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SOLVING THE KEPLER EQUATION USING ITERATIONS

Abstract: The Kepler equation, which describes how planets and other celestial bodies move in gravitational fields, is a key concept in celestial mechanics. For many uses in space research and engineering, an accurate and speedy solution of the Kepler equation is necessary. In this study, we provide an iterative approach to resolving the Kepler equation. The technique is based on a numerical methodology that iteratively converges to a solution and yields precise and trustworthy findings. The outcomes show how effective our iteration method is in resolving the Kepler equation.

Keywords: Kepler Equation, Celestial Mechanics, Iteration Method.

Mathematical Subject Classification (2010) No.: 83C40, 83C45.

1. Introduction

The Kepler equation, which derives from Newton's law of gravitation, describes the connection between a celestial body's orbital components and the passage of time during its orbit. The equation is stated as

$$E = M + e\sin E \tag{1}$$

Here, the mean anomaly is denoted by M, eccentric anomaly by E, and orbit's eccentricity by e.

The Kepler equation links the nonlinear relative angular position of two entities in Keplerian motion around their centers of mass to the linearly rising time.

$$M(t) = E(t) - e\sin E(t) \tag{2}$$

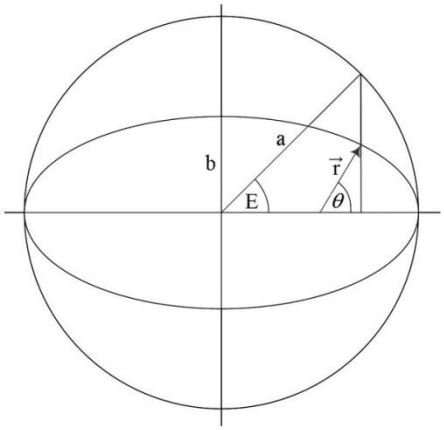


Figure 1. The geometric connection between the eccentric anomaly E and the actual anomaly θ

where the mean anomaly is $M(t) = n(t - \iota)$ and $n = \sqrt{\frac{G(m_1 + m_2)}{a^3}}$ is two-body mean motion with α being the semimajor axis of the orbital ellipse and $\sqrt{}$time of pericenter passage. Fig. 1 depicts the geometric correlation between the actual and eccentric anomalies E.

Here, we shall go through how a numerical approach to solving the Kepler equation was created. The Kepler equation is transcendental, iterative solutions are required. Therefore, each numerical process has two tasks. The 1st step is the iterative loop, which repeats a refining process until a desired convergence is obtained. In general, fewer repeated passes are required the higher the algorithm's rank.

The iterative loop's starting value must be chosen as the second task. The loop will converge more quickly the better the starting approximation. It is not necessary for the starting value method to be identical to or even closely related to the iteration technique.

2. An Initial Value Approach

The more precise initial value is provided to the iterative loop seems reasonable given that we must solve the Kepler equation repeatedly, at least till the expression complexity gets disagreeable. If, we only have $E = M$ in the limit of zero

eccentricity. Thus, eq. (1) proposes a straightforward iterative method for enhancing the initial approximation $E = M$. It forms as:

$$E_k = M + e\sin E_{k-1} \tag{3}$$

Here $E_0 = M$. The recursive phrase eq. (3) can then be repeated to any number of orders in e. The third-order approximation, for instance, is given by

$$E = M + e\sin M + e^2 \sin M \cos M + \frac{1}{2}e^3 \sin M(3\cos^2 M - 1) \tag{4}$$

3. Using Iterations

Either numerically or analytically, eq. (2) must be solved iteratively because it is a transcendental equation. Therefore, we must look for an expression that, given an error-containing value of E, provides an approximate value that minimizes the error. It must converge as well. To do this, write equation (2) in the following expression

$$f(x) = x - e\sin x - M \tag{5}$$

Here, solution to $f(x) = 0$ is $x = E$. Let $\varphi = x - E$ be approximation error of E provided by x. $x = E$ is the center of the Taylor expansion.

$$f(E) = f(x - e) = x - e\sin x - M - (1 - e\cos x)\varphi + \frac{1}{2}\varphi^2 e\sin x - \frac{1}{6}e\cos x + \ldots \tag{6}$$

assuming $\prod$ is very small. We may find

$$\varphi = \frac{x - e\sin x - M}{1 - e\cos x} \tag{7}$$

by solving this first-order term in equation (6) for $\prod$. This serves as the basis for a first-order iterative strategy. Let's assume that our original hunch is that $x = x_0$. In such case, for E, $x_1 = x_0 + \varphi$ ought to be a more accurate estimation than x_0, and so on. As a result, we propose the 1st order iterative method

$$\varphi_{n+1} = \frac{x_n - e\sin x_n - M}{1 - e\cos x_n} \tag{8}$$

where a more detailed discussion of the initial value for x_0 will follow. Equation (8) provides a first-order iterative single-step calculation method i.e. $E_{n+1} = E_n - \varphi_n$.

At second order Equation (6) in form of Horner is

$$f(x-\varphi) = x - e\sin x - M - \left(1 - e\cos x - \frac{1}{2}e\sin x\varphi\right)\varphi \tag{9}$$

We can rearrange the expression $f(x-\varphi) = 0$, then equation (9) written as

$$\varphi = \frac{x - e\sin x - M}{1 - e\cos x - \frac{1}{2}e\sin x} \tag{10}$$

So, let's write to establish a 2nd-order iterative system in the analogy to (8)

$$\varphi_{n+1} = \frac{x_n - e\sin x_n - M}{1 - e\cos x_n - \frac{1}{2}\varphi_n e\sin x_n} \tag{11}$$

This can be turned into a two-step iterative process by initially determining φ_n as expressed in equation (8) and then φ_{n+1} as presented in equation (11). By replacing φ_n in equation (11), simply substituting equation (7) for φ_n, we can also omit the intermediary step. The single-step iteration is then

$$\varphi_{n+1} = \frac{x_n - e\sin x_n - M}{1 - e\cos x_n - \frac{1}{2}e\sin x_n \frac{x_n - e\sin x_n - M}{1 - e\cos x_n}} \tag{12}$$

The third-order approximation of $f(E) = f(x-\varphi)$ in Horner form is

$$f(x-\varphi) = x - e\sin x - M - \left(1 - e\cos x - \left(\frac{1}{2}e\sin x - \frac{1}{6}e\cos x\varphi\right)\varphi\right)\varphi \tag{13}$$

After solving the "top-level" on the right and setting this to zero, and deduced to

$$\varphi_{n+1} = \frac{x_n - e\sin x_n - M}{1 - e\cos x_n - \frac{1}{2}\left(e\sin x_n - \frac{1}{3}e\cos x_n\varphi_n\right)\varphi_n} \tag{14}$$

For a two-step technique, we can utilize equation (12); for a three-step method, equation (8); and the resulting equation (14) for φ_n. As an alternative, we may simply put equation (12) for φ_n a one-step third-order technique in equation (14).

4. Conclusion

In this paper, we proposed an iterative approach for solving the Kepler equation that yields accurate and efficient answers for a wide variety of orbital conditions.

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