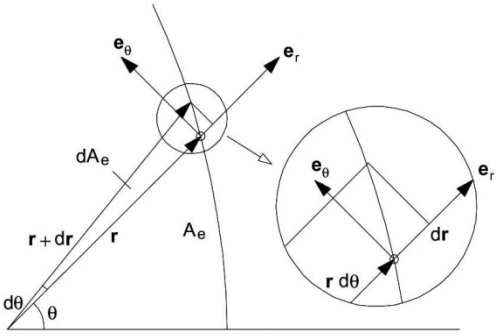


Fig. 1

Reformulate the equation in term of L,

$$\mathbf{r}\cdot\left(\times L\right)\frac{d\mathbf{r}}{dt}=\frac{dr}{dt}e_r+r\frac{d\theta}{dt}e_\theta \quad \text{(as depicted in Figure 1)} \tag{8}$$

$$L = \mathbf{r} \times \frac{d\mathbf{r}}{dt} = \mathbf{r} \times \left(\frac{dr}{dt} e_r + r \frac{d\theta}{dt} e_\theta \right) = \frac{dr}{dt} \mathbf{r} \times e_r + r^2 \frac{d\theta}{dt} e_r \times e_\theta \tag{9}$$

In the provided equation, the first term on the right-hand side equates to zero due to the collinearity of vectors r and er . Consequently, Equation (8) can be reformulated as follows:

$$L = r^2 \frac{d\theta}{dt} \tag{10}$$

A visual examination of Figure 1 reveals that.

$$r^2 d\theta = 2dA_e, \tag{11}$$

In other words, it signifies twice the area covered by the radius vector within a specific timeframe. Given the constancy of L, it can be deduced that the satellite traverses equivalent areas over equal time intervals. Thus, we get Kepler's 2nd equation of planetary motion.

Solution of Kepler's 3rd Law equation

Equation (7) amount of semi-latus rectus p. So,

$$\frac{L^2}{\mu} = p = a\left(1 - e^2\right),$$

or,
$$L = \left[\mu a \left(1 - e^2\right) \right]^{\frac{1}{2}} \tag{12}$$

By combining equation (10) and (11), the following conclusion can be drawn

$$L = 2 \frac{dA_e}{dt}$$

Since L is constant, it follows that

$$A_e(t) = \frac{L}{2} t = \frac{\left[\mu a \left(1 - e^2\right) \right]^{\frac{1}{2}}}{2} t \tag{13}$$

When t = T, or one orbital period,

$$A_e = \pi ab = \pi a^2 \left(1-e^2\right)^{\frac{1}{2}} \tag{14}$$

Combining eqn. (13) and (14), it follows that

$$\frac{\left[\mu a\left(1-e^2\right)\right]^{\frac{1}{2}}}{2} T = \pi a^2 \left(1-e^2\right)^{\frac{1}{2}}$$

or,

$$T = 2\pi \frac{a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}} \tag{15}$$

Thus, we get Kepler's 3rd equation of planetary motion.

REFERENCES

[1] Bochner, S. (1966): The Role of Mathematics in the Rise of Science, Princeton University Press.

[2] Broucke, R. (1980): On Kepler's Equation and Strange Attractors, *J. Astronautical Sciences*.

[3] Danby, J. M. A. and Burkhardt, T. M. (1983): The Solution of Kepler's Equation, I, *Celestial Mechanics*.

[4] Danby, J.M.A. (1987): The solution of Kepler's equation, III, *Celestial Mech*.

[5] Evans, J. C. (1998): Newton's Laws and Planetary Motion. <http://physics.gmu.edu>.

[5] Serafin, R. A. (1986): Bounds on the Solution to Kepler's Equation, *Celestial Mechanics*.

[6] Simmons, G. (1972): Differential Equations, with Applications and Historical Notes. New York, McGraw-Hill.

Research Scholar,
Department of Mathematics,
V.K.S. University, Ara
E-mail:amitabhkumar8102@gmail.com

(*Received*, August 3, 2024)