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A FAMILY OF GIBONACCI SUMS: GENERALIZATIONS AND CONSEQUENCES

Abstract: We explore a generalization of an infinite sum involving a large family of gibonacci polynomial squares and its Pell and Jacobsthal consequences.

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1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*.

They can also be defined by the *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 3].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively [3]. Pell numbers P_n and Pell-Lucas numbers Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [3].

Suppose $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal Lucas polynomial*. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$ and $j_n(1) = L_n$ [2, 3, 6].

Gibonacci and Jacobsthal polynomials are linked by the relationships $J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$ and $j_n(x) = x^{n/2} l_n(1/\sqrt{x})$ [3, 6].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, and $b_n = p_n$ or q_n; $\Delta = \sqrt{x^2 + 4}$ and $2\alpha = x + \Delta$. Gibonacci and Pell polynomials are linked by the relationship $b_n(x) = g_n(2x)$.

1.1 Fundamental Gibonacci Identities: Gibonacci polynomials satisfy the following properties [3, 5, 7]:

$$g_{n+k}g_{n-k} - g_n^2 = \begin{cases} (-1)^{n+k+1} f_k^2, & \text{if } g_n = f_n \\ (-1)^{n+k} \Delta^2 f_k^2, & \text{otherwise;} \end{cases} \tag{1}$$

$$g_{n+k+r}g_{n-k} - g_{n+k}g_{n-k+r} = \begin{cases} (-1)^{n+k+1} f_r f_{2k}, & \text{if } g_n = f_n \\ (-1)^{n+k} \Delta^2 f_r f_{2k}, & \text{otherwise;} \end{cases} \tag{2}$$

$$g_{n+k+r}g_{n-k} + g_{n+k}g_{n-k+r} = \begin{cases} \frac{1}{\Delta^2}[2l_{2n+r} - (-1)^{n+k} l_{2k} l_r], & \text{if } g_n = f_n \\ 2l_{2n+r} + (-1)^{n+k} l_{2k} l_r, & \text{otherwise,} \end{cases} \tag{3}$$

where k and r are positive integers. These properties can be confirmed using the Binet-like formulas.

Identity (3) is a polynomial extension of *d'Ocagne identity* [3, 5].

Consequently, we have

$$g_{n+k+r}^2\ g_{n-k}^2 - g_{n+k}^2 g_{n-k+r}^2 = \begin{cases} \frac{(-1)^{n+k+1}}{\Delta^2}[2l_{2n+r} - (-1)^{n+k} l_{2k} l_r] f_{2k} f_r, & \text{if } g_n = f_n \\ (-1)^{n+k}\Delta^2[2l_{2n+r} + (-1)^{n+k} l_{2k} l_r] f_{2k} f_r, & \text{otherwise.} \end{cases} \tag{4}$$

Again, in the interest of brevity and clarity, we now let

$$A = 2l_{2(2pn+t-p)k+r} - (-1)^{tk} l_{2pk} l_r; \text{ and } B = 2l_{2(2pn+t-p)k+r} + (-1)^{tk} l_{2pk} l_r.$$

It then follows by identities (1) and (4) that [7]

$$g_{(2pn+t)k} g_{(2pn+t-2p)k} - g_{(2pn+t-p)k}^2 = \begin{cases} (-1)^{tk+1} f_{pk}^2, & \text{if } g_n = f_n \\ (-1)^{tk}\Delta^2 f_{pk}^2, & \text{otherwise;} \end{cases} \tag{5}$$

$$g_{(2pn+t)k+r}^2 g_{(2pn+t-2p)k}^2 - g_{(2pn+t)k}^2 g_{(2pn+t-2p)k+r}^2 = \begin{cases} \frac{(-1)^{tk+1}}{\Delta^2} A f_{2pk} f_r, & \text{if } g_n = f_n \\ (-1)^{tk}\Delta^2 B f_{2pk} f_r, & \text{otherwise,} \end{cases} \tag{6}$$

where k, p, r, and t are positive integers, and $t \le 2p$.

2. A Telescoping Gibonacci Sum

With recursion, we established the following telescoping gibonacci sum in [7]. For the sake of conciseness and expediency, we omit its proof here.

Lemma 1: *Let k, p, r, t, and λ be positive integers, where $t \le 2p$. Then*

$$\sum_{n=1}^{\infty} \left[\frac{g_{(2pn+t-2p)k+r}^{\lambda}}{g_{(2pn+t-2p)k}^{\lambda}} - \frac{g_{(2pn+t)k+r}^{\lambda}}{g_{(2pn+t)k}^{\lambda}} \right] = \frac{g_{tk+r}^{\lambda}}{g_{tk}^{\lambda}} - \alpha^{\lambda r}. \tag{7}$$

3. Gibonacci Sums

Coupled with identities (5) and (6), Lemma 1 with $\lambda = 2$ plays a major role in our explorations. To this end, in the interest of brevity, we first let

$$\mu = \begin{cases} 1, & \text{if } g_n = f_n \\ \Delta^2, & \text{otherwise;} \end{cases} \qquad \mu^* = \begin{cases} \frac{1}{\Delta^2} & \text{if } g_n = f_n \\ \Delta^2, & \text{otherwise;} \end{cases}$$

$$\nu = \begin{cases} -1, & \text{if } g_n = f_n \\ 1, & \text{otherwise;} \end{cases} \quad \text{and} \quad \nu = \begin{cases} 1, & \text{if } g_n = f_n \\ -1, & \text{otherwise.} \end{cases}$$

With these tools at our finger tips, we now showcase the main result in the following theorem.

Theorem 1: *Let k, p, r, and t be positive integers, where $t \le 2p$. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk}\mu^*\nu^*[2l_{2(2pn+t-p)k+r} + (-1)^{tk}\nu l_{2pk}l_r]f_{2pk}f_r}{[g^2_{(2pn+t-p)k} + (-1)^{tk}\mu\nu f^2_{pk}]} = \frac{g^2_{tk+r}}{g^2_{tk}} - \alpha^{2r}. \quad (8)$$

Proof: Suppose $g_n = f_n$. With identities (5) and (6), Lemma 1 then yields

$$\frac{(-1)^{tk+1}Af_{2pk}f_r}{\Delta^2[f^2_{(2pn+t-p)k} - (-1)^{tk}f^2_{pk}]^2} = \frac{f^2_{(2pn+t)k+r}f^2_{(2pn+t-2p)k} - f^2_{(2pn+t)k}f^2_{(2pn+t-2p)k+r}}{f^2_{(2pn+t)k}f^2_{(2pn+t-2p)k+r}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk}Af_{2pk}f_r}{\Delta^2[f^2_{(2pn+t-p)k} - (-1)^{tk}f^2_{pk}]^2} = \sum_{n=1}^{\infty}\left[\frac{f^2_{(2pn+t-2p)k+r}}{f^2_{(2pn+t-2p)k}} - \frac{f^2_{(2pn+t)k+r}}{f^2_{(2pn+t)k}}\right]$$

$$= \frac{f^2_{tk+r}}{f^2_{tk}} - \alpha^{2r}$$

On the flip side, suppose $g_n = l_n$. It then follows by the same two identities and Lemma 1 that

$$\frac{(-1)^{tk+1}\Delta^2 B f_{2pk} f_r}{[l^2_{(2pn+t-p)k} + (-1)^{tk}\Delta^2 f^2_{pk}]^2} = \frac{l^2_{(2pn+t)k} l^2_{(2pn+t-2p)k+r} - l^2_{(2pn+t)k+r} l^2_{(2pn+t-2p)k}}{l^2_{(2pn+t)k} l^2_{(2pn+t-2p)k}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk+1}\Delta^2 B f_{2pk} f_r}{[l^2_{(2pn+t-p)k} + (-1)^{tk}\Delta^2 f^2_{pk}]^2} = \sum_{n=1}^{\infty} \left[\frac{l^2_{(2pn+t-2p)k+r}}{l^2_{(2pn+t-2p)k}} - \frac{l^2_{(2pn+t)k+r}}{l^2_{(2pn+t)k}} \right]$$

$$= \frac{l^2_{tk+r}}{l^2_{tk}} - \alpha^{2r}.$$

Combining the two cases yields the given result, as desired. □

In particular, the theorem yields

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk}[2L_{2(2pn+t-p)k+r} - (-1)^{tk} L_{2pk} L_r] F_{2pk} F_r}{5[F^2_{(2pn+t-p)k} - (-1)^{tk} F^2_{pk}]^2} = \frac{F^2_{tk+r}}{F^2_{tk}} - \alpha^{2r}; \tag{9}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk+1} 5[2L_{2(2pn+t-p)k+r} + (-1)^{tk} L_{2pk} L_r] F_{2pk} F_r}{[F^2_{(2pn+t-p)k} + (-1)^{tk} F^2_{pk}]^2} = \frac{L^2_{tk+r}}{L^2_{tk}} - \alpha^{2r}. \tag{10}$$

With $p \le 2$, $k \le 3$, and $r = 1 = t$, it then follows that

$$\sum_{n=1}^{\infty} \frac{2L_{4n+1} + 3}{(F^2_{2n} + 1)^2} = \frac{5}{2} + \frac{5\sqrt{5}}{2}; \qquad \sum_{n=1}^{\infty} \frac{2L_{4n+1} - 3}{(L^2_{2n} - 5)^2} = \frac{3}{2} - \frac{\sqrt{5}}{10};$$

$$\sum_{n=1}^{\infty} \frac{2L_{8n+1} - 7}{(F^2_{4n} - 1)^2} = \frac{25}{6} - \frac{5\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{2L_{8n+1} + 7}{(L^2_{4n} + 5)^2} = -\frac{1}{54} + \frac{\sqrt{5}}{30};$$

$$\sum_{n=1}^{\infty} \frac{2L_{12n+1} + 9}{(F^2_{6n} + 4)^2} = -\frac{15}{64} + \frac{5\sqrt{5}}{32}; \qquad \sum_{n=1}^{\infty} \frac{2L_{12n+1} - 9}{(L^2_{6n} - 20)^2} = \frac{5}{256} - \frac{\sqrt{5}}{160};$$

$$\sum_{n=1}^{\infty} \frac{2L_{8n-1}+7}{(F_{4n-1}^2+1)^2} = \frac{5}{6} + \frac{5\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{2L_{8n-1}-7}{(L_{4n-1}^2-5)^2} = \frac{1}{2} - \frac{\sqrt{5}}{30};$$

$$\sum_{n=1}^{\infty} \frac{2L_{16n-3}-47}{(F_{8n-2}^2-9)^2} = \frac{25}{42} - \frac{5\sqrt{5}}{42}; \qquad \sum_{n=1}^{\infty} \frac{2L_{16n-3}+47}{(L_{8n-2}^2+45)^2} = -\frac{1}{378} + \frac{\sqrt{5}}{210};$$

$$\sum_{n=1}^{\infty} \frac{2L_{24n-5}+161}{(F_{12n-3}^2+64)^2} = -\frac{5}{384} + \frac{5\sqrt{5}}{576}; \quad \sum_{n=1}^{\infty} \frac{2L_{24n-5}-161}{(L_{12n-3}^2-320)^2} = \frac{5}{4,608} - \frac{\sqrt{5}}{2,880}.$$

We now encourage gibonacci enthusiasts to explore the corresponding sums with $p, k, r, t \le 3$.

Next, we explore the Pell version of Theorem 1, and its implications.

4. Pell Version and Consequences

With the Pell-gibonacci relationship $b_n(x) = g_n(2x)$ and letting $E = \sqrt{x^2+1}$;

$$A^* = 2q_{2(2pn+t-p)k+r} - (-1)^{tk} q_{2pk} q_r; \text{ and } B^* = 2q_{2(2pn+t-p)k+r} + (-1)^{tk} q_{2pk} q_r.$$

Theorem 1 yields the following the Pell versions:

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk} A^* p_{2pk} p_r}{4E^2[p_{(2pn+t-p)k}^2 - (-1)^{tk} p_{pk}^2]^2} = \frac{p_{tk+r}^2}{p_{tk}^2} - \gamma^{2r};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk+1} 4E^2 B^* p_{2pk} p_r}{[q_{(2pn+t-p)k}^2 + (-1)^{tk} 4E^2 p_{pk}^2]^2} = \frac{q_{tk+r}^2}{q_{tk}^2} - \gamma^{2r},$$

where $\gamma = x + E$.

Consequently, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk}[Q_{2(2pn+t-p)k+r} - (-1)^{tk} Q_{2pk}Q_r]P_{2pk}P_r}{2[P^2_{(2pn+t-p)k} - (-1)^{tk} P^2_{pk}]^2} = \frac{P^2_{tk+r}}{P^2_{tk}} - \gamma^{2r};$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk+1}2[Q_{2(2pn+t-p)k+r} + (-1)^{tk} Q_{2pk}Q_r]P_{2pk}P_r}{[Q^2_{(2pn+t-p)k} + (-1)^{tk} 2P^2_{pk}]^2} = \frac{Q^2_{tk+r}}{Q^2_{tk}} - \gamma^{2r}.$$

With $p \le 2$ and $k = r = t = 1$, it then follows that

$$\sum_{n=1}^{\infty} \frac{Q_{4n+1} + 3}{(P^2_{2n} + 1)^2} = -1 + 2\sqrt{2}; \qquad \sum_{n=1}^{\infty} \frac{Q_{4n+1} - 3}{(Q^2_{2n} - 2)^2} = 3 - \sqrt{2};$$

$$\sum_{n=1}^{\infty} \frac{Q_{8n-1} + 7}{(P^2_{4n-1} + 4)^2} = -\frac{1}{6} + \frac{\sqrt{2}}{3}; \qquad \sum_{n=1}^{\infty} \frac{Q_{8n-1} - 17}{(Q^2_{4n-1} - 8)^2} = \frac{1}{4} - \frac{\sqrt{2}}{12}.$$

We now pursue the Jacobsthal counterpart of Theorem 1 and its consequences.

5. Jacobsthal Version and Implications

Recall from Section 1 that Jacobsthal and Gibonacci polynomials are linked by the relationships

$$f_n(1/\sqrt{x}) = \frac{J_n(x)}{x^{(n-1)/2}} \quad \text{and} \quad l_n(1/\sqrt{x}) = \frac{j_n(x)}{x^{(n/2)}} \text{ [3, 4].}$$

Again, in the interest of conciseness, clarity, and convenience, we let $c_n = J_n$ or j_n, $D = \sqrt{4x+1}$, and $2\omega = 1 + D$. Then $\alpha(1/\sqrt{x}) = \dfrac{1+D}{\sqrt{x}} = \dfrac{\omega}{\sqrt{x}}$ [6].

In addition, we let L denote the fractional expression on the left side of the given gibonacci equation and R that on its right side; and LHS and RHS the left-hand side and right-hand side of the corresponding Jacobsthal equation, as in [6, 4].

With this brief background, we now begin our exploration.

Proof: Case 1. Suppose $g_n = f_n$.

We have $L = \dfrac{(-1)^{tk}[2l_{2(2pn+t-p)k+r} - (-1)^{tk} l_{2pk} l_r] f_{2pk} f_r}{\Delta^2[f^2_{(2pn+t-p)k} - (-1)^{tk} f^2_{pk}]^2}$.

Replacing x with $1/\sqrt{x}$, and A as in Subsection 1.1, we get

$$f_{2pk} f_r = \frac{J_{2pk} J_r}{x^{(2pk+r)/2-1}};$$

$$l_{2pk} l_r = \frac{j_{2pk} j_r}{x^{(2pk+r)/2}};$$

$$2l_{2(2pn+t-p)k+r} = \frac{2j_{2(2pn+t-p)k+r}}{x^{(2pn+t-p)k+r/2}};$$

$$Af_{2pk} f_r = [2j_{2(2pn+t-p)k+r} - (-1)^{tk} x^{(2pn+t-2p)k} j_{2pk} j_r] J_{2pk} J_r;$$

$$\Delta^2[f^2_{(2pn+t-p)k} - (-1)^{tk} f^2_{pk}]^2 = \frac{D^2[J^2_{(2pn+t-p)k}] - (-1)^{tk} x^{(2pn+t-2p)k} J^2_{pk}}{x^{2(2pn+t-p)k-1}},$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Consequently, we have

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{(-1)^{tk} x^{(2pn+t-2p)k} [2j_{2(2pn+t-p)k+r} - (-1)^{tk} x^{(2pn+t-2p)k} j_{2pk} j_r] J_{2pk} J_r}{D^2 x^r [J^2_{(2pn+t-p)k} - (-1)^{tk} x^{(2pn+t-2p)k} J^2_{pk}]^2},$$

where $c_n = c_n(x)$.

Now, turn to the right side; we have $R = \dfrac{f^2_{tk+r}}{f^2_{tk}} - \alpha^{2r}$. Replacing x with $1/\sqrt{x}$, and then multiplying the numerator and denominator with x^{tk+r-1}, this yields

$$R = \frac{[x^{(tk+r-1)/2} f_{tk+r}]^2}{x^r [x^{(tk-1)/2} f_{tk}]^2} - \frac{\omega^2 r}{x^r};$$

$$\text{RHS} = \frac{J_{tk+r}^2}{x^r J_{tk}^2} - \frac{\omega^{2r}}{x^r},$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Combining the two sides, we get the desired Jacobsthal version of equation (8):

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk} x^{(2pn+t-2p)k} [2j_{2(2pn+t-p)k+r} - (-1)^{tk} x^{(2pn+t-2p)k} j_{2pk} j_r] J_{2pk} J_r}{D^2 [J_{(2pn+t-p)k}^2 - (-1)^{tk} x^{(2pn+t-2p)k} J_{pk}^2]^2} = \frac{J_{tk+r}^2}{J_{tk}^2} - \omega^{2r}, \tag{11}$$

where $c_n = c_n(x)$.

Next, we pursue the Jacobsthal-Lucas version of theorem 1.

Case 2: With $g_n = l_n$, we have

$$L = \frac{(-1)^{tk+1} \Delta^2 [2l_{2(2pn+t-p)k+r} + (-1)^{tk} l_{2pk} l_r] f_{2pk} f_r}{[l_{(2pn+t-p)k}^2 + (-1)^{tk} \Delta^2 f_{pk}^2]^2}.$$

Again, replace x with $1/\sqrt{x}$; as above, we then get

$$Bf_{2pk} f_r = \frac{D^2 [2j_{2(2pn+t-p)k+r} + (-1)^{tk} x^{(2pn+t-2p)k} j_{2pk} j_r] J_{2pk} J_r}{x^{(2pn+t)k+r}};$$

$$[l_{(2pn+t-p)k}^2 + (-1)^{tk} \Delta^2 f_{pk}^2]^2 = \frac{[j_{(2pn+t-p)k}^2 + (-1)^{tk} D^2 x^{(2pn+t-2p)k} J_{pk}^2]^2}{x^{2(2pn+t-p)k}},$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$. Thus,

$$\text{LHS} = \frac{(-1)^{tk+1} D^2 x^{(2pn+t-2p)k} [2j_{2(2pn+t-p)k+r} + (-1)^{tk} x^{(2pn+t-2p)k} j_{2pk} j_r] J_{2pk} J_r}{x^r [j^2_{(2\ pn+t-p)k} + (-1)^{tk} D^2 x^{(2pn+t-2p)k} J^2_{pk}]^2},$$

where $c_n = c_n(x)$.

Turning to the right side, we have $R = \dfrac{l^2_{tk+r}}{l^2_{tk}} - \alpha^{2r}$. Replacing x with $1/\sqrt{x}$, and then multiplying the numerator and denominator with x^{tk+r} yields

$$R = \frac{[x^{(tk+r)/2} l_{tk+r}]^2}{x^r [x^{tk/2} l_{tk}]^2} - \frac{\omega^{2r}}{x^r};$$

$$\text{RHS} = \frac{j^2_{tk+r}}{x^r j^2_{tk}} - \frac{\omega^{2r}}{x^r},$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Merging the two sides gives the Jacobsthal-Lucas version of equation (8):

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk+1} D^2 x^{(2pn+t-2p)k} [2j_{2(2pn+t-p)k+r} + (-1)^{tk} x^{(2pn+t-2p)k} j_{2pk} j_r] J_{2pk} J_r}{[j^2_{(2\ pn+t-p)k} + (-1)^{tk} D^2 x^{(2pn+t-2p)k} J^2_{pk}]^2} = \frac{j^2_{tk+r}}{j^2_{tk}} - \omega^{2r}, \tag{12}$$

where $c_n = c_n(x)$.

This equation, coupled with formula (11), yields the desired Jacobsthal version of Theorem 1, as in the next theorem. To this end, we modify the definitions of μ, μ^*, ν and ν^* to fit the context:

$$\mu = \begin{cases} 1, & \text{if } c_n = J_n \\ D^2, & \text{otherwise;} \end{cases} \qquad \mu^* = \begin{cases} \frac{1}{D^2} & \text{if } c_n = J_n \\ D^2, & \text{otherwise;} \end{cases}$$

$$\nu = \begin{cases} -1, & \text{if } c_n = J_n \\ 1, & \text{otherwise;} \end{cases} \quad \text{and} \quad \nu = \begin{cases} 1, & \text{if } c_n = J_n \\ -1, & \text{otherwise.} \end{cases}$$

We are now ready to showcase the desired Jacobsthal version.

Theorem 2: *Let k, p, r, and t be positive integers, where $t \le 2p$. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk+1}\mu^*\nu^* x^{(2pm+t-2p)k}[2j_{2(2pm+t-p)k+r} + (-1)^{tk}\nu x^{(2pm+t-2p)k} j_{2pk} j_r] J_{2pk} J_r}{[c^2_{(2\ pm+t-p)k} + (-1)^{tk}\mu\nu x^{(2pm+t-2p)k} J^2_{pk}]^2} = \frac{j^2_{tk+r}}{j^2_{tk}} - \omega^{2r}, \tag{13}$$

□

In particular, equations (9) and (10) follow from this. Additionally, with $p, k \le 2$ and $r = 1 = t$, we get:

$$\sum_{n=1}^{\infty} \frac{2^n(2j_{4n+1} + 5\cdot 2^{n-1})}{(J^2_{2n} + 2^{n-1})^2} = 27; \qquad \sum_{n=1}^{\infty} \frac{2^n(2j_{4n+1} - 5\cdot 2^{n-1})}{(j^2_{2n} - 9\cdot 2^{n-1})^2} = \frac{7}{3};$$

$$\sum_{n=1}^{\infty} \frac{2^{4n-2}(2j_{8n+1} - 17\cdot 24^{n-2})}{(J^2_{4n} - 2^{4n-2})^2} = 9; \quad \sum_{n=1}^{\infty} \frac{2^{4n-2}(2j_{8n+1} + 17\cdot 2^{4n-2})}{(j^2_{4n} + 9\cdot 2^{4n-2})^2} = \frac{51}{1,125};$$

$$\sum_{n=1}^{\infty} \frac{2^{4n-3}(2j_{8n-1} + 17\cdot 2^{4n-3})}{(J^2_{4n-1} + 2^{4n-3})^2} = \frac{27}{5}; \quad \sum_{n=1}^{\infty} \frac{2^{4n-3}(2j_{8n-1} - 17\cdot 2^{4n-3})}{(j^2_{4n-1} - 9\cdot 2^{4n-3})^2} = \frac{7}{15};$$

$$\sum_{n=1}^{\infty} \frac{2^{8n-6}(2j_{16n-3} - 257\cdot 2^{8n-6})}{(J^2_{8n-2} - 25\cdot 2^{8n-6})^2} = \frac{9}{17}; \quad \sum_{n=1}^{\infty} \frac{2^{8n-6}(2j_{16n-3} + 257\cdot 2^{8n-6})}{(j^2_{8n-2} + 225\cdot 2^{8n-6})^2} = \frac{51}{19,125}.$$

Finally, we encourage giboancci enthusiasts to compute the Jacobsthal sums with $p, k, r, t \in \{2, 3\}$.

6. Acknowledgment

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