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HARDY-ROGERS-TYPE MAPPINGS IN RELATIONAL METRIC SPACES AND FIXED POINT THEOREMS

Abstract: This work aims to present an analogous version of the fixed point theorem of Hardy and Rogers in relational metric spaces. We extend the notions of several mappings and some well-known classical fixed point theorems in the settings of relational metric spaces. Some examples are also provided which illustrate the results. This investigation offers generalization and unification of several known results.

Keywords: Metric Space; Relation-Theoretic Contraction Principle; Fixed Point.

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1. Introduction

Fixed point theory, of course, entails the search for a combination of conditions on a set X and a mapping $T : X \to X$ which in turn assures that T leaves at least one point of X fixed, i.e. $x = Tx$ for some $x \in X$. Banach [2] proved the famous Banach contraction principle or Banach's fixed point theorem in 1922. Banach's fixed point result has laid down the foundation of modern fixed point theory for Banach contractions in the metric and Banach spaces. A drawback of Banach contraction is that it is continuous (indeed, uniformly continuous) and cannot characterize the completeness of the space. These drawbacks were considered by Kannan [6, 7], and he introduced a new type of mapping known as Kannan type mapping. Such mappings are not necessarily continuous as shown by Kannan. Subrahmanyam [16] showed that the completeness of metric spaces can be

characterized by Kannan mappings. Reich [13, 14] introduced a new type of mappings called the Reich type mapping and generalized and unified the notions of Banach contractions and Kannan type mappings. Chatterjea [4] introduced another type of mapping which is independent of the mappings considered by Banach [2] and Kannan [6, 7]. Hardy and Rogers [5] established a generalization of the fixed point theorem of Reich. The mappings introduced by Hardy and Rogers [5] were a unification and generalization of the classes of mappings considered by Banach [2], Kannan [6, 7], Reich [13, 14], and Chatterjea [4].

In 2003, Ran and Reurings [12] introduced the fixed point theory in metric spaces endowed with a partial order and proved an analogue of Banach's fixed point theorem in partially ordered sets. They found uses of such results in finding the solutions of linear and nonlinear matrix equations. Very recently, Alam and Imdad [1] introduced a novel variant of Banach's fixed point theorem on a complete metric space endowed with a binary relation which under universal relation reduces into the Banach contraction principle. Further, they derived the results of Ran and Reurings [12], Nieto, R. Rodríguez-López [10, 11], and Turinici [17, 19] from the relation-theoretic contraction principle.

In this work, we have introduced the notion of relation theoretic Hardy-Rogers- type contractions and proved some fixed point results for such mappings. These results generalize, unify, and extend the results of Banach [2], Kannan [6, 7], Chatterjea [4], Reich [13, 14], Hardy and Rogers [5] and Alam and Imdad [1] in relational metric spaces.

2. Preliminaries

This section presents basic definitions, propositions, and relevant necessary relation theoretic analogues of standard metrical notions.

Definition 1 (Hardy-Rogers-type mapping): Let (X, d) be a metric space and $T : X \to X$ be a mapping. The mapping T is called a Hardy-Rogers-type mapping, if there exist nonnegative constants a_i satisfying $\sum_{i=1}^{5} a_i < 1$ such that:

$$d(Tx, Ty) \leq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx) \qquad (1)$$

for all $x, y \in X$. Without loss of generality, we may assume that $a_2 = a_3$ and $a_4 = a_5$ (see [5]). Therefore, the condition (1) is equivalent to the following condition:

$$d(Tx, Ty) \leq a_1 d(x, y) + a_2[d(x, Tx) + d(y, Ty)] + a_3[d(x, Ty) + d(y, Tx)] \quad (2)$$

for all $x, y \in X$, where $a_1 + 2a_2 + 2a_3 < 1$.

Definition 2 (see [9]): Let X be a nonempty set. A subset $\mathcal{R}$ of $X \times X$ is called a binary relation on X. Notice that for each pair $x, y \in X$ one of the following conditions holds:

1. $(x, y) \in \mathcal{R}$ which amounts $\mathcal{R}$ to saying that "x is $\mathcal{R}$-related to y" or "x relates to y under $\mathcal{R}$ ". Sometimes, we write $x\mathcal{R}y$ instead of $(x, y) \in \mathcal{R}$.

2. $(x, y) \notin \mathcal{R}$ which means that "x is not $\mathcal{R}$-related to y" or "x does not relate to y under $\mathcal{R}$ ".

Trivially, $X \times X$ and being subsets of $X \times X$ are binary relations on X, which are respectively called the universal relation (or full relation) and empty relation. Another important relation of this kind is the relation:

$$\Delta_X = \{(x,x) : x \in X\}$$

and called the identity relation or the diagonal relation on X. Throughout the discussion, $\mathcal{R}$ stands for a nonempty binary relation, but for the sake of simplicity, we write only "binary relation" instead of "nonempty binary relation".

Definition 3 (see [1]): Let $\mathcal{R}$ be a binary relation defined on a nonempty set X and $x, y \in X$. We say that x and y are $\mathcal{R}$-comparative if either $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$. We denote it by $[x, y] \in \mathcal{R}$.

Definition 4 (see [9]): Let X be a nonempty set and $\mathcal{R}$ a binary relation on X.

1. The inverse, transpose, or dual relation of $\mathcal{R}$, denoted by $\mathcal{R}^{-1}$, is defined by

$$\mathcal{R}^{-1} = \{(x, y) \in X \times X : (y, x) \in \mathcal{R}\}.$$

2. The reflexive closure of $\mathcal{R}$, denoted by $\mathcal{R}^{\#}$, is defined to be the set $\mathcal{R} \cup \Delta_X$ (i.e., $\mathcal{R}^{\#} := \mathcal{R} \cup \Delta_X$). Indeed $\mathcal{R}^{\#}$ is the smallest reflexive relation on X containing $\mathcal{R}$.

3. The symmetric closure of $\mathcal{R}$, denoted by $\mathcal{R}_s$, is defined to be the set $\mathcal{R} \cup \mathcal{R}^{-1}$ (i.e., $\mathcal{R}_s := \mathcal{R} \cup \mathcal{R}^{-1}$). Indeed $\mathcal{R}_s$ is the smallest symmetric relation on X containing $\mathcal{R}$.

Proposition 1 (see [1]): For a binary relation $\mathcal{R}$ defined on a nonempty set X

$$(x, y) \in \mathcal{R}_s \Leftrightarrow [x, y] \in \mathcal{R}.$$

Definition 5 (see [1]): Let X be a nonempty set and $\mathcal{R}$ a binary relation on X.

(a) A sequence $\{x_n\}$ in X is called $\mathcal{R}$-preserving if $(x_n, x_{n+1}) \in \mathcal{R}$ for all $n \in \mathbb{N}$.

(b) If $T : X \to X$ is a mapping, then $\mathcal{R}$ is called T-closed if for every $(x, y) \in \mathcal{R}$, we have $(Tx, Ty) \in \mathcal{R}$.

Proposition 2 (see [1]): Let X, T, and $\mathcal{R}$ be the same as in Definition 5. If $\mathcal{R}$ is T-closed, then $\mathcal{R}_s$ is also T-closed.

If (X, d) is a metric space such that $\mathcal{R}$ is a binary relation on X, then we say that the space X is a relational space with distance d and relation $\mathcal{R}$.

Definition 6 (see [1]): Let (X, d) be a metric space with a binary relation $\mathcal{R}$. Then (X, d) is called $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence in X, converges to a point in X.

Every complete metric space is $\mathcal{R}$-complete for any binary relation

$\mathcal{R}$. Particularly, under the universal relation the notion of $\mathcal{R}$-completeness coincides with the usual completeness.

Definition 7 (see [1]): Let (X, d) be a metric space with a binary relation $\mathcal{R}$ and $x \in X$. A self-mapping T on X is called $\mathcal{R}$-continuous at x if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ such that $x_n \to x$, we have $Tx_n \to Tx$. Moreover, T is called $\mathcal{R}$-continuous on X if it is $\mathcal{R}$-continuous at each point of X. The relation $\mathcal{R}$ is called d-self-closed if whenever $\{x_n\}$ is an $\mathcal{R}$-preserving sequence and $x_n \to x$ then there exists a subsequence $\{x_{n_k}\}$ such that $[x_{n_k}, x] \in \mathcal{R}$ for all $k \in \mathbb{N}$.

The sequence $\{x_n\}$ is called $\mathcal{R}$-convergent to x if $(x_n, x_{n+1}) \in \mathcal{R}$ for all $n \in \mathbb{N}$ and $x_n \to x$ implies that $(x_n, x) \in \mathcal{R}_s$.

Every continuous mapping is $\mathcal{R}$-continuous for any binary relation $\mathcal{R}$. Particularly, under the universal relation the notion of $\mathcal{R}$-continuity coincides with the usual continuity.

Definition 8 (see [8, 15]): Let X be a nonempty set and $\mathcal{R}$ a binary relation on X.

(a) A subset E of X is called $\mathcal{R}$-directed if for each $x, y \in E$, there exists $z \in X$ such that $(x, z) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$.

(b) For $x, y \in X$, a path of length r (where r is a natural number) in $\mathcal{R}$ from x to y is a finite sequence $\{z_i\}_{i=0}^{r} \subset X$ satisfying the following conditions:

(i) $z_0 = x$ and $z_r = y$;

(ii) $(z_i, z_{i+1}) \in \mathcal{R}$ for each $0 \le i \le r-1$.

(c) A subset E of X is called $\mathcal{R}$-connected if for each pair $x, y \in E$, there exists a path in $\mathcal{R}$ from x to y.

Following [1] we use the following notation:

(i) $F(T) =$ the set of all fixed points of T ;

(ii) $X(T, \mathcal{R}) := \{x \in X : (x, Tx) \in \mathcal{R}\}$.

In the next section, we state the findings of this work.

3. Main results

In this section, first, we define the generalized relation-theoretic Hardy-Rogers-type mappings and prove the existence and uniqueness of fixed points of such types of mappings in relational metric spaces.

Definition 9 (Relation theoretic Hardy-Rogers-type mapping): Let X denote a relational metric space with distance function d and relation $\mathcal{R}$, and $T : X \to X$ be a mapping. The mapping T is called a relation theoretic Hardy-Rogers-type mapping if there exist nonnegative constants a_i satisfying $a_1 + 2a_2 + 2a_3 < 1$ such that

$$d(Tx, Ty) \leq a_1 d(x, y) + a_2[d(x, Tx) + d(y, Ty)] + a_3[d(x, Ty) + d(y, Tx)] \quad (3)$$

for all $x, y \in X$ with $(x, y) \in \mathcal{R}$.

Relation theoretic Hardy-Rogers-type mappings are weaker than the Hardy-Rogers-type mappings in the sense that relation theoretic Hardy-Rogers-type mappings satisfies the contractive condition (2) for only those pairs $(x, y) \in X \times X$ which are members of $\mathcal{R}$. In contrast, Hardy-Rogers-type mapping must satisfy the contractive condition for all the pairs $(x, y) \in X \times X$.

Example 1: Let $X = \mathbb{R}$ be the metric space with ordinary distance and define a binary relation $\mathcal{R}$ on X by $\mathcal{R} = \{(x, y) : x, y \in \mathbb{Q} \cap [0, \infty)\}$, where $\mathbb{Q}$ is the set of all rational numbers. Define a mapping $T : X \to X$ by:

$$Tx = \begin{cases} \dfrac{x}{2} & \text{if} \quad x \in \mathbb{Q} \cap [0, \infty); \\ 1, & \text{otherwise.} \end{cases}$$

Then T is not a Hardy-Rogers-type mapping, for example at $x = 1$ and $y = 1 + \frac{\pi}{n}$ (where $n \in \mathbb{N}$) and for large enough n the contractive condition (2) fails. At the same time, it is easy to see that T is relation theoretic Hardy-Rogers-type mapping with $a_1 = \frac{1}{2}$ and arbitrary a_2, a_3 such that $0 < a_2 + a_3 < \frac{1}{4}$.

The following theorem is an existence result for a relation theoretic Hardy-Roger-type mapping.

Theorem 1: Let (X, d) be a metric space, $\mathcal{R}$ a binary relation on X such that (X, d) is $\mathcal{R}$-complete. Let $T : X \to X$ be a relation theoretic Hardy-Rogers-type mapping and the following conditions hold:

1. $X(T, \mathcal{R})$ is nonempty;

2. is T-closed;

3. T is $\mathcal{R}$-continuous on X; or $\mathcal{R}$ is d-self-closed.

Then T has a fixed point, i.e., there exists $z \in X$ such that $Tz = z$.

Proof: Because $X(T, \mathcal{R}) \neq \varnothing$, there exists $x_0 \in X(T, \mathcal{R})$, i.e., $(x_0, Tx_0) \in \mathcal{R}$. Define a sequence $\{x_n\}$ of Picard iterates, i.e.

$$x_n = T^n x_0 \text{ for all } n \in \mathbb{N}_0 \tag{4}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is the set of whole numbers. Since, $(x_0, Tx_0) \in \mathcal{R}$ using T-closedness of $\mathcal{R}$, we have:

$$(T^n x_0, T^{n+1} x_0) \in \mathcal{R}, \text{ i.e. } (x_n, x_{n+1}) \in \mathcal{R} \text{ for all } n \in \mathbb{N}_0. \tag{5}$$

Thus, the sequence $\{x_n\}$ is $\mathcal{R}$-preserving.

Therefore, using (3) with $x = x_{n-1}$, $y = x_n$, we have

$$d(x_n, x_{n+1}) = d(Tx_{n-1},\ Tx_n),$$

$$\leq a_1 d(x_{n-1}, x_n) + a_2[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)]$$

$$+ a_3[d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})]$$

$$= a_1 d(x_{n-1}, x_n) + a_2[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]$$

$$+ a_3[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)].$$

This shows that;

$$d(x_n, x_{n+1}) \leq r d(x_{n-1}, x_n) \tag{6}$$

where $r = \frac{a_1 + a_2 + a_3}{1 - a_2 - a_3}$. Since $a_1 + 2a_2 + 2a_3 < 1$, hence, $r < 1$. By induction process:

$$d(x_n, x_{n+1}) \leq r^n d(x_0, x_1) \text{ for all } n \in \mathbb{N}. \tag{7}$$

So, for any positive integers $m, n \in \mathbb{N}$ with $m > n$, we have

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m)$$
$$\leq (r^n + r^{n-1} + \cdots + r^{m-1}) d(x_0, x_1)$$
$$\leq \frac{r^n}{1 - r}(x_0, x_1).$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is $\mathcal{R}$-complete metric space, there exists $z \in X$ such that:

$$\lim_{n \to \infty} x_n = z. \tag{8}$$

First suppose that T is $\mathcal{R}$-continuous. Then by (4) and (8), we have $x_{n+1} = Tx_n \to Tz$. By uniqueness of the limit, we must have $Tz = z$, i.e. z is a fixed point of T.

Now suppose that $\mathcal{R}$ is d-self-closed, then there exists a subsequence $\{x_{n_k}\}$ such that $[x_{n_k}, x] \in \mathcal{R}$ for all $k \in \mathbb{N}$. If $[x_{n_k}, z] \in \mathcal{R}$ (the proof for the case $(z, x_{n_k}) \in \mathcal{R}$ will be the same), then as T is Hardy-Rogers-type mapping we have:

$$d(x_{n_k+1}, Tz) = d(Tx_{n_k}, Tz)$$

$$\leq a_1 d(x_{n_k}, z) + a_2[d(x_{n_k}, Tx_{n_k}) + d(z, Tz)] + a_3[d(x_{n_k}, Tz) + a_5 d(z, Tx_{n_k})]$$

$$= a_1 d(x_{n_k}, z) + a_2[d(x_{n_k}, x_{n_k+1}) + d(z, Tz)] + a_3[d(x_{n_k}, Tz) + a_5 d(z, x_{n_k+1})].$$

Letting $k \to \infty$ we get:

$$d(z, Tz) \leq a_1 \cdot 0 + a_2[0 + d(z, Tz)] + a_3[d(z, Tz) + 0]$$
$$= (a_2 + a_3) d(z, Tz).$$

Since, $a_2 + a_3 < 1$, we must have $d(z, Tz) = 0$, i.e. $Tz = z$. This proves the result.

Example 2: Let $X_1 = [0, 1/2)$ and $X_2 = [1/2, 1]$, $X = X_1 \cup X_2$ and d is the usual distance defined on X. Define a relation $\mathcal{R}$ on X such that:

$$\mathcal{R} = \{(x, y) : x, y \in X_1 \text{ with } y < x\} \cup \{(x, y) : x, y \in X_2\}.$$

Define a mapping $T : X \to X$ by:

$$Tx = \begin{cases} \dfrac{x}{2} & \text{if } x \in X_1 \\ \dfrac{1}{2} & \text{if } x \in X_2 \end{cases}$$

Then, we note that T is not a Hardy-Rogers-type mapping on X. Indeed, for $x = 0$ and $y = 1/2$ the condition (2) is not satisfied. On the other hand, it is easy to verify that T is a relation theoretic Hardy-Rogers-type mapping on X with

$a_1 = \frac{1}{2}, a_2 = a_3 = \frac{1}{9}$. Also, for any $x_0 \in X_1$ we have $Tx_0 = \frac{x_0}{2} \in X_1$, hence, $(x_0, Tx_0) \in \mathcal{R}$ for all $x_0 \in X_1$ so $X(T, \mathcal{R}) \neq \varnothing$. If $(x, y) \in \mathcal{R}$, then we must have $(x, y) \in X_1 \times X_1$, $y < x$ or $(x, y) \in X_2 \times X_2$. If $(x, y) \in X_1 \times X_1, y < x$, then by definition of T we have $Tx, Ty \in X_1, Ty < Tx$, i.e. $(Tx, Ty) \in \mathcal{R}$. If $(x, y) \in X_2 \times X_2$, then we have $Tx = Ty = \frac{1}{2}$, so $(Tx, Ty) = (\frac{1}{2}, \frac{1}{2}) \in \mathcal{R}$. Therefore, $\mathcal{R}$ is T-closed. Finally, it is easy to see that T is $\mathcal{R}$-continuous on X. Hence, by Theorem 1, T must have a fixed point. Indeed, T has two fixed points, namely 0 and 1/2.

In the next theorem, we provide a sufficient condition for the uniqueness of the fixed point of a relation theoretic Hardy-Rogers-type mapping.

Theorem 2: Suppose that all the conditions of Theorem 1 are satisfied and that $\Delta_X \subseteq \mathcal{R}$ and $X(T, \mathcal{R})$ is $\mathcal{R}_s$-connected. Then T has a unique fixed point.

Proof: By Theorem 1 we have $F(T) \neq \varnothing$. Let x and y be two fixed points of T, i.e. $x, y \in F(T)$, then for all $n \in \mathbb{N}_0$, we have:

$$T^n x = x, T^n y = y . \tag{9}$$

Since, $\Delta_X \subseteq \mathcal{R}$, hence, $x, y \in X(T, \mathcal{R})$. By $\mathcal{R}_s$-connectedness of $X(T, \mathcal{R})$, there exists a path (say $z_0, z_1, \ldots z_k$) of some finite length k in $\mathcal{R}_s$ from x to y so that:

$$z_0 = x, z_k = y \text{ and } [z_i, z_{i+1}] \in \mathcal{R} \text{ for each } i \in \{0, 1, \ldots, k-1\} . \tag{10}$$

By T-closedness of $\mathcal{R}$ we have

$$[T^n z_i, T^n z_{i+1}] \in \mathcal{R} \text{ for each } i \in \{0, 1, \ldots, k-1\} \text{ and for each } n \in \mathbb{N} \cup \{0\} . \tag{11}$$

Hence, using (3), for each $i \in \{0, 1, \ldots, k\}$ and for all $n \in \mathbb{N}$ we get

$$d(T^n z_i, T^{n+1} z_i) \leq a_1 d(T^{n-1} z_i, T^n z_i) + a_2 [d(T^{n-1} z_i, T^n z_i) + d(T^n z_i, T^{n+1} z_i)]$$

$$+ a_3[d(T^{n-1}z_i, T^{n+1}z_i) + d(T^n z_i, T^n z_i)]$$

$$\leq (a_1 + a_2 + a_3)d(T^{n-1}z_i, T^n z_i) + (a_2 + a_3)d(T^n z_i, T^{n+1}z_i)\,.$$

Hence,

$$d(T^n z_i, T^{n+1}z_i) \leq r\,d(T^{n-1}z_i, T^n z_i)$$

Where $r = \frac{a_1+a_2+a_3}{1-a_2-a_3} < 1$ (as $a_1 + a_2 + a_3 < 1$). Repetition of the above process gives:

$$d(T^n z_i, T^{n+1}z_i) \leq r^n d(z_i, Tz_i) \tag{12}$$

for all $n \in \mathbb{N}$ and for each $i \in \{0,1,\ldots,k\}$. Again

$$d(T^n z_i, T^n z_{i+1}) \leq a_1 d(T^{n-1}z_i, T^{n-1}z_{i+1}) + a_2[d(T^{n-1}z_i, T^n z_i) + d(T^{n-1}z_{i+1}, T^n z_{i+1})]$$

$$+ a_3[d(T^{n-1}z_i, T^n z_{i+1}) + d(T^{n-1}z_{i+1}, T^n z_i)]$$

$$\leq a_1 d(T^{n-1}z_i, T^n z_i) + a_1 d(T^n z_i, T^n z_{i+1}) + a_1 d(T^n z_{i+1}, T^{n-1}z_{i+1})$$

$$+ a_2[d(T^{n-1}z_i, T^n z_i) + a(T^{n-1}z_{i+1}, T^n z_{i+1})] + a_3[d(T^{n-1}z_i, T^n z_i)$$

$$+ d(T^n z_i, T^n z_{i+1}) + d(T^{n-1}z_{i+1}, T^n z_{i+1}) + d(T^n z_{i+1}, T^n z_i)]$$

so that

$$(1 - a_1 - 2a_3)d(T^n z_i, T^n z_{i+1})$$

$$\leq (a_1 + a_2 + a_3)d(T^{n-1}z_i, T^n z_i) + (a_1 + a_2 + a_3)d(T^n z_{i+1}, T^{n-1}z_{i+1})\,.$$

Since, $1 - a_1 - 2a_2 > 0$, hence using (12) in the above inequality we obtain:

$$\lim_{n\to\infty} d(T^n z_i, T^n z_{i+1}) = 0 \text{ for each } i \in \{0,1,\ldots,k-1\}\,.$$

Finally, making use of triangular inequality and the above equation we obtain

$$d(x,y) = d(T^n z_0, T^n z_k) \leq \sum_{i=0}^{k-1} d(T^n z_i, T^n z_{i+1}) \to 0 \text{ as } n \to \infty .$$

Hence, T has a unique fixed point.

In the next theorem, we replace the $\mathcal{R}$ -completeness of the relational space by imposing another condition on the mapping T.

Theorem 3: Let (X,d) be a metric space, $\mathcal{R}$ a binary relation on X and $T : X \to X$ be a relation theoretic Hardy-Rogers-type mapping and the following conditions hold:

1. $X(T,\mathcal{R})$ is nonempty;

2. $\mathcal{R}$ is T-closed;

3. there exists $u \in X(T,\mathcal{R})$ such that the following condition holds:

$$d(u,Tu) \leq d(x,Tx) \text{ for all } x \in X(T,\mathcal{R}). \tag{13}$$

Then T has a fixed point. If $\Delta_X \subseteq \mathcal{R}$ and $X(T,\mathcal{R})$ is $\mathcal{R}_s$ -connected, then T has a unique fixed point.

Proof: Let us denote:

$$D(x) = d(x,Tx), x \in X .$$

Since, $X(T,\mathcal{R}) \neq \varnothing$, by (13) we have:

$$D(u) \leq D(x) \text{ for all } x \in X(T,\mathcal{R}). \tag{14}$$

We claim that $Tu = u$. On the contrary, suppose that $Tu \neq u$. Since, $u \in X(T,\mathcal{R})$, by T-closedness of $\mathcal{R}$ we have $Tu \in X(T,\mathcal{R})$. As T is a relational theoretic Hardy-Rogers- type mapping we have:

$$D(Tu) = d(Tu, TTu)$$

$$\leq a_1 d(u, Tu) + a_2[d(u, Tu) + d(Tu, TTu)] + a_3[d(u, TTu) + d(Tu, Tu)]$$

$$= a_1 D(u) + a_2[D(u) + D(Tu)] + a_3[d(u, TTu) + 0]$$

$$= a_1 D(u) + a_2[D(u) + D(Tu)] + a_3[d(u, Tu) + d(Tu, TTu)]$$

$$= (a_1 + a_2 + a_3)D(u) + (a_2 + a_3)D(Tu) .$$

This shows that $(1 - a_2 - a_3)D(Tu) \leq (a_1 + a_2 + a_3)D(u)$, i.e.

$$D(Tu) \leq \frac{a_1 + a_2 + a_3}{1 - a_2 - a_3} D(u) = rD(u)$$

where $r = \frac{a_1 + a_2 + a_3}{1 - a_2 - a_3} < 1$. Hence, we must have $D(Tu) < D(u)$ and $Tu \in X(T, \mathcal{R})$.

This contradicts inequality (14). Hence, we must $Tu = u$, i.e. u is a fixed point of T.

The uniqueness of the fixed point can be proved by following the same process used in the proof of Theorem 2.

4. Consequences

Let X denotes a relational metric space with distance function d and relation $\mathcal{R}$, and $T : X \to X$ be a mapping. The mapping T is called a relation theoretic contraction, if there exists nonnegative constant a satisfying $a < 1$ such that

$$d(Tx, Ty) \leq ad(x, y)$$

for all $x, y \in X$ with $(x, y) \in \mathcal{R}$.

The mapping T is called a relation theoretic Kannan-type mapping, if

there exists nonnegative constant a satisfying $a < \frac{1}{2}$ such that

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$ with $(x, y) \in \mathcal{R}$.

The mapping T is called a relation theoretic Reich-type mapping, if there exist nonnegative constants a_1, a_2 satisfying $a_1 + 2a_2 < 1$ such that

$$d(Tx, Ty) \leq a_1 d(x, y) + a_2[d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$ with $(x, y) \in \mathcal{R}$.

The mapping T is called a relation theoretic Chaterjea-type mapping, if there exists nonnegative constant a satisfying $a < \frac{1}{2}$ such that:

$$d(Tx, Ty) \leq a[d(x, Ty) + d(y, Tx)]$$

for all $x, y \in X$ with $(x, y) \in \mathcal{R}$.

Assuming the following values of constants a_1, a_2 and a_3 in Theorem 2:

(A) $a_1 = a, a_2 = a_3 = 0$;

(B) $a_1 = a_3 = 0, a_2 = a$;

(C) $a_3 = 0$;

(D) $a_1 = a_2 = 0, a_3 = a$;

we obtain the following fixed point results respectively:

Corollary 4: Let (X, d) be a metric space, $\mathcal{R}$ a binary relation on X such

that (X, d) is $\mathcal{R}$ -complete. Let $T : X \to X$ be at least one of the following:

(A) a relation theoretic contraction;

(B) a relation theoretic Kannan-type mapping;

(C) a relation theoretic Reich-type mapping;

(D) a relation theoretic Chterjea-type mapping;

Suppose, the following conditions hold:

1. $X(T, \mathcal{R})$ is nonempty;

2. $\mathcal{R}$ is T-closed;

3. T is $\mathcal{R}$ -continuous on X; or $\mathcal{R}$ is d-self-closed.

Then T has a fixed point, i.e., there exists $x \in X$ such that $Tz = z$. In addition, suppose that $\Delta_X \subseteq \mathcal{R}$ and $X(T, \mathcal{R})$ is $\mathcal{R}_s$ -connected, then T has a unique fixed point.

We have used a general binary relation $\mathcal{R}$ and proved the existence of relation theoretic fixed point of Hardy-Rogers-type mappings. Hence, by assuming $\mathcal{R}$ with some properties, we can obtain several fixed point results and their generalizations.

1. With $\mathcal{R} = X \times X$ (universal relation) in Theorem 2, we get the fixed point result for Hardy-Rogers-type mapping [5] in a metric space.

2. Assuming $\mathcal{R}$ to be a partial order relation, we can get a generalized and improved version of the results of Nieto and Rodriguez-Lopez [10].

3. If we assume $\mathcal{R}$ as a preorder (i.e., reflexive and transitive) in Theorem 2, we get a generalized and improved version of Theorem 1 of Turinici [18].

4. If $\mathcal{R}$ is assumed to be transitive in Theorem 2, we obtain a generalized and improved version of the results of Ben-El-Mechaiekh [3].

5. With $\mathcal{R}$ as a tolerance relation (see, [1]) in Theorem 2, we obtain a generalized and improved version of the theorems of Turinici [17, 18] and Ran and Reurings [12].

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