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NEW ALGEBRAIC STRUCTURE OF MULTI SPHERICAL FUZZY SETS

Abstract: The main aim of the present paper is to introduce the concept of multi spherical fuzzy (briefly, MSF) Lie sub algebras and multi spherical fuzzy (briefly, MSF) Lie ideals of Lie algebra. Some of their fundamental properties and operations like intersection and Cartesian product are investigated. Moreover, the relationship between multi spherical fuzzy Lie sub algebras and multispherical fuzzy Lie ideals are also established. Moreover, the images and the inverse images of both multi spherical fuzzy Lie sub algebras and multi spherical fuzzy Lie ideals under Lie homomorphisms are also studied.

Key words and phrases: Lie Algebra, Lie Subalgebra, Multi Spherical Fuzzy Set, Multi Spherical Fuzzy Lie Subalgebras, Multi Spherical Fuzzy Lie Ideals.

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1. Introduction

The idea of Lie algebra was first introduced by Sophus Lie (1842-1899) in an effort to categorise certain smooth subgroups of general linear groups [13].Lie algebras, so named because they were invented by Sophus Lie, are a specific instance of general linear algebra. Following the introduction of this theory, Lie groups were used in mathematics and physics to categorise smooth subgroups. In [[3], [16], [17], [21], [22], [23]], Lie subalgebras and their properties were created and explored in more detail. Zadeh [15] proposed the concept of fuzzy set (FS) in situations that are vague, imprecise and uncertain. As a generalisation of fuzzy set, Atanassov [14] created intuitionistic fuzzy set (IFS) in 1986. His theory thereafter became widely

acknowledged as an essential resource in the fields of science, technology, engineering, medicine, etc. In 1995, neutrosophic set (in short, NS) was introduced by Smarandache [8] as a generalization of not only intuitionistic fuzzy set but also Pythagorean fuzzy set, Farmatean fuzzy set, spherical fuzzy set and so on. In 2023, Smarandache [7] proposed the model of multi spherical fuzzy set which is the genralization of multi pythagorean fuzzy set as well as spherical fuzzy set model which has wider scope of applications in multi criteria decision making problems where as spherical fuzzy set has great importance in real life application for single criteria decision making problems.

One such application is due to Ajoy, Ganeshsree, Aldring, Pham Huy, Hoang, Cong [2]. In order to create model for vague and imprecise information, the model of Pythagorean fuzzy set (PFS) was introduced by Yager [19, 20] which is some how different from IFS model since it involves the condition $0 \leq \mathrm{T}^2 + \mathrm{F}^2 \leq 1$ where T and F stand for membership and non-membership function respectably. In decision making problems, PFS model has significant application proposed by Garg [11, 12]. The application range of solving real life problems such as decision making problems in spherical fuzzy set model is popularly increasing than that of the PFS model because number of triplets satisfying the condition $0 \leq \tau^2 + \lambda^2 + \eta^2 \leq 1$ is higher than that of the pair satisfying the condition $0 \leq \tau^2 + \eta^2 \leq 1$. Recently, Debnath and Smarandache [4] introduced the notion of Multi Intuitionistic Fuzzy Lie Algebras and Debnath [5]initiated Multi Fermatean Fuzzy Lie Ideals of Lie algebras. In the modern age of artificial intelligence (AI) , science and technology significantly deal with intricate phenomena and processes for which there is inadequate information. In such kind of situations, mathematical models are created for dealing with different types of systems that have uncertain and imprecise components. One of such models is multi spherical fuzzy set which was built on the extensions of standard set theory. In this paper we cover the core features of a multi spherical fuzzy Lie subalgebra and Lie ideals of Lie algebra. The numerous domains along with signal processing, artificial intelligence, multiagent systems, computer networks, robotics, genetic algorithms, expert systems, neural networks, decision making, medical diagnosis and automata theory shall be benefited with the acquired outcomes.

We organized the work as follows: we presented the introduction and literature review in the first section. Section 2 focuses into common definitions and preliminaries. Section 3 describes the concept of multi spherical fuzzy Lie subalgebras and multi spherical fuzzy Lie ideals of Lie algebra. Some of their fundamental properties and operations like intersection and generalized cartesian product are investigated. Moreover, the relationship between multi spherical fuzzy Lie subalgebras and multi spherical fuzzy Lie ideals are also established. In section 4,

we investigate the images and the inverse images of multi spherical fuzzy Lie subalgebras and multi spherical fuzzy Lie ideals under Lie homomorphisms. In section 5, we give the conclusion of the newly defined concept of multi spherical fuzzy Lie subalgebras and Lie ideals.

2. Preliminaries

This section consists of some common notations and definitions which have been involved in the course of the paper.

A Lie algebra is a vector space $\mathcal{L}$ over the field $\mathcal{F}(\mathcal{L} = \mathrm{R} \text{ or } \mathrm{C})$ on which $\mathcal{L} \times \mathcal{L} \to \mathcal{L}$ defined by $(\gamma, \mu) = [\gamma, \mu]$ for all $\gamma, \mu \in \mathcal{L}$, where, $[\gamma, \mu]$ is called Lie Bracket satisfying the following conditions:

(1) $[\gamma, \mu]$ is bilinear

(2) $[\gamma, \gamma] = 0$, for all $\gamma \in \mathcal{L}$

(3) $[[\gamma, \mu], \lambda] + [[\mu, \lambda], \gamma] + [[\lambda, \gamma], \mu] = 0$, for all $\gamma, \mu, \lambda \in \mathcal{L}$. (This is called Jacobi identity).

Throughout the paper $\mathcal{L}$ will denote Lie algebra and also we note that the operation Lie bracket $[.,.]$ is neither associative nor commutative i.e., $[[\gamma, \mu], \lambda] \neq [[\mu, \lambda], \gamma]$ and $[\gamma, \mu] \neq [\mu, \gamma]$. But the operation Lie bracket $[.,.]$ is anti-commutative i.e., $[\gamma, \mu] = -[\mu, \gamma]$.

A subspace H of $\mathcal{L}$ is called a Lie subalgebra if it is closed under $[.,.]$.

A subspace I of $\mathcal{L}$ is called a Lie ideal if $[\mathrm{I}, \mathrm{L}] \subset \mathrm{I}$. It is always true that every Lie ideal is Lie subalgebra.

Definition 2.1: [15] Let U be a universe of discourse. Then the fuzzy set on U is described as $\mathrm{F} = \{(x, \mu(x)) : x \in \mathrm{X}$, where, $\mu(x) \in [0, 1]$, denotes the degree of membership of $x \in \mathrm{X}\}$.

Definition 2.2: [14] Let U be a universe of discourse. Then the intuitonistic fuzzy set on U is described as $\mathrm{F} = \{(x, \mu(x), \nu(x)) : x \in \mathrm{X}$, where, $\mu(x)$, $\nu(x) \in [0, 1]$, indicating the degrees of membership and non-membership respectively such that $0 \leq \mu_A(x) + \nu_A(x) \leq 1\}$.

Definition 2.3: [8] Let X be a universe of discourse. Then the neutrosophic set is defined by $\mathrm{N} = \{(x, \tau(x), \lambda(x), \eta(x)), x \in \mathrm{X}$, where, $\tau, \lambda, \eta \in [0,1]$, indicating the degrees of truth, indeterminacy and falsehood respectively that satisfy $0 \leq \inf(\tau) + \inf(\lambda) + \inf(\eta) \leq \sup(\tau) + \sup(\lambda) + \sup(\eta) \leq 3\}$.

Definition 2.4: [14] Let U be a universe of discourse. Then the Pythagorean fuzzy set on U is described as, $\mathrm{P} = \{(x, \mu(x), \nu(x)) : x \in \mathrm{X}$ where, $\mu(x), \nu(x) \in [0,1]$, indicating the degrees of membership and non-membership respectively satisfying $0 \leq \mu_P^2(x) + \nu_P^2(x) \leq 1\}$.

Definition 2.5: [10] Let X be a universe of discourse. Then the spherical fuzzy set is defined by $\mathrm{S} = \{(x, \tau(x), \lambda(x), \eta(x)), x \in \mathrm{X}$, where, $\tau, \lambda, \eta \in [0,1]$, indicating the degrees of truth, indeterminacy and falsehood respectively that satisfy $0 \leq \mu_S^2(x) + \lambda_S^2(x) + \nu_S^2(x) \leq 1\}$.

Definition 2.6: [21] A neutrosophic set $\mathrm{W} = (\tau, \lambda, \eta)$ on $\mathcal{L}$ is said to be neutrosophic subalgebra if, $\forall \gamma, \mu \in \mathcal{L}; \forall \mathrm{c} \in \mathcal{F}$, the following assumptions hold good:

(1) $\tau_W(\gamma + \mu) \geq \mathrm{Min}\{\tau_W(\gamma), \tau_W(\mu)\}$, $\lambda_W(\gamma + \mu) \geq \mathrm{Min}\{\lambda_W(\gamma), \lambda_W(\mu)\}$, $\eta_W(\gamma + \mu) \leq \mathrm{Max}\{\eta_W(\gamma), \eta_W(\mu)\}$.

(2) $\tau_W(c\gamma) \geq \tau_W(\gamma)$, $\lambda_W(c\gamma) \geq \lambda_W(\gamma)$, $\eta_W(c\gamma) \leq \eta_W(\gamma)$.

(3) $\tau_W[\gamma, \mu] \geq \mathrm{Min}\{\tau_W(\gamma), \tau_W(\mu)\}$, $\lambda_W[\gamma, \mu] \geq \mathrm{Min}\{\lambda_W(\gamma), \lambda W(\mu)\}$, $\eta_W[\gamma, \mu] \leq \mathrm{Max}\{\eta_W(\gamma), \eta_W(\mu)\}$

Definition 2.7: [21] A neutrosophic set $\mathrm{W} = (\tau, \lambda, \eta)$ on $\mathcal{L}$ is said to be neutrosophic Lie ideal if, $\forall \gamma, \mu \in \mathcal{L}; \forall \mathrm{c} \in \mathcal{F}$, the conditions (1) and (2) of definition (2.4) and the conditions: $\tau_W([\gamma, \mu]) \geq \tau_W(\gamma)$, $\lambda_W([\gamma, \mu]) \geq \lambda_W(\gamma)$, $\eta_W([\gamma, \mu]) \leq \eta_W(\gamma)$, hold good.

Definition 2.8: [7] Let X be a universe of discourse. Then, a multi neutrosophic set on X is defined by

$$\mathrm{M} = \{(x, x(\tau_1, \tau_2, ..., \tau_p \lambda_1, \lambda_2, ..., \lambda q; \eta_1, \eta_2, ..., \eta_s)) : x \in \mathrm{X},$$

where p,q,s are integers ≥ 0, $\mathrm{p} + \mathrm{q} + \mathrm{s} = \mathrm{n} \geq 2$, and at least one of p,q,s is ≥ 2, in order to ensure the existence of multiplicity of at least one neutrosophic component: truth, indeterminacy, or falsehood; all subsets $\tau_1, \tau_2, ..., \tau_p$; $\lambda_1, \lambda_2, ..., \lambda_q$; $\eta_1, \eta_2, ..., \eta_s \subseteq [0, 1]$; $0 \leq \sum_{i=1}^{p} \inf(\tau_i) + \sum_{j=1}^{q} \inf(\lambda_j) + \sum_{k=1}^{s} \inf(\eta_k) \leq \sum_{i=1}^{p} \sup(\tau_i) + \sum_{j=1}^{q} \sup(\lambda_j) + \sum_{k=1}^{s} \sup(\eta_k) \leq n\}$.

Definition 2.9: [7] Let X be a universe of discourse. Then, a multi spherical fuzzy set on X is defined by

$$\mathrm{M} = \{(x, x(\tau_1, \tau_2, ..., \tau_p; \lambda_1, \lambda_2, ..., \lambda_q; \eta_1, \eta_2, ..., \eta_s)) : x \in \mathrm{X},$$

where p, q, s are integers ≥ 1, $p + q + s \geq 4$, in order to ensure the existence of multiplicity of at least one neutrosophic component: truth, indeterminacy, or falsehood; all subsets $\tau_1, \tau_2, ..., \tau_p$; $\lambda_1, \lambda_2, ..., \lambda_q$; $\eta_1, \eta_2, ..., \eta_s \subseteq [0, 1]$; $0 \leq \tau_i^2 + \lambda_j^2 + \eta_k^2 \leq 1$, $R_{ijk} = (1 - \tau_i^2 - \lambda_j^2 - \eta_k^2)^{\frac{1}{2}}$, for all $i = 1, 2, 3, ..., p$; $j = 1, 2, ..., q$ and $k = 1, 2, 3, ..., s\}$.

3. Multi Spherical Fuzzy Lie Algebraic Structure and its Properties

In this section, the notion of multi spherical fuzzy Lie subalgebra is initiated as a generalized version of spherical fuzzy Lie subalgebra. Some characterizations, counter examples, and basic properties are also investigated.

Definition 3.1: Let $\mathrm{N}_l = \{1, 2, ..., 1\}$, $\mathrm{N}_m = \{1, 2, ..., m\}$, $\mathrm{N}_n = \{1, 2, ..., n\}$ and $\mathcal{L}$ be a Lie Algebra of vectors over the field $\mathcal{F}$. A multi spherical fuzzy set $\mathrm{W} = (\tau_W^1, \tau_W^2, ..., \tau_W^l; \lambda_W^1, \lambda_W^2 ...., \lambda_W^m; \eta_W^1, \eta_W^2, ..., \eta_W^n)$ on $\mathcal{L}$ is said to be multi spherical fuzzy Lie subalgebra over $\mathcal{L}$ if, $\forall \gamma, \mu \in \mathcal{L}$; $\forall \mathrm{i} \in \mathrm{N}_l$, $\forall \mathrm{j} \in \mathrm{N}_m$, $\forall \mathrm{k} \in \mathrm{N}_n$, $\forall \mathrm{c} \in \mathcal{F}$, the following assumptions hold good:

(1) $\tau_W^i(\gamma+\mu) \geq \text{Min}\{\tau_W^i(\gamma), \tau_W^i(\mu)\}$, $\lambda_W^j(\gamma+\mu) \geq \text{Min}\{\lambda_W^j(\gamma), \lambda_W^j(\mu)\}$, $\eta_W^k(\gamma+\mu) \leq \text{Max}\{\eta_W^k(\gamma), \eta_W^k(\mu)\}$

(2) $\tau_W^i(c\gamma) \geq \tau_W^i(\gamma)$, $\lambda_W^j(c\gamma) \geq \lambda_W^j(\gamma)$, $\eta_W^k(c\gamma) \leq \eta_W^k(\gamma)$

(3) $\tau_W^i[\gamma,\mu] \geq \text{Min}\{\tau_W^i(\gamma), \tau_W^i(\mu)\}$, $\lambda_W^j[\gamma,\mu] \geq \text{Min}\{\lambda_W^j(\gamma), \lambda_W^j(\mu)\}$, $\eta_W^k[\gamma,\mu] \leq \text{Max}\{\eta_W^k(\gamma), \eta_W^k(\mu)\}$

Definition 3.2: Let $\text{N}_l = \{1, 2, \ldots., 1\}$, $\text{N}_m = \{1, 2, \ldots., m\}$, $\text{N}_n = \{1, 2, \ldots., n\}$ and $\mathcal{L}$ be a Lie Algebra of vectors over the field $\mathcal{F}$. A multi spherical fuzzy set $\text{W} = (\tau_W^1, \tau_W^2, \ldots, \tau_W^l; \lambda_W^1, \lambda_W^2 \ldots., \lambda_W^m; \eta_W^1, \eta_W^2, \ldots, \eta_W^n)$ on $\mathcal{L}$ is said to be multi spherical fuzzy Lie ideal if, $\forall \gamma, \mu \in \mathcal{L}$; $\forall \text{i} \in \text{N}_l$, $\forall \text{j} \in \text{N}_m$, $\forall \text{k} \in \text{N}_n$, $\forall \text{c} \in \mathcal{F}$, the conditions (1)-(2) of definition 3.1 along with the following conditions are satisfied:

(4) $\tau_W^i[\gamma,\mu] \geq \tau_W^i(\gamma), \lambda_W^j[\gamma,\mu] \geq \lambda_W^j(\gamma), \eta_W^k[\gamma,\mu] \leq \eta_W^k(\gamma)$

It follows from condition (2) that

(5) $\tau_W^i(0) \geq \tau_W^i(\gamma), \lambda_W^j(0) \geq \lambda_W^j(\gamma), \eta_W^k(0) \leq \eta_W^k(\gamma)$

(6) $\tau_W^i(-\gamma) \geq \tau_W^i(\gamma), \lambda_W^j(-\gamma) \geq \lambda_W^j(\gamma), \eta_W^k(-\gamma) \leq \eta_W^k(\gamma)$

Theorem 3.3: *Let* $\text{M} = (\mathcal{L}, \tau_M^1, \tau_M^2, \ldots, \tau_M^l; \lambda_M^1, \lambda_M^2 \ldots., \lambda_M^m; \eta_M^1, \eta_M^2, \ldots, \eta_M^n)$ *be a multi spherical fuzzy Lie ideal over* $\mathcal{L}$ *and let* $N_l = \{1, 2, \ldots., 1\}$, $\text{N}_m = \{1, 2, \ldots., m\}$, $\text{N}_n = \{1, 2, \ldots., n\}$, *then* $\tau_M^i(0) = \text{Sup}\{\tau_M^i(\gamma) : \gamma \in \mathcal{L}\}$, $\lambda_M^j(0) = \text{Sup}\{\lambda_M^j(\gamma) : \gamma \in \mathcal{L}\}$ *and* $\eta_M^k(0) = \text{Inf}\{\eta_M^k(\gamma) : \gamma \in \mathcal{L}\}$, $\forall \text{i} \in \text{N}_l$, $\forall \text{j} \in \text{N}_m$, $\forall \text{k} \in \text{N}_n$.

Proof: From condition (5) of definition 3.2, we have,

$$\tau_M^i(0) \geq \tau_M^i(\gamma) \tag{1}$$

$$\lambda_M^j(0) \geq \lambda_M^j(\gamma) \tag{2}$$

$$\eta_M^k(0) \leq \eta_M^k(\gamma) \tag{3}$$

As γ runs over $\mathcal{L}$, the results follow just taking supremum on both sides of above inequalities (1) and (2) and infimum on (3). □

Theorem 3.4: *Let* $\mathrm{M} = (\mathcal{L}, \{\tau_M^i\}_{i=1}^l, \{\lambda_M^j\}_{j=1}^m, \{\eta_M^k\}_{k=1}^n)$ *be a multi spherical fuzzy Lie ideal over* $\mathcal{L}$. *Then for each,* $\psi, \xi, \sigma \in [0,1]$ *satisfying* $\tau_M^i(0) \geq \psi, \lambda_M^j(0) \geq \xi, \eta_M^k(0) \leq \sigma$ *and* $0 \leq \psi + \xi + \sigma \leq 1$, *the* (ψ, ξ, σ) *-level subset* $\mathcal{L}_M^{(\psi;\xi;\sigma)}$ *is a multi spherical fuzzy Lie ideal of* $\mathcal{L}$.

Proof: Straight forward from definition. □

Theorem 3.5: *If* δ *is a fixed element of* $\mathcal{L}$ *and* $\mathrm{W} = (\mathcal{L}, \{\tau_W^i\}_{i=1}^l, \{\lambda_W^j\}_{j=1}^m, \{\eta_W^k\}_{k=1}^n)$ *is a multi spherical fuzzy Lie ideal of* $\mathcal{L}$. *Then the set defined by* $\mathrm{W}^\delta = \{\gamma \in \mathcal{L} : \tau_W^i(\gamma) \geq \tau_W^i(\delta), \lambda_W^j(\gamma) \geq \lambda_W^j(\delta), \eta_W^k(\gamma) \leq \eta_W^k(\delta)\}$ *is a multi spherical fuzzy Lie ideal of* $\mathcal{L}$.

Proof: Suppose that $\gamma, \mu \in \mathrm{W}^\delta$, $\mathrm{i} \in \mathrm{N}_l$, $\mathrm{j} \in \mathrm{N}_m$, $\mathrm{k} \in \mathrm{N}_n$.

Then $\forall \gamma, \mu \in \mathrm{W}^\delta$, $\forall \mathrm{i} \in \mathrm{N}_l$, $\forall \mathrm{j} \in \mathrm{N}_m$, $\forall \mathrm{k} \in \mathrm{N}_n$

$$\tau_W^i(\gamma + \mu) \geq \mathrm{Min}\{\tau_W^i(\gamma), \tau_W^i(\mu)\} \geq \tau_W^i(\delta);$$

$$\lambda_W^j(\gamma + \mu) \geq \mathrm{Min}\{\lambda_W^j(\gamma), \lambda_W^j(\mu)\} \geq \lambda_W^i(\delta);$$

$$\eta_W^k(\gamma + \mu) \leq \mathrm{Max}\{\eta_W^k(\gamma), \eta_W^k(\mu)\} \leq \eta_W^i(\delta)$$

This implies that $\gamma + \mu \in \mathrm{W}^\delta$.

Now, $\forall \gamma, \mu \in \mathrm{W}^{\delta}$, $\forall \mathrm{i} \in \mathrm{N}_l$, $\forall \mathrm{j} \in \mathrm{N}_m$, $\forall \mathrm{k} \in \mathrm{N}_n$, $\forall \mathrm{c} \in \mathcal{F}$,

$$\tau_W^i(c\gamma) \geq \tau_W^i(\gamma) \geq \tau_W^i(\delta); \lambda_W^j(c\gamma) \geq \lambda_W^j(\gamma) \geq \lambda_W^i(\delta);$$

$$\eta_W^k(c\gamma) \leq \eta_W^k(\gamma) \leq \eta_W^i(\delta) \Rightarrow c\gamma \in \mathrm{W}^{\delta}$$

Also for every $\gamma \in \mathrm{W}^{\delta}$ and for every $\mu \in \mathrm{W}^{\delta}$, $\forall \mathrm{i} \in \mathrm{N}_l$, $\forall \mathrm{j} \in \mathrm{N}_m$, $\forall \mathrm{k} \in \mathrm{N}_n$

$$\tau_W^i[\gamma, \mu] \geq \mathrm{Min}\{\tau_W^i(\gamma), \tau_W^i(\mu)\} \geq \tau_W^i(\delta);$$

$$\lambda_W^j[\gamma, \mu] \geq \mathrm{Min}\{\lambda_W^j(\gamma), \lambda_W^j(\mu)\} \geq \lambda_W^i(\delta);$$

$\eta_W^k[\gamma, \mu] \leq \mathrm{Max}\{\eta_W^k(\gamma), \eta_W^k(\mu)\} \leq \eta_W^i(\delta)$, which shows that $[\gamma, \eta] \in \mathrm{W}^{\delta}$

Hence, W^{δ} is a multi spherical fuzzy Lie ideal of $\mathcal{L}$. □

Theorem 3.6: *If* $\mathrm{W} = (\mathcal{L}, \{\tau_W^i\}_{i=1}^l, \{\lambda_W^j\}_{j=1}^m, \{\eta_W^k\}_{k=1}^n)$ *is a multi spherical fuzzy Lie ideal of* $\mathcal{L}$. *Then the set defined by* $\mathrm{W}^0 = \{\gamma \in L : \tau_W^i(\gamma) \geq \tau_W^i(0)$, $\lambda_W^j(\gamma) \geq \lambda_W^j(0)$, $\eta_W^k(\gamma) \leq \eta_W^k(0)$, $\forall \mathrm{i} \in \mathrm{N}_l$, $\forall \mathrm{j} \in \mathrm{N}_m$, $\forall \mathrm{k} \in \mathrm{N}_n\}$ *is a multi spherical fuzzy Lie ideal of* $\mathcal{L}$.

Proof: Straight forward. □

Theorem 3.7: *Let* $\mathrm{W} = (\{\tau_W^i\}_{i=1}^l, \{\lambda_W^j\}_{j=1}^m, \{\eta_W^k\}_{k=1}^n)$ *be a multi spherical fuzzy Lie subalgebra of a Lie algebra* $\mathcal{L}$ *and* $\mathcal{R} \subseteq \mathcal{L} \times \mathcal{L}$ *be a binary relation on* $\mathcal{L}$ *defined by* $\mathcal{R} = \{(\gamma, \mu) \in \mathcal{L} \times \mathcal{L} \mid \tau_W^i(\gamma - \mu) = \tau_W^i(0)$, $\lambda_W^j(\gamma - \mu) = \lambda_W^j(0)$, $\eta_W^k(\gamma - \mu) = \eta_W^k(0)$, $\gamma, \mu \in \mathcal{L}$, $\mathrm{i} \in \mathrm{N}_l$, $\mathrm{j} \in \mathrm{N}_m$, $\mathrm{k} \in \mathrm{N}_n\}$, *then* $\mathcal{R}$ *is a congruence relation on* $\mathcal{L}$.

Proof: First of all we need to prove that the relation $\mathcal{R}$ is equivalence relation on $\mathcal{L}$.

(i) **Reflexivity:** Since $\forall \gamma \in \mathcal{L}$, $\tau_W^i(\gamma - \gamma) = \tau_W^i(0)$, $\lambda_W^j(\gamma - \gamma) = \lambda_W^j(0)$, $\eta_W^k(\gamma - \gamma) = \eta_W^k(0)$, thus, $(\gamma, \gamma) \in \mathcal{R}$, $\forall \gamma \in \mathcal{L}$, $\forall \mathrm{i} \in \mathrm{N}_l$, $\forall \mathrm{j} \in \mathrm{N}_m$, $\forall \mathrm{k} \in \mathrm{N}_n$, and consequently $\mathcal{R}$ is reflexive relation on $\mathcal{L}$.

(ii) **Symmetric:** Let, $(\gamma, \mu) \in \mathcal{R}$. Then $\tau_W^i(\gamma - \mu) = \tau_W^i(0)$

$$\Rightarrow \tau_W^i(-(\mu - \gamma)) \geq \tau_W^i((\mu - \gamma)) = \tau_W^i(0),$$

$$\lambda_W^j(\gamma - \mu) = \lambda_W^j(0) \Rightarrow \lambda_W^j(-(\mu - \gamma)) \geq \lambda_W^j((\mu - \gamma)) = \lambda_W^j(0),$$

$$\eta_W^k(\gamma - \mu) = \eta_W^k(0), \Rightarrow \eta_W^k(-(\mu - \gamma)) \leq \eta_W^j((\mu - \gamma)) = \eta_W^j(0),$$

Thus, $(\mu, \gamma) \in \mathcal{R}$, $\forall \gamma, \mu \in \mathcal{L}$, $\forall \mathrm{i} \in \mathrm{N}_l$, $\forall \mathrm{j} \in \mathrm{N}_m$, $\forall \mathrm{k} \in \mathrm{N}_n$, so that $\mathcal{R}$ is symmetric relation on L.

(iii) **Transitive:** Let, $(\gamma, \mu), (\mu, \sigma) \in \mathcal{R}$. Then $\tau_W^i(\gamma - \mu) = \tau_W^i(0)$,

$$\tau_W^i(\mu - \sigma) = \tau_W^i(0)$$

$$\lambda_W^j(\gamma - \mu) = \lambda_W^j(0),\ \lambda_W^j(\mu - \sigma) = \lambda_W^j(0)$$

$$\eta_W^k(\gamma - \mu) = \eta_W^k(0),\ \eta_W^k(\mu - \sigma) = \eta_W^k(0)$$

From which we have,

$$\tau_W^i(\gamma - \sigma) = \tau_W^i\{(\gamma - \mu) + (\mu - \sigma)\} \geq \mathrm{Min}\{\tau_W^i(\gamma - \mu), \tau_W^i(\mu - \sigma)\} = \tau_W^i(0)\ ,$$

$$\lambda_W^j(\gamma - \sigma) = \lambda_W^j\{(\gamma - \mu) + (\mu - \sigma)\} \geq \mathrm{Min}\{\lambda_W^k(\gamma - \mu), \lambda_W^j(\mu - \sigma)\} = \lambda_W^j(0),$$

$$\eta_W^k(\gamma - \sigma) = \eta_W^k\{(\gamma - \mu) + (\mu - \sigma)\} \leq \mathrm{Max}\{\eta_W^k(\gamma - \mu), \eta_W^k(\mu - \sigma)\} = \eta_W^k(0).$$

Hence, $(\gamma, \sigma) \in \mathcal{L}$, $\forall \mathrm{i} \in \mathrm{N}_l$, $\forall \mathrm{j} \in \mathrm{N}_m$, $\forall \mathrm{k} \in \mathrm{N}_n$ and consequently, $\mathcal{R}$ is transitive relation on $\mathcal{L}$.

Hence, $\mathcal{R}$ is an equivalence relation on $\mathcal{L}$.

We now verify that $\mathcal{R}$ is an congruence relation on $\mathcal{L}$ and for that let us take $(\gamma, \mu), (\mu, \sigma) \in \mathcal{R}$. Then

$$\tau_W^i(\gamma - \mu) = \tau_W^i(0), \tau_W^i(\mu - \sigma) = \tau_W^i(0)$$

$$\lambda_W^j(\gamma - \mu) = \lambda_W^j(0), \lambda_W^j(\mu - \sigma) = \lambda_W^j(0)$$

$$\eta_W^k(\gamma - \mu) = \eta_W^k(0), \eta_W^k(\mu - \sigma) = \eta_W^k(0)$$

Now if $\gamma_1, \gamma_2, \mu_1, \mu_2 \in \mathcal{R}$, then we must have,

$$\tau_W^i\{(\gamma_1 + \gamma_2) - (\mu_1 + \mu_2)\} = \tau_W^i\{(\gamma_1 - \mu_1) + (\gamma_2 - \mu_2)\},$$
$$\geq \mathrm{Min}\{\tau_W^i(\gamma_1 - \mu_1), \tau_W^i(\gamma_2 - \mu_2)\} = \tau_W^i(0)$$

$$\lambda_W^j\{(\gamma_1 + \gamma_2) - (\mu_1 + \mu_2)\} = \lambda_W^j\{(\gamma_1 - \mu_1) + (\gamma_2 - \mu_2)\},$$
$$\geq \mathrm{Min}\{\lambda_W^j(\gamma_1 - \mu_1), \lambda_W^j(\gamma_2 - \mu_2)\} = \lambda_W^j(0)$$

$$\eta_W^k\{(\gamma_1 + \gamma_2) - (\mu_1 + \mu_2)\} = \eta_W^k\{(\gamma_1 - \mu_1) + (\gamma_2 - \mu_2)\},$$
$$\leq \mathrm{Max}\{\eta_W^k(\gamma_1 - \mu_1), \eta_W^k(\gamma_2 - \mu_2)\} = \eta_W^k(0)$$

$$\tau_W^i(c\gamma_1 - c\mu_1) = \tau_W^i\{c(\gamma_1 - \mu_1)\} \geq \tau_W^i(\gamma_1 - \mu_1) = \tau_W^i(0),$$

$$\lambda_W^j(c\gamma_1 - c\mu_1) = \lambda_W^j\{c(\gamma_1 - \mu_1)\} \geq \lambda_W^j(\gamma_1 - \mu_1) = \lambda_W^i(0),$$

$$\eta_W^k(c\gamma_1 - c\mu_1) = \eta_W^k\{c(\gamma_1 - \mu_1)\} \geq \eta_W^k(\gamma_1 - \mu_1) = \eta_W^k(0),$$

$$\tau_W^i\{[\gamma_1, \gamma_2] - [\mu_1, \mu_2]\} = \tau_W^i[(\gamma_1 - \mu_1), (\gamma_2 - \mu_2)]$$
$$\geq \mathrm{Min}\{\tau_W^i(\gamma_1 - \mu_1), \tau_W^i(\gamma_2 - \mu_2)\} = \tau_W^i(0),$$

$$\lambda_W^j\{[\gamma_1,\gamma_2]-[\mu_1,\mu_2]\} = \lambda_W^j[(\gamma_1-\mu_1),(\gamma_2-\mu_2)]$$
$$\geq \operatorname{Min}\{\lambda_W^j(\gamma_1-\mu_1),\lambda_W^j(\gamma_2-\mu_2)\} = \lambda_W^j(0),$$
$$\eta_W^k\{[\gamma_1+\gamma_2]-[\mu_1+\mu_2]\} = \eta_W^k[(\gamma_1-\mu_1),(\gamma_2-\mu_2)]$$
$$\leq \operatorname{Max}\{\eta_W^k(\gamma_1-\mu_1),\eta_W^k(\gamma_2-\mu_2)\} = \eta_W^k(0),$$

Thus, $(\gamma_1+\gamma_2)\ \mathcal{R}\ (\mu_1+\mu_2), c\gamma_1\ \mathcal{R}\ c\mu_1$ and $[\gamma_1,\gamma_2]\ R\ [\mu_1,\mu_2]$.

Hence, $\mathcal{R}$ is a congruence relation on $\mathcal{L}$. □

Theorem 3.8: *Every multi spherical fuzzy Lie ideal is multi spherical fuzzy Lie sub-algebra.*

Proof: Straight forward from definition. □

The converse of the above theorem (3.8) is not true which can be seen from the following example:

Example 3.9: Suppose that $\mathrm{F} = \mathbb{R}$, the set of real numbers and $\mathcal{L} = \{(\alpha,\beta,\gamma) : \alpha,\beta,\gamma \in \mathbb{R}\}$ be the Lie algebra. Let us define the mapping $\mathcal{L}\times\mathcal{L}\to\mathcal{L}$ by $[u,v] = u\times v + v$, where $\times$ denotes the vector product (or cross product). Consider the multi spherical fuzzy set $\mathrm{S} = (\tau_S^1,\tau_S^2,\tau_S^3,\lambda_S^1,\lambda_S^2,:$ $\eta_S^1,\eta_S^2,\ldots,\eta_S^4)\ \mathcal{L}\to[0,1]\times[0,1]\times[0,1]\times[0,1]\times[0,1]\times[0,1]\times[0,1]\times[0,1]\times[0,1]$ described by $\tau_S^i(\alpha,\beta,\gamma) = \{\frac{o.8}{i}$, if $\alpha=\beta=\gamma=0, \frac{o.6}{i}$, if $\alpha\neq 0,\ \beta=\gamma=0,\ 0,$ otherwise$\}$ for $i = 1,2,3$.

$\lambda_S^j(\alpha,\beta,\gamma) = \{\frac{o.3}{j}$, if $\alpha=\beta=\gamma=0, \frac{o.2}{j}$, if $\alpha\neq 0,\ \beta=\gamma=0,\ 0$, otherwise$\}$ for $j = 1,2$.

$\eta_S^k(\alpha,\beta,\gamma) = \{0$, if $\alpha=\beta=\gamma=0, \frac{o.4}{k}$, if $\alpha\neq 0,\ \beta=\gamma=0, \frac{1}{k}$, otherwise$\}$ for $\mathrm{k} = 1,2,3,4$.

Then it is easy to verify that S is a multi spherical fuzzy Lie subalgebra of $\mathcal{L}$ but it is not multi spherical fuzzy Lie ideal of $\mathcal{L}$ because $\lambda_S^j[(2,0,0),(j,-j,j)]$ $= \lambda_S^j(j,-3j,-j) = 0 \not\geq \frac{o.2}{j} = \lambda_S^j(2,0,0)$, for all $j = 1,2$. □

Theorem 3.10: *Let* $\mathrm{N}_l = \{1,2,....,1\}, \mathrm{N}_m = \{1,2,....,m\}, \mathrm{N}_n = \{1,2,....,n\}$ *and* $\mathcal{L}$ *be a Lie Algebra of vectors over the field* $\mathcal{F}$. *The necessary and sufficient condition for a multi spherical fuzzy set* $\mathrm{W} = (\tau_W^1, \tau_W^2, ..., \tau_W^l; \lambda_W^1, \lambda_W^2...., \lambda_W^m; \eta_W^1, \eta_W^2, ..., \eta_W^n)$ *to be a multi spherical fuzzy Lie subalgebra over* $\mathcal{L}$ *is that,* $\forall \mathrm{r} \in [0,1]$, *non-empty upper* r*-level cut* $\mathrm{U}_\mathrm{r}(\tau_W^i) = \{\gamma \in \mathcal{L} : \tau_W^i(\gamma) \geq \mathrm{r}, \quad \forall \mathrm{i} \in \mathrm{N}_l\}$, $\mathrm{U}_r(\lambda_W^j) = \{\gamma \in \mathcal{L} : \lambda_W^j(\gamma) \geq \mathrm{r}, \forall \mathrm{j} \in \mathrm{N}_m\}$,*and non-empty lower r-level cut* $\mathrm{V}_r(\eta_W^k) = \{\gamma \in \mathcal{L} : \eta_W^k(\gamma) \leq \mathrm{r}, \forall \mathrm{k} \in \mathrm{N}_n\}$ *are Lie subalgebra over* $\mathcal{L}$.

Proof: Suppose that $\mathrm{W} = (\tau_W^1, \tau_W^2, ..., \tau_W^l ; \lambda_W^1, \lambda_W^2...., \lambda_W^m; \eta_W^1, \eta_W^2, ..., \eta_W^n)$ is a multi spherical fuzzy Lie subalgebra over $\mathcal{L}$ and $r \in [0,1]$ is such that $\mathrm{U}_r(\tau_W^i) \neq \phi$. Let $\gamma, \mu \in \mathrm{U}_r(\tau_W^i)$.

Then $\forall \gamma, \mu \in \mathcal{L}; \forall \mathrm{i} \in \mathrm{N}_l, \forall \mathrm{j} \in \mathrm{N}_m, \forall \mathrm{k} \in \mathrm{N}_n, \forall \mathrm{c} \in \mathcal{F}$,

$\tau_W^i(\gamma + \mu) \geq \mathrm{Min}\{\tau_W^i(\gamma), \tau_W^i(\mu)\} \geq \mathrm{r}, \lambda_W^j(\gamma + \mu) \geq \mathrm{Min}\{\lambda_W^j(\gamma), \lambda_W^j(\mu)\} \geq \mathrm{r}$,

$\eta_W^k(\gamma + \mu) \leq \mathrm{Max}\{\eta_W^k(\gamma), \eta_W^k(\mu)\} \leq \mathrm{r}$

$\tau_W^i(c\gamma) \geq \tau_W^i(\gamma) \geq \mathrm{r}, \lambda_W^j(c\gamma) \geq \lambda_W^j(\gamma) \geq \mathrm{r}, \eta_W^k(c\gamma) \leq \eta_W^k(\gamma) \leq \mathrm{r}$,

$\tau_W^i[\gamma, \mu] \geq \mathrm{Min}\{\tau_W^i(\gamma), \tau_W^i(\mu)\} \geq \mathrm{r}$

$\lambda_W^j[\gamma, \mu] \geq \mathrm{Min}\{\lambda_W^j(\gamma), \lambda_W^j(\mu)\} \geq \mathrm{r}$

$\eta_W^k[\gamma, \mu] \leq \mathrm{Max}\{\eta_W^k(\gamma), \eta_W^k(\mu)\} \leq \mathrm{r}$.

Thus,

$\gamma + \mu, c\gamma, [\gamma, \mu] \in \mathrm{U}_r(\tau_W^i), \gamma + \mu, c\gamma, [\gamma, \mu] \in \mathrm{U}_r(\lambda_W^j), \gamma + \mu, c\gamma, [\gamma, \mu] \in \mathrm{V}r(\eta_W^k)$.

Thus, $\mathrm{U}_r(\tau_W^i), \mathrm{U}_r(\lambda_W^j)$ and $\mathrm{V}_r(\eta_W^k)$ constitute Lie sub algebra over $\mathcal{L}$.

Conversely, suppose that $\forall \mathrm{i} \in \mathrm{i} \in \mathrm{N}_l$ and $\forall \mathrm{r} \in [0,1]$, $\mathrm{U}_r(\tau_W^i) \neq \phi$ is a Lie sub algebra over $\mathcal{L}$ and if possible suppose that $\tau_W^i(\gamma + \mu)\, \mathrm{Min}\{\tau_W^i(\gamma), \tau_W^i(\mu)\}$ for some $\gamma, \mu \in \mathcal{L}$. If we choose $\mathrm{r}_0 = \frac{1}{2}(\tau_W^i(\gamma + \mu) + \mathrm{Min}\{\tau_W^i(\gamma), \tau_W^i(\mu)\})$, the by properties of inequality we must have, $\tau_W^i(\gamma + \mu) < \mathrm{r}_0 < \mathrm{Min}\{\tau_W^i(\gamma), \tau_W^i(\mu)\}$.

This implies that, $\gamma + \mu = \notin \mathrm{U}_r(\tau_W^i), \gamma, \mu \in \mathrm{U}_r(\tau_W^i)$, which is a contradiction. Hence, $\tau_W^i(\gamma + \mu) \geq \mathrm{Min}\{\tau_W^i(\gamma), \tau_W^i(\mu)\}, \forall\ \gamma, \mu \in \mathcal{L}$.

In a similar manner we can prove that $\tau_W^i(c\gamma) \geq \tau_W^i(\gamma)$ and $\tau_W^i[\gamma, \mu] \geq \mathrm{Min}\{\tau_W^i(\gamma), \tau_W^i(\mu)\}, \forall \mathrm{c} \in \mathcal{F}, \forall \mathrm{i} \in \mathrm{N}_l$.

The proof is similar for cases $\mathrm{U}_r(\lambda_W^j)$ and $\mathrm{V}_r(\eta_W^k) \precsim$. This completes the proof. □

Theorem 3.11: *Let* $\mathrm{N}_l = \{1, 2, ..., \mathrm{l}\}$, $\mathrm{N}_m = \{1, 2, ..., \mathrm{m}\}$, $\mathrm{N}_n = \{1, 2, ..., \mathrm{n}\}$ *and* $\mathcal{L}$ *be a Lie Algebra of vectors over the field* $\mathcal{F}$. *If* $\mathrm{V} = (\tau_V^1, \tau_V^2, ..., \tau_V^l; \lambda_V^1, \lambda_V^2 ...., \lambda_V^m; \eta_V^1, \eta_V^2, ..., \eta_V^n)$; *and* $\mathrm{W} = (\tau_W^1, \tau_W^2, ..., \tau_W^l; \lambda_W^1, \lambda_W^2 ...., \lambda_W^m; \eta_W^1, \eta_W^2, ..., \eta_W^n)$ *are two multi spherical fuzzy Lie subalgebra over* $\mathcal{L}$, *then their intersection* $\mathrm{V} \cap \mathrm{W} = \mathrm{H} = (\tau_H^1, \tau_H^2, ..., \tau_H^l; \lambda_H^1, \lambda_H^2 ...., \lambda_H^m; \eta_H^1, \eta_H^2, ..., \eta_H^n)$ *is also multi spherical fuzzy Lie subalgebra over* $\mathcal{L}$.

Proof: Suppose that $\gamma, \mu \in \mathcal{L}$ be arbitrary. Then $\forall \mathrm{i} \in \mathrm{N}_l$, $\forall \mathrm{j} \in \mathrm{N}_m$, $\forall \mathrm{k} \in \mathrm{N}_n$, $\forall \mathrm{c} \in \mathcal{F}$, we have,

$\tau_H^i(\gamma+\mu) = \text{Min}\{\tau_V^i(\gamma+\mu), \tau_W^i(\gamma+\mu)\}$

$\geq \text{Min}\{\text{Min}\{\tau_V^i(\gamma), \tau_V^i(\mu)\}, \text{Min}\{\tau_W^i(\gamma), \tau_W^i(\mu)\}\}$

$= \text{Min}\{\text{Min}\{\tau_V^i(\gamma), \tau_W^i(\gamma)\}, \text{Min}\{\tau_V^i(\mu), \tau_W^i(\mu)\}\} = \text{Min}\{\tau_H^i(\gamma), \tau_H^i(\mu)\}$

$\lambda_H^j(\gamma+\mu) = \text{Min}\{\lambda_V^j(\gamma+\mu), \lambda_W^j(\gamma+\mu)\}$

$\geq \text{Min}\{\text{Min}\{\lambda_V^j(\gamma), \lambda_V^j(\mu)\}, \text{Min}\{\lambda_W^j(\gamma), \lambda_W^j(\mu)\}\}$

$= \text{Min}\{\text{Min}\{\lambda_V^j(\gamma), \lambda_W^j(\gamma)\}, \text{Min}\{\lambda_V^i(\mu), \lambda_W^i(\mu)\}\} = \text{Min}\{\lambda_H^j(\gamma), \lambda_H^j(\mu)\}$

$\eta_H^k(\gamma+\mu) = \text{Max}\{\eta_V^k(\gamma+\mu), \eta_W^k(\gamma+\mu)\} \leq \text{Max}\{\text{Max}\{\eta_V^k(\gamma), \eta_V^k(\mu)\},$

$\text{Max}\{\eta_W^k(\gamma), \eta_W^k(\mu)\}\}$

$= \text{Max}\{\text{Max}\{\eta_V^k(\gamma), \eta_W^k(\gamma)\}, \text{Max}\{\eta_V^k(\mu), \eta_W^k(\mu)\}\} = \text{Max}\{\eta_H^k(\gamma), \eta_H^k(\mu)\}$

$\tau_H^i(c\gamma) = \text{Min}\{\tau_V^i(c\gamma), \tau_W^i(c\gamma)\} \geq \{\text{Min}\{\tau_V^i(\gamma), \tau_W^i(\gamma)\} = \tau_H^i(\gamma)$

$\lambda_H^j(c\gamma) = \text{Min}\{\lambda_V^j(c\gamma), \lambda_W^j(c\gamma)\} \geq \{\text{Min}\{\lambda_V^j(\gamma), \lambda_W^j(\gamma)\} = \lambda_H^j(\gamma)$

$\eta_H^k(c\gamma) = \text{Max}\{\eta_V^k(c\gamma), \eta_W^k(c\gamma)\} \leq \{\text{Max}\{\eta_V^k(\gamma), \eta_W^k(\gamma)\} = \eta_H^k(\gamma)$

$\tau_H^i[\gamma, \mu] = \text{Min}\{\tau_V^i[\gamma, \mu], \tau_W^i[\gamma, \mu]\}$

$\geq \text{Min}\{\text{Min}\{\tau_V^i(\gamma), \tau_V^i(\mu)\}, \text{Min}\{\tau_W^i(\gamma), \tau_W^i(\mu)\}\}$

$= \text{Min}\{\text{Min}\{\tau_V^i(\gamma), \tau_W^i(\gamma)\}, \text{Min}\{\tau_V^i(\mu), \tau_W^i(\mu)\}\} = \text{Min}\{\tau_H^i(\gamma), \tau_H^i(\mu)\}$

$\lambda_H^j[\gamma, \mu] = \text{Min}\{\lambda_V^j[\gamma, \mu], \lambda_W^j[\gamma, \mu]\}$

$\geq \text{Min}\{\text{Min}\{\lambda_V^j(\gamma), \lambda_V^j(\mu)\}, \text{Min}\{\lambda_W^j(\gamma), \lambda_W^j(\mu)\}\}$

$= \text{Min}\{\text{Min}\{\lambda_V^j(\gamma), \lambda_W^j(\gamma)\}, \text{Min}\{\lambda_V^j(\mu), \lambda_W^j(\mu)\}\} = \text{Min}\{\lambda_H^j(\gamma), \lambda_H^j(\mu)\}$

$\eta_H^k[\gamma, \mu] = \text{Max}\{\eta_V^k[\gamma, \mu], \eta_W^k[\gamma, \mu]\}$

$\leq \text{Max}\{\text{Max}\{\eta_V^k(\gamma), \eta_V^k(\mu)\}, \text{Max}\{\eta_W^k(\gamma), \eta_W^k(\mu)\}\}$

$= \text{Max}\{\text{Max}\{\eta_V^k(\gamma), \eta_W^k(\gamma)\}, \text{Max}\{\eta_V^k(\mu), \eta_W^k(\mu)\}\} = \text{Max}\{\eta_H^k(\gamma), \eta_H^k(\mu)\}$

Hence, $\text{V} \cap \text{W} = \text{H}$ is multi spherical fuzzy Lie subalgebra over $\mathcal{L}$. □

Definition 3.12: *Let* $\text{N}_l = \{1, 2, \ldots., \text{l}\}$, $\text{N}_m = \{1, 2, \ldots., \text{m}\}$,

$N_n = \{1, 2, ..., n\}$ *and* $\mathcal{L}$ *be a Lie Algebra of vectors over the field* $\mathcal{F}$. *If* $V = (\tau_V^1, \tau_V^2, ..., \tau_V^l; \lambda_V^1, \lambda_V^2 ...., \lambda_V^m; \eta_V^1, \eta_V^2, ..., \eta_V^n)$ *and* $W = (\tau_W^1, \tau_W^2, ..., \tau_W^l; \lambda_W^1, \lambda_W^2 ...., \lambda_W^m; \eta_W^1, \eta_W^2, ..., \eta_W^n)$ *are two multi spherical fuzzy sets on* $\mathcal{L}$, *then the product* $H = V \times W$ *defined on* $\mathcal{L} \times \mathcal{L}$ *will be known as generalized cartasian product if* $(V \times W)(\gamma, \mu) = [(\tau_V^1, \tau_V^2, ..., \tau_V^l; \lambda_V^1, \lambda_V^2 ...., \lambda_V^m; \eta_V^1, \eta_V^2, ..., \eta_V^n) \times (\tau_W^1, \tau_W^2, ..., \tau_W^l; \lambda_W^1, \lambda_W^2 ...., \lambda_W^m; \eta_W^1, \eta_W^2, ..., \eta_W^n)](\gamma, \mu) = (\tau_H^1(\gamma, \mu), \tau_H^2(\gamma, \mu), ..., \tau_H^l(\gamma, \mu); \lambda_H^1(\gamma, \mu), \lambda_H^2(\gamma, \mu), ..., \lambda_H^m(\gamma, \mu); \eta_H^1(\gamma, \mu), \eta_H^2(\gamma, \mu), ..., \eta_H^n(\gamma, \mu)), \forall (\gamma, \mu) \mathcal{L} \times \mathcal{L}$ *where,* $\tau_H^i(\gamma, \mu) = (\tau_V^i \times \tau_W^i)(\gamma, \mu) = \text{Min}\{\tau_V^i(\gamma), \tau_W^i(\mu)\}, \forall i \in N_l$.

$$\lambda_H^j(\gamma, \mu) = (\lambda_V^j \times \lambda_W^j)(\gamma, \mu) = \text{Min}\{\lambda_V^j(\gamma), \lambda_W^j(\mu)\}, \forall j \in N_m.$$
$$\eta_H^k(\gamma, \mu) = (\eta_V^k \times \eta_W^k)(\gamma, \mu) = \text{Max}\{\eta_V^k(\gamma), \eta_W^k(\mu)\}, \forall k \in N_n.$$

Evidently the generalized cartasian product $(V \times W)$ is multi spherical fuzzy set on $\mathcal{L} \times \mathcal{L}$ if $0 \le \{\tau_H^i(\gamma, \mu)\}^2 + \lambda_H^i(\gamma, \mu)\}^2 + \{\eta_H^k(\gamma, \mu)\}^2 \le 1$
i.e.,
$0 \le \{\text{Min}\{\tau_V^i(\gamma), \tau_W^i(\mu)\}\}^2 + \text{Min}\{\lambda_V^i(\gamma), \tau_W^i(\mu)\}\}^2 + \{\text{Max}\{\eta_V^k(\gamma), \eta_W^k(\mu)\}\}^2 \le 1$,
where, $i \in N_l$, $j \in N_m$, $k \in N_n$.

Theorem 3.13: *Let* $\mathcal{L}$ *be the Lie Algebra of vectors over the field* $\mathcal{F}$. *If* $V = (\tau_V^1, \tau_V^2, ..., \tau_V^l; \lambda_V^1, \lambda_V^2 ...., \lambda_V^m; \eta_V^1, \eta_V^2, ..., \eta_V^n)$ *and* $W = (\tau_W^1, \tau_W^2, ..., \tau_W^l; \lambda_W^1, \lambda_W^2 ...., \lambda_W^m; \eta_W^1, \eta_W^2, ..., \eta_W^n)$ *are two multi spherical fuzzy Lie subalgebra of* $\mathcal{L}$, *then the generalized cartasian product* $(V \times W)$ *is multi spherical fuzzy Lie subalgebra of* $\mathcal{L} \times \mathcal{L}$.

Proof: Let $N_l = \{1, 2, ..., l\}$, $N_m = \{1, 2, ..., m\}$, $N_n = \{1, 2, ..., n\}$ and $\mathcal{L}$ be a Lie Algebra of vectors over the field $\mathcal{F}$. Then, $\forall i \in N_l$, $\forall j \in N_m$, $\forall k \in N_n$, $\forall \gamma = (\gamma_1, \gamma_2)$, $\mu = (\mu_1, \mu_2) \in \mathcal{L} \times \mathcal{L}$ and $c \in \mathcal{F}$, we have,
$(\tau_V^i \times \tau_W^i)(\gamma + \mu) = (\tau_V^i \times \tau_W^i)((\gamma_1, \gamma_2) + (\mu_1, \mu_2))$

$= (\tau_V^i \times \tau_W^i)\ ((\gamma_1 + \mu_1),(\gamma_2 + \mu_2)) = \text{Min}\{\tau_V^i(\gamma_1 + \mu_1), \tau_W^i(\gamma_2 + \mu_2)\}$

$\geq \text{Min}\{\text{Min}\{\tau_V^i(\gamma_1), \tau_V^i(\mu_1)\}, \text{Min}\{\tau_W^i(\gamma_2), \tau_W^i(\mu_2)\}\}$

$= \text{Min}\{\text{Min}\{\tau_V^i(\gamma_1), \tau_W^i(\gamma_2)\}, \text{Min}\{\tau_V^i(\mu_1), \tau_W^i(\mu_2)\}\}$

$= \text{Min}\{(\tau_V^i \times \tau_W^i)(\gamma_1, \gamma_2), (\tau_V^i \times \tau_W^i)(\mu_1, \mu_2)\} = \text{Min}\{(\tau_V^i \times \tau_W^i)(\gamma), (\tau_V^i \times \tau_W^i)(\mu)\}$

$(\lambda_V^j \times \lambda_W^j) = (\gamma + \mu) = (\lambda_V^j \times \lambda_W^j)((\gamma_1, \gamma_2) + (\mu_1, \mu_2))$

$= (\lambda_V^j \times \lambda_W^j)\ ((\gamma_1 + \mu_1), (\gamma_2 + \mu_2))$

$= \text{Min}\{\lambda_V^j(\gamma_1 + \mu_1), \lambda_W^j(\gamma_2 + \mu_2)\}$

$\geq \text{Min}\{\text{Min}\{\lambda_V^j(\gamma_1), \lambda_V^j(\mu_1)\}, \text{Min}\{\lambda_W^j(\gamma_2), \lambda_W^j(\mu_2)\}\}$

$= \text{Min}\{\text{Min}\{\lambda_V^j(\gamma_1), \lambda_W^j(\gamma_2)\}, \text{Min}\{\lambda_V^j(\mu_1), \lambda_W^j(\mu_2)\}\}$

$= \text{Min}\{(\lambda_V^j \times \lambda_W^j)(\gamma_1, \gamma_2), (\lambda_V^j \times \lambda_W^j)(\mu_1, \mu_2)\}\} = \text{Min}\{(\lambda_V^j \times \lambda_W^j)(\gamma), (\lambda_V^j \times \lambda_W^j)(\mu)\}$

$(\eta_V^k \times \eta_W^k)(\gamma + \mu) = (\eta_V^k \times \eta_W^k)((\gamma_1, \gamma_2) + (\mu_1, \mu_2)) = (\eta_V^k \times \eta_W^k)((\gamma_1 + \mu_1), (\gamma_2 + \mu_2))$

$= \text{Max}\{\eta_V^k(\gamma_1 + \mu_1), \eta_W^k(\gamma_2 + \mu_2)\}$

$\leq \text{Max}\{\text{Max}\{\eta_V^k(\gamma_1), \eta_V^k(\mu_1)\}, \text{Min}\{\eta_W^k(\gamma_2), \eta_W^k(\mu_2)\}\}$

$= \text{Max}\{\text{Max}\{\eta_V^k(\gamma_1), \eta_W^k(\gamma_2)\}, \text{Max}\{\eta_V^k(\mu_1), \eta_W^k(\mu_2)\}\}$

$= \text{Max}\{(\eta_V^k \times \eta_W^k)(\gamma_1, \gamma_2), (\eta_V^k \times \eta_W^k)(\mu_1, \mu_2)\}\}$

$= \text{Max}\{(\eta_V^k \times \eta_W^k)(\gamma), (\eta_V^k \times \eta_W^k)(\mu)\}$

$(\tau_V^i \times \tau_W^i)(c\gamma) = (\tau_V^i \times \tau_W^i)\{c(\gamma_1, \gamma_2)\} = (\tau_V^i \times \tau_W^i)(c\gamma_1, c\gamma_2)$

$= \text{Min}\{\tau_V^i(c\gamma_1), \tau_W^i(c\gamma_2)\}$

$\geq \text{Min}\{\tau_V^i(\gamma_1), \tau_W^i(\gamma_2)\} = (\tau_V^i \times \tau_W^i)(\gamma_1, \gamma_2) = (\tau_V^i \times \tau_W^i)(\gamma)$

$(\lambda_V^j \times \lambda_W^j)(c\gamma) = (\lambda_V^j \times \lambda_W^j)\{c(\gamma_1, \gamma_2)\} = (\lambda_V^j \times \lambda_W^j)(c\gamma_1, c\gamma_2)$

$= \text{Min}\{\lambda_V^j(c\gamma_1), \lambda_W^j(c\gamma_2)\}$

$\geq \text{Min}\{\lambda_V^j(\gamma_1), \lambda_W^j(\gamma_2)\} = (\lambda_V^j \times \lambda_W^j)(\gamma_1, \gamma_2) = (\lambda_V^j \times \lambda_W^j)(\gamma)$

$(\eta_V^k \times \eta_W^k)\{(c\gamma)\} = (\eta_V^k \times \eta_W^k)\{c(\gamma_1, \gamma_2)\} = (\eta_V^k \times \eta_W^k)(c\gamma_1, c\gamma_2)$

$= \text{Max}\{\eta_V^k(c\gamma_1), \eta_W^k(c\gamma_2)\} \leq \text{Max}\{\eta_V^k(\gamma_1), \eta_W^k(\gamma_2)\} = (\eta_V^k \times \eta_W^k)(\gamma_1, \gamma_2)$

$= (\eta_V^k \times \eta_W^k)(\gamma)\ (\tau_V^i \times \tau_W^i)[\gamma, \mu] = (\tau_V^i \times \tau_W^i)[(\gamma_1, \gamma_2), (\mu_1, \mu_2)]$

$\geq \text{Min}\{\text{Min}\{\tau_V^i(\gamma_1), \tau_W^i(\gamma_2)\}, \{\text{Min}\{\tau_V^i(\mu_1), \tau_W^i(\mu_2)\}\}$

$= \text{Min}\{(\tau_V^i \times \tau_W^i)(\gamma_1, \gamma_2)\}, \{(\tau_V^i \times \tau_W^i)(\mu_1, \mu_2)\}$

$= \text{Min}\{(\tau_V^i \times \tau_W^i)(\gamma), (\tau_V^i \times \tau_W^i)(\mu)\}$

$(\lambda_V^j \times \lambda_W^j)[\gamma, \mu] = (\lambda_V^j \times \lambda_W^j)[(\gamma_1, \gamma_2), (\mu_1, \mu_2)]$

$\geq \text{Min}\{\text{Min}\{\lambda_V^j\ (\gamma_1), \lambda_W^j\ (\gamma_2)\}, \{\text{Min}\{\lambda_V^j\ (\mu_1), \lambda_W^j\ (\mu_2)\}\}$

$= \text{Min}\{(\lambda_V^j \times \lambda_W^j)(\gamma_1, \gamma_2)\}, \{(\lambda_V^j \times \lambda_W^j)(\mu_1, \mu_2)\} = \text{Min}\{(\lambda_V^j \times \lambda_W^j)(\gamma), (\lambda_V^j \times \lambda_W^j)(\mu)\}$

$(\eta_V^k \times \eta_W^k)[\gamma, \mu] = (\eta_V^k \times \eta_W^k)[(\gamma_1, \gamma_2), (\mu_1, \mu_2)]$

$\leq \text{Max}\{\text{Max}\{\eta_V^k(\gamma_1), \eta_W^k(\gamma_2)\}, \{\text{Max}\{\eta_V^k(\mu_1), \eta_W^k(\mu_2)\}\}$

$= \text{Max}\{(\eta_V^k \times \eta_W^k)(\gamma_1, \gamma_2)\}, \{(\eta_V^k \times \eta_W^k)(\mu_1, \mu_2)\} = \text{Max}\{(\eta_V^k \times \eta_W^k)(\gamma), (\eta_V^k \times \eta_W^k)(\mu)\}$

Hence, $(\text{V} \times \text{W})$ is multi spherical fuzzy Lie subalgebra of $\mathcal{L} \times \mathcal{L}$. □

4. Multi Spherical Fuzzy Lie Algebra Homomorphisms

In this section, the properties of multi spherical fuzzy Lie subalgebras and multi spherical fuzzy Lie ideals under homomorphisms of Lie algebras are investigated. Also the preservation aspects are examined.

Definition 4.1: Let L_1 and L_2 be two Lie algebras over the field $\mathcal{F}$. A linear transformation $\varphi : \text{L}_1 \to \text{L}_2$ is said to be Lie homomorphism if the relationship $\varphi([\gamma, \mu]) = [\varphi(\gamma), \varphi(\mu)]$ is true, $\forall \gamma, \mu \in \text{L}_1$.

Definition 4.2: Let L_1 and L_2 be two Lie algebras over the field $\mathcal{F}$. Then a Lie homomorphism $\varphi : \text{L}_1 \to \text{L}_2$ is said have natural extension $\varphi : \text{I}^{L_1} \to \text{I}^{L_2}$ if $\forall \text{W} = (\tau_W^1 \tau_W^2, \ldots, \tau_W^l; \lambda_W^1, \lambda_W^2 \ldots., \lambda_W^m; \eta_W^1, \eta_W^2, \ldots, \eta_W^n) \in \text{I}^{L_1}$ and $\mu \in \text{L}_2$, the followings hold:

$\varphi(\tau_W^i)(\mu) = \text{Sup}\{\tau_W^i(\gamma) : \gamma \in \varphi^{-1}(\mu), \gamma \in \text{L}_1\}$, for all $\text{i} = 1, 2, 3, \ldots, \text{l}$.

$\varphi(\lambda_W^j)(\mu) = \text{Sup}\{\tau_W^j(\gamma) : \gamma \in \varphi^{-1}(\mu), \gamma \in \text{L}_1\}$, for all $\text{j} = 1, 2, 3, \ldots, \text{m}$.

$\varphi(\eta_W^k)(\mu) = \text{Inf}\{\tau_W^k(\gamma) : \gamma \in \varphi^{-1}(\mu), \gamma \in \text{L}_1\}$, for all $\text{k} = 1, 2, 3, ..., \text{n}$.

Theorem 4.3: *Let* $\text{W} = (\tau_W^1, \tau_W^2, ..., \tau_W^l; \lambda_W^1, \lambda_W^2, ..., \lambda_W^m; \eta_W^1, \eta_W^2, ..., \eta_W^n) \in \text{I}^{L_1}$ *be multi spherical fuzzy Lie subalgebras and* $\varphi : \text{L}_1 \to \text{L}_2$ *be Lie homomorphism between* L_1 *and* L_2.

Then $\varphi(\text{W})$ is multi spherical fuzzy Lie subalgebras of L_2.

Proof: Suppose that $\mu_1, \mu_2 \in \text{L}_2$. Then $\{\gamma : \gamma \in \varphi^{-1}(\mu_1 + \mu_2)\} \supseteq \{\gamma_1 + \gamma_2 : \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\}$. Now, for all $\text{i} = 1, 2, 3, ..., \text{l}$, we have,

$\varphi(\tau_W^i)(\mu_1 + \mu_2) = \text{Sup}\{\tau_W^i(\gamma) : \gamma \in \varphi^{-1}(\mu_1 + \mu_2), \gamma \in \text{L}_1\}$

$\geq \{\tau_W^i(\gamma_1 + \gamma_2) : \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\}$

$\geq \text{Sup}\{\text{Min}\{\tau_W^i(\gamma_1), \tau_W^i(\gamma_2)\} : \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\}$

$= \text{Min}\{\text{Sup}\{\tau_W^i(\gamma_1) : \gamma_1 \in \varphi^{-1}(\mu_1)\}, \text{Sup}\{\tau_W^i(\gamma_2) : \gamma_2 \in \varphi^{-1}(\mu_2)\}\}$

$= \text{Min}\{\varphi(\tau_W^i)(\mu_1), \varphi(\tau_W^i)(\mu_2)\}$.

Similarly, for all $\text{j} = 1, 2, 3, ..., \text{m}$, we can easily prove that,

$\varphi(\lambda_W^j)(\mu_1 + \mu_2) = \text{Sup}\{\lambda_W^j(\gamma) : \gamma \in \varphi^{-1}(\mu_1 + \mu_2), \gamma \in \text{L}_1\}$

$\geq \text{Min}\{\varphi(\lambda_W^j)(\mu_1), \varphi(\lambda_W^j)(\mu_2)\}$ and for all $\text{k} = 1, 2, 3, ..., \text{n}$, we have,

$\varphi(\eta_W^k)(\mu_1 + \mu_2) = \text{Inf}\{\eta_W^k(\gamma) : \gamma \in \varphi^{-1}(\mu_1 + \mu_2), \gamma \in \text{L}_1\}$

$\leq \{\eta_W^k(\gamma_1 + \gamma_2) : \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\}$

$\leq \text{Inf}\{\text{Max}\{\eta_W^k(\gamma_1), \eta_W^k(\gamma_2)\} : \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\}$

$= \text{Max}\{\text{Inf}\{\eta_W^k(\gamma_1) : \gamma_1 \in \varphi^{-1}(\mu_1)\}, \text{Inf}\{\eta_W^k(\gamma_2) : \gamma_2 \in \varphi^{-1}(\mu_2)\}\}$

$= \text{Max}\{\varphi(\eta_W^k)(\mu_1), \varphi(\eta_W^k)(\mu_2)\}$

For $\mu \in \text{L}_2$ and $\text{c} \in \mathcal{F}$, we have, $\{\gamma : \gamma \in \varphi^{-1}(\text{c}\mu_1)\} \supseteq \{\text{c}\gamma : \gamma \in \varphi^{-1}(\mu)\}$.

Now, for all $i = 1, 2, 3, \ldots, l$, we have,

$$\varphi(\tau_W^i)(c\mu) = \text{Sup}\{\tau_W^i(c\gamma) : \gamma \in \varphi^{-1}(\mu), \gamma \in L_1\}$$
$$\geq \text{Sup}\{\tau_W^i(c\gamma) : \gamma \in \varphi^{-1}(c\mu), \gamma \in L_1\}$$
$$\geq \text{Sup}\{\tau_W^i(\gamma) : \gamma \in \varphi^{-1}(\mu), \gamma \in L_1\} = \varphi(\tau_W^i)(\mu)$$

Similarly, for all $j = 1, 2, \ldots, m$ and $k = 1, 2, 3, \ldots, m$, we can we prove that

$$\varphi(\lambda_W^j)(c\mu) = \text{Sup}\{\lambda_W^j(c\gamma) : \gamma \in \varphi^{-1}(\mu), \gamma \in L_1\} \geq \text{Sup}\{\lambda_W^j(c\gamma) : \gamma \in \varphi^{-1}(c\mu), \gamma \in L_1\}$$
$$\geq \text{Sup}\{\lambda_W^j(\gamma) : \gamma \in \varphi^{-1}(\mu), \gamma \in L_1\} = \varphi(\lambda_W^j)(\mu)$$
$$\varphi(\eta_W^k)(c\mu) = \text{Inf}\{\eta_W^k(c\gamma) : \gamma \in \varphi^{-1}(\mu), \gamma \in L_1\} \leq \text{Inf}\{\eta_W^k(c\gamma) : \gamma \in \varphi^{-1}(c\mu), \gamma \in L_1\}$$
$$\leq \text{Inf}\{\eta_W^k(\gamma) : \gamma \in \varphi^{-1}(\mu), \gamma \in L_1\} = \varphi(\eta_W^k)(\mu)$$

For, $\mu_1, \mu_2 \in L_2$, then

$$\{\gamma : \gamma \in \varphi^{-1}(\mu_1 + \mu_2)\} \supseteq \{\gamma_1 + \gamma_2 : \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\}.$$

Now, for all $i = 1, 2, 3, \ldots, l$, we have,

$$\varphi(\tau_W^i)([\mu_1, \mu_2]) = \text{Sup}\{\tau_W^i(\gamma) : \gamma \in \varphi^{-1}([\mu_1, \mu_2]), \gamma \in L_1\}$$
$$\geq \text{Sup}\{\tau_W^i([\gamma_1, \gamma_2]) : \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\}$$
$$\geq \text{Sup}\{\text{Min}\{\tau_W^i(\gamma_1), \tau_W^i(\gamma_2)\} : \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\}$$
$$= \text{Min}\{\text{Sup}\{\tau_W^i(\gamma_1) : \gamma_1 \in \varphi^{-1}(\mu_1)\}, \text{Sup}\{\tau_W^i(\gamma_2) : \gamma_2 \in \varphi^{-1}(\mu_2)\}$$
$$= \text{Min}\{\varphi(\tau_W^i)(\mu_1), \varphi(\tau_W^i)(\mu_2)\}$$

Similarly, for all $j = 1, 2, \ldots, m$ and $k = 1, 2, 3, \ldots, m$, we can we prove that

$$\varphi(\lambda_W^j)([\mu_1, \mu_2]) = \text{Sup}\{\lambda_W^j(\gamma) : \gamma \in \varphi^{-1}([\mu_1, \mu_2]), \gamma \in L_1\}$$
$$\geq \text{Sup}\{\lambda_W^j([\gamma_1, \gamma_2]) : \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\}$$
$$\geq \text{Sup}\{\text{Min}\{\lambda_W^j(\gamma_1), \lambda_W^j(\gamma_2)\} : \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\}$$

$= \mathrm{Min}\{\mathrm{Sup}\{\lambda_W^j(\gamma_1) : \gamma_1 \in \varphi^{-1}(\mu_1)\}, \mathrm{Sup}\{\lambda_W^j(\gamma_2) : \gamma_2 \in \varphi^{-1}(\mu_2)\}\}$

$= \mathrm{Min}\{\varphi(\lambda_W^j)(\mu_1), \varphi(\lambda_W^j)(\mu_2)\}$

$\varphi(\eta_W^k)([\mu_1, \mu_2]) = \mathrm{Sup}\{\eta_W^k(\gamma) : \gamma \in \varphi^{-1}([\mu_1, \mu_2]), \gamma \in \mathrm{L}_1\}$

$\geq \mathrm{Sup}\{\eta_W^k([\gamma_1, \gamma_2])\ \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\}$

$\geq \mathrm{Sup}\{\mathrm{Min}\{\eta_W^k(\gamma_1), \tau_W^k(\gamma_2)\} : \gamma_1 \in \varphi^{-1}(\mu_1), \gamma_2 \in \varphi^{-1}(\mu_2)\}$

$= \mathrm{Min}\{\mathrm{Sup}\{\eta_W^k(\gamma_1) : \gamma_1 \in \varphi^{-1}(\mu_1)\}, \mathrm{Sup}\{\eta_W^i(\gamma_2) : \gamma_2 \in \varphi^{-1}(\mu_2)\}\}$

$= \mathrm{Min}\{\varphi(\eta_W^k)(\mu_1), \varphi(\eta_W^k)(\mu_2)\}$.

Hence, $\varphi(\mathrm{W})$ is multi spherical fuzzy Lie subalgebras of L_2.

□

Theorem 4.4: *Let* $\mathrm{W} = (\tau_W^1, \tau_W^2, ..., \tau_W^l; \lambda_W^1, \lambda_W^2, ..., \lambda_W^m;\ \eta_W^1, \eta_W^2, ..., \eta_W^n) \in \mathrm{I}^{L_1}$ *be multispherical fuzzy Lie ideal and* $\varphi : \mathrm{L}_1 \to \mathrm{L}_2$ *be Lie homomorphism between* L_1 *and* L_2. *Then* $\varphi(\mathrm{W})$ *is multi spherical fuzzy Lie ideal of* L_2.

Proof: The proof is similar to the proof of theorem (4.3) □

Theorem 4.5: *Let* $\mathrm{W} = (\tau_W^1, \tau_W^2, ..., \tau_W^l; \lambda_W^1, \lambda_W^2, ..., \lambda_W^m; \eta_W^1, \eta_W^2, ..., \eta_W^n) \in \mathrm{I}^{L_2}$ *be multi spherical fuzzy Lie subalgebras and* $\varphi : \mathrm{L}_1 \to \mathrm{L}_2$ *be Lie homomorphism between* L_1 *and* L_2.

Then $\varphi^{-1}(W)$ is multi spherical fuzzy Lie subalgebra of L_1.

Proof: Suppose that $\mu_1, \mu_2 \in \mathrm{L}_1$. Now, for all $\mathrm{i} = 1, 2, 3, ..., \mathrm{l}$; $\mathrm{j} = 1, 2, 3, ..., \mathrm{m}$; $\mathrm{k} = 1, 2, 3, ..., \mathrm{n}$; we have,

$\varphi^{-1}(\tau_W^i)(\mu_1 + \mu_2) = \tau_W^i[\varphi(\mu_1 + \mu_2)] = \tau_W^i[\varphi(\mu_1) + \varphi(\mu_2)]$

$\geq \text{Min}\{\tau_W^i(\varphi(\mu_1)), \tau_W^i(\varphi(\mu_2))\}$

$= \text{Min}\{\varphi^{-1}(\tau_W^i)(\mu_1), \varphi^{-1}(\tau_W^i)(\mu_2)\}$

$\varphi^{-1}(\lambda_W^j)(\mu_1 + \mu_2) = \lambda_W^j[\varphi(\mu_1 + \mu_2)] = \lambda_W^j[\varphi(\mu_1) + \varphi(\mu_2)]$

$\geq \text{Min}\{\lambda_W^j(\varphi(\mu_1)), \lambda_W^j(\varphi(\mu_2))\}$

$= \text{Min}\{\varphi^{-1}(\lambda_W^j)(\mu_1), \varphi^{-1}(\lambda_W^j)(\mu_2)\}$

$\varphi^{-1}(\eta_W^k)(\mu_1 + \mu_2) = \eta_W^k[\varphi(\mu_1 + \mu_2)] = \eta_W^k[\varphi(\mu_1) + \varphi(\mu_2)]$

$\leq \text{Max}\{\eta_W^k(\varphi(\mu_1)), \eta_W^k(\varphi(\mu_2))\}$

$= \text{Max}\{\varphi^{-1}(\eta_W^k)(\mu_1), \varphi^{-1}(\eta_W^k)(\mu_2)\}$

For all $\mu \in \text{L}_1$ and $\text{c} \in \mathcal{F}$, we have,

$\varphi^{-1}(\tau_W^i)(\text{c}\mu) = \tau_W^i[\varphi(\text{c}\mu)] = \tau_W^i[\text{c}\varphi(\mu)] \geq \tau_W^i(\varphi(\mu)) = \varphi^{-1}(\tau_W^i)(\mu)$

$\varphi^{-1}(\lambda_W^i)(\text{c}\mu) = \lambda_W^i[\varphi(\text{c}\mu) = \lambda_W^i[\text{c}\varphi(\mu)] \geq \lambda_W^i(\varphi(\mu)) = \varphi^{-1}(\lambda_W^i)(\mu)$

$\varphi^{-1}(\eta_W^i)(\text{c}\mu) = \eta_W^i[\varphi(\text{c}\mu)] = \eta_W^i[\text{c}\varphi(\mu)] \leq \eta_W^i(\varphi(\mu)) = \varphi^{-1}(\eta_W^i)(\mu)$

For all $\mu_1, \mu_2 \in \text{L}_1$.

$\varphi^{-1}(\tau_W^i)[\mu_1, \mu_2] = \tau_W^i(\varphi[\mu_1, \mu_2]) = \tau_W^i[\varphi(\mu_1), \varphi(\mu_2)]$

$\geq \text{Min}\{\tau_W^i(\varphi(\mu_1)), \tau_W^i(\varphi(\mu_2))\}$

$= \text{Min}\{\varphi^{-1}(\tau_W^i)(\mu_1), \varphi^{-1}(\tau_W^i)(\mu_2)\}$

$\varphi^{-1}(\lambda_W^i)[\mu_1, \mu_2] = \lambda_W^i(\varphi[\mu_1, \mu_2]) = \lambda_W^i[\varphi(\mu_1), \varphi(\mu_2)]$

$\geq \text{Min}\{\lambda_W^i(\varphi(\mu_1)), \lambda_W^i(\varphi(\mu_2))\}$

$= \text{Min}\{\varphi^{-1}(\lambda_W^i)(\mu_1), \varphi^{-1}(\lambda_W^i)(\mu_2)\}$

$\varphi^{-1}(\eta_W^i)[\mu_1, \mu_2] = \eta_W^i(\varphi[\mu_1, \mu_2]) = \eta_W^i[\varphi(\mu_1), \varphi(\mu_2)]$

$\leq \text{Max}\{\eta_W^i(\varphi(\mu_1)), \eta_W^i(\varphi(\mu_2))\}$

$= \text{Max}\{\varphi^{-1}(\eta_W^i)(\mu_1), \varphi^{-1}(\eta_W^i)(\mu_2)\}$

Hence, $\varphi^{-1}(\mathrm{W})$ is multi spherical fuzzy Lie subalgebra of L_1. □

Theorem 4.6: *Let* $\mathrm{W} = (\tau_W^1, \tau_W^2, ..., \tau_W^l; \lambda_W^1, \lambda_W^2, ..., \lambda_W^m; \eta_W^1, \eta_W^2, ..., \eta_W^n) \in \mathrm{I}^{L_2}$ *be multi spherical fuzzy Lie ideal and* $\varphi : \mathrm{L}_1 \to \mathrm{L}_2$ *be Lie homomorphism between* L_1 *and* L_2. *Then* $\varphi^{-1}(\mathrm{W})$ *is multi spherical fuzzy Lie ideal of* L_1.

Proof: The proof is similar to the proof of theorem (4.5) □

5. Conclusion

Multi spherical fuzzy set theory is the generalization of Multi Pythagorean fuzzy set theory as well as spherical fuzzy set theory. In this paper we introduce the concept of multi spherical fuzzy Lie subalgebras and multi spherical fuzzy Lie ideals of Lie algebra. Some of their fundamental properties and basic operations are investigated. Moreover, the relationship between multi spherical fuzzy Lie subalgebras and multi spherical fuzzy Lie ideals are established. Lastly, the image and inverse image of multi spherical fuzzy Lie subalgebras (multi neutrosophic Lie ideals) under Lie homomorphisms are also studied. In future we shall investigate more properties of multi spherical fuzzy Lie subalgebras and Lie ideals. The outcome of the proposed work is applicable in multicriterion decision making problem, pattern recognition medical diagnosis and classification problems etc.

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