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ALGORITHMIC APPROACH FOR SPLIT DOMINATION NUMBER OF GRAPH

Abstract: This article presents a novel algorithm for determining the split domination number of various graph classes such as Cartesian product of Complete graph and Path graph, its Line graph, Tadpole graph, Line graph of Tadpole graph, web graph and its dual graph etc. To enable the functioning of algorithm, the vertices of these graphs are divided into sets resembling some specific graph types such as $K_1, K_2, K_{1,2}, K_{1,3}$ etc.. By examining these divisions, the algorithm ascertains the bare minimum of vertices required to dominate every other vertex, thereby disconnecting the graph upon removal. This process, known as split domination, is crucial in understanding the connectivity and the structure of graph. The algorithm is versatile and can be applied to any graph that allows for partitioning vertices in the similar fashion.

Keywords: Partitions, Domination Number, Split Domination Number, Split Adjacency Matrix.

Mathematics Subject Classification (2020) No.: 05C69, 05C85, 05C90, 68R10.

1. Introduction

A key idea in graph theory [4] that aids in comprehending the composition and characteristics of graphs is the notion of domination number $\gamma(G)$ of G [1]. The domination number of a graph is the smallest number of vertices required to dominate all other vertices in the graph. It is said that a vertex

dominates all of the vertices around it.

In 1997 [2], V R Kulli *et al*. introduced the concepts Split domination number and afterwards studied for research. The notions of the dominance number and vertices connectivity are combined to define the split dominating set and split dominating number. The dominating set D of a graph is known as a split dominating set if on removal of D graph becomes disconnected, and the split domination number $\gamma_s(G)$ of G is the minimum cardinality of the split dominating set.

The computation of the Split Domination Number of $K_2 \times P_n$, Line graph of $K_2 \times P_n$, Tadpole graph $T(m, n)$ and Line graph of Tadpole graph is analysed in [5, 6]. We looked at the split domination number of some graphs' dual graphs in [10].

In [7] Authors presents present two algorithms for finding a dominating set of the $m \times n$ complete toroidal grid graph $C_m \times C_n$. Algorithmic aspects of the k-domination problem in graphs is provided in [8].

To exactly find the split domination number for any graph is known to be NP-hard problem. Some approximation algorithms and heuristic methods are available, which can be used to find or estimate the split domination number of specific graph.

In this article, we present a new algorithmic approach to obtain the split domination number of some specific graphs, such as $K_2 \times P_n$, line graph of $K_2 \times P_n$, Tadpole graph $T(m, n)$, line graph of tadpole graph $L(T(m, n))$, Web graph, and Dual of Web graph by partitioning the vertex set in terms of $K_1, K_2, K_{1,n}$, with n in ascending order.

The proposed algorithm can be applied to any graph that allows for defined graph partitioning.

Here we are using the concept of graph partitioning, which is a fundamental problem in computer science and mathematics that involves dividing a graph into subsets of vertices, with the goal of optimizing certain objectives or constraint. Graph partitioning is performed by considering all the vertices or it is possible to perform graph partitioning without considering all vertices in certain contexts or in specific algorithms.

We are following Constraint-Based Partitioning approach in our algorithm by considering partitions in form of $K_2, K_1, K_{1,n}$ specifically $K_{1,2}, K_{1,3}$.

The paper is organized as: section 2 contains some elementary theory and statements defined by us, section 3 contains an algorithm for determining split dominance number, and section 4 justifies our findings with some illustrative examples. Section 5 contains our work's concluding remarks.

2. Preliminaries

Definition 2.1 Line Graph: Graph $G's$ Line graph $L(G)$ [3] is a graph with its vertices as points, which are mapped by the edges of G. If two edges of G have a common vertex, then the two vertices of $L(G)$ that are mapped to those two edges of G are connected.

Definition 2.2 Cartesian product of graph $K_2 \times P_n$: Complete graph K_2 and path graph P_n are graphs with vertices set $V(K_2) = \{u_1, u_2\}$ and $V(P_n) = \{\nu_1, \nu_2, \cdots, \nu_n\}$ respectively.

The Cartesian product of K_2 and P_n denoted by $K_2 \times P_n$ is the graph with vertex set $V(K_2 \times P_n) = V(K_2) \times V(P_n) = \{(u_i, \nu_j) \mid u_i \epsilon V(K_2), \nu_j \epsilon V(P_n)\}$, edge set $E = \{e\}$ such that ' e ' is an edge of $K_2 \times P_n$ if and only if $e = \langle (u_i, \nu_j), (u_k, \nu_l) \rangle$

Where either

1. $i = k$ and $\nu_j \nu_l \epsilon E(P_n)$
2. $j = l$ and $\nu_j \nu_l \epsilon E(K_2)$

Definition 2.3 Graph Partition: To generalize the algorithm, we define a graph partition by considering all the vertices of the graph, as follows: Let $P^1, P^2, \ldots, P^n$ be any of the induced sub graphs of G, like $K_1, K_2, K_{1,n}, n \geq 2$, in ascending order $A = \{P^1, P^2, \ldots, P^n\}$ is the partition of Graph G, if A is minimum satisfying following conditions

1. $V(P^1) \cap V(P^2) \cap \ldots \cap V(P^n) = \phi$

2. $E(P^i) \cap E(P^j) = \phi$ for any $i \neq j$.

3. $\forall v_i \in G, \exists j : v_i \in P^j$ but not all $e_j \in P^i$.

Illustration

To explain the partition consider graph of $K_2 \times P_n$: The observed partition follows a pattern as shown in Figure 1.

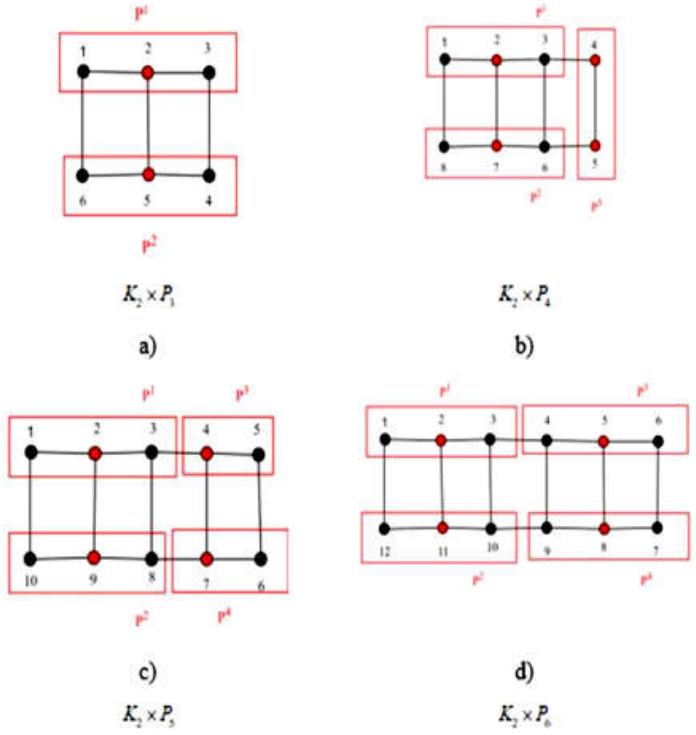


Figure 1 Partition of $K_2 \times P_n$

(a) $K_2 \times P_3 : \{K_{1,2}, K_{1,2}\}$ as $\{P^1, P^2\}$

(b) $K_2 \times P_4 : \{K_{1,2}, K_{1,2}, K_2\}$ as $\{P^1, P^2, P^3\}$

(c) $K_2 \times P_5 : \{K_{1,2}, K_{1,2}, K_2, K_2\}$ as $\{P^1, P^2, P^3, P^4\}$

(d) $K_2 \times P_6 : \{K_{1,2}, K_{1,2}, K_{1,2}, K_{1,2}\}$ as $\{P^1, P^2, P^3, P^4\}$

Similarly for $K_2 \times P_7 : \{K_{1,2}, K_{1,2}, K_{1,2}, K_{1,2}, K_2\}$ as $\{P^1, P^2, P^3, P^4, P^5\}$ and so on.

Such partition can be observed for many other classes of graph.

Table 1 illustrate the observed partitions for some of the observed graph classes, used by us for practical purpose. Where $T(m, n)$ is the tadpole graph and $L(T(m, n))$ is $T(m, n)'s$ line graph. $K_2 \times P_n$ is graph of Cartesian product of K_2 and P_n, $L(K_2 \times P_n)$ is Line graph of Cartesian product graph of complete graph K_2 and path P_n.

Such partition patterns can also be obtained for other graphs like cycle, path, wheel and web etc. along with their line and dual graphs.

Definition 2.4 Partition Matrix: For a graph G where partition of graph is done as per definition 2.3, we define the partition matrix $P = [P_{ij}]$ in which the rows corresponds to the partition sets and the columns to the vertices of the graph defined as follows:

$$P_{ij} = \begin{cases} d_j \text{ if } i^{th} \text{ partition contains vertex which is of degree } d_j \\ 0, \text{ Otherwise} \end{cases}$$

It is required to specify that d_j is degree of vertex v_j in the induced partition, not in the graph.

Also for specific partition $\boldsymbol{K}_1$, the degree of that vertex is considered as one for algorithmic purpose.

Table 1: Partition set of graph

Some graph and their obtained Partitioned set as per definition 2.3

Graph	Partitioned set
$K_2 X P_3$	$K_{1,2}, K_{1,2}$ as P^1, P^2
$K_2 X P_4$	$K_{1,2}, K_{1,2}, K_2$ as P^1, P^2, P^1
$L(K_2 \times P_3)$	$K_1, K_{1,2}, K_{1,2}$ as P^1, P^2, P^3
$L(K_2 \times P_4)$	$K_1, K_{1,2}, K_{1,2}, K_{1,2}$ as P^1, P^2, P^3, P^4
$L(K_2 \times P_n)$	$K_1, K_{1,2}, K_{1,2}, K_{1,2} \ldots K_{1,2}$ as $P^1, P^2, P^3, P^4, \ldots, P^n$
$T(4,4)$	$K_1, K_{1,3}, K_{1,2}$ as P^1, P^2, P^3
$T(4,5)$	$K_1, K_{1,3}, K_{1,2}, K_1$ as P^1, P^2, P^3, P^4
$T(4,6)$	$K_1, K_{1,3}, K_{1,2}, K_{1,2}$ as P^1, P^2, P^3, P^4
$T(4,7)$	$K_1, K_{1,3}, K_{1,2}, K_{1,2}, K_1$ as P^1, P^2, P^3, P^4, P^5
$T(4,n)$	$\begin{cases} K_1, K_{1,3}, K_{1,2}, \ldots, K_1 \text{ as } P^1, P^2, \ldots, P^{n-2} & \text{for n odd} \\ K_1, K_{1,3}, K_{1,2}, K_{1,2}, \ldots, K_{1,2} \text{ as } P^1, P^2, \ldots, P^{n-2} & \text{for n even} \end{cases}$
$L(T(3,1))$	K_2, K_2 as P^1, P^2
$L(T(3,2))$	$K_2, K_{1,2}$ as P^1, P^2
$L(T(3,3))$	$K_2, K_{1,2}, K_1$ as P^1, P^2, P^3
$L(T(3,4))$	$K_2, K_{1,2}, K_2$ as P^1, P^2, P^3
$L(T(3,5))$	$K_2, K_{1,2}, K_{1,2}$ as P^1, P^2, P^3
$L(T(3,6))$	$K_2, K_{1,2}, K_{1,2}, K_1$ as P^1, P^2, P^3, P^4
$L(T(3,7))$	$K_2, K_{1,2}, K_{1,2}, K_2$ as P^1, P^2, P^3, P^4
$L(T(3,8))$	$K_2, K_{1,2}, K_{1,2}, K_{1,2}$ as P^1, P^2, P^3, P^4
$L(T(3,9))$	$K_2, K_{1,2}, K_{1,2}, K_{1,2}, K_1$ as P^1, P^2, P^3, P^4, P^5
$L(T(3,10))$	$K_2, K_{1,2}, K_{1,2}, K_{1,2}, K_2$ as P^1, P^2, P^3, P^4, P^5

From figure 1 (c)

The partition set of the graph $K_2 \times P_5$: $A = \{P^1, P^2, P^3, P^4\}$

Partition Matrix P obtained as follows

	1	2	3	4	5	6	7	8	9	10
P^1	1	2	1	0	0	0	0	0	0	0
P^2	0	0	0	0	0	0	0	1	2	1
P^3	0	0	0	1	1	0	0	0	0	0
P^4	0	0	0	0	0	1	1	0	0	0

(1)

2.5 Deduction from Partition matrix: ***2.5.1 Dominating Set D*:** Select the vertices with maximum degree from each row of Partition matrix this gives the set D may be defines mathematically as

$$D = \{\nu_i \in \langle V(Pi)\rangle, P_i \in A \,/\, \nu_i, \; is\; of \max \deg of \; \langle V(P_i)\rangle\}$$

For Specified matrix (1) of 2.4 section

Using maximum degree in each row of P, we get

Dominating set $D = \{2, 9, 4, 7\}$ (2)

2.5.2 Split Adjacency Matrix: Find out the Adjacency matrix $A = [a_{ij}]$ of dimension $n \times n$ between vertices of graph G, defined with

$$a_{ij} \begin{cases} 1, & \text{if } j^{\text{th}} \text{ vertex is adjacent to } i^{\text{th}} \text{ vertex} \\ 0, & \text{Otherwise} \end{cases}.$$

Here graph G under consideration have no parallel edges.

Now update ${a_{ij}}^{th}$ element of matrix A by replacing one by zero if vertex ν_i is adjacent to ν_j and $\nu_j \in D$ the reduced matrix is split dominating matrix given by $S = [S_{ij}]$ of dimension $n \times n$ between vertices of graph G, where

$$S_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is adjacent to } v_j \text{ and } vj \notin D \\ 0, & \text{Otherwise} \end{cases}.$$

Using matrix (1) and dominating set (2), the Split Adjacency Matrix S corresponding to figure 1(c), given by

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array} \begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \left[\begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \tag{3}$$

3. Algorithm to find the Split domination number

Step 1: Divide the specified graph by partitioning the vertex set in terms of $K_1, K_2, K_{1,2}, K_{1,3}$ and so on, to obtain the partition set A as mentioned definition 2.3.

Step 2: Generate the partition matrix P as per definition 2.4.

Step 3: Find out the dominating Set D as defined in deduction 2.5.1.

Step 4: Create Split adjacency matrix S as discussed in deduction 2.5.2.

Step 5: Update the Matrix S by deleting rows and columns corresponding to vertices in set D.

Step 6: After deleting the corresponding row and columns, we get a adjacency matrix between vertices of $G - D$, containing block of matrices or

containing row with all zero that indicates the isolated components.

In either case the resultant matrix represents the disconnected graph [9]. The total of number of blocks and the zeros rows say N represents that on removal of D, graph get splits into N components.

Step 7: Display the split domination number of $G : \gamma_s(G) = |D|$, splited components: N.

Step 8: END

Illustrative Example

3.1 Split domination number of $L(K_2 \times P_5)$

Find out the partition set $A = \{P^1, P^2, P^3, P^4, P^5\}$ as specified in section 2.

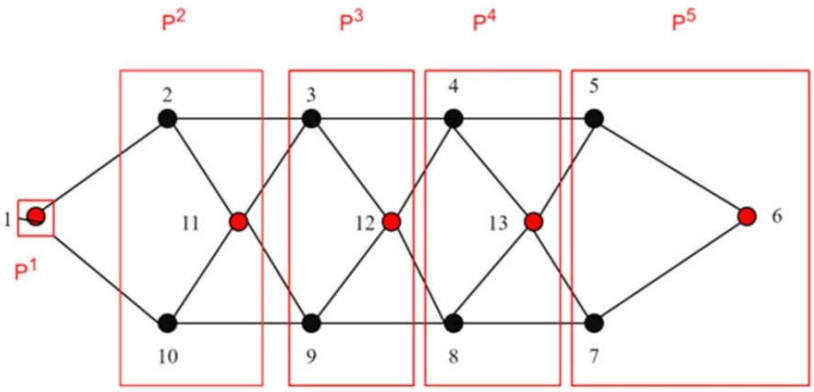


Figure 2: Line graph of $K_2 \times P_5$

Creat the Partition Matrix P

	1	2	3	4	5	6	7	8	9	10	11	12	13
P^1	1	0	0	0	0	0	0	0	0	0	0	0	0
P^2	0	1	0	0	0	0	0	0	0	1	2	0	0
P^3	0	0	1	0	0	0	0	0	1	0	0	2	0
P^4	0	0	0	1	0	0	0	1	0	0	0	0	2
P^5	0	0	0	0	1	2	1	0	0	0	0	0	0

Using maximum degree in each row of P, we get

Dominating set $D = \{1, 11, 12, 13, 6\}$

Obtain the Split adjacency matrix S as

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
1	0	1	0	0	0	0	0	0	0	1	0	0	0
2	0	0	1	0	0	0	0	0	0	0	0	0	0
3	0	1	0	1	0	0	0	0	0	0	0	0	0
4	0	0	1	0	1	0	0	0	0	0	0	0	0
5	0	0	0	1	0	0	0	0	0	0	0	0	0
6	0	0	0	0	1	0	1	0	0	0	0	0	0
7	0	0	0	0	0	0	0	1	0	0	0	0	0
8	0	0	0	0	0	0	1	0	1	0	0	0	0
9	0	0	0	0	0	0	0	1	0	1	0	0	0
10	0	0	0	0	0	0	0	0	1	0	0	0	0
11	0	1	1	0	0	0	0	0	1	1	0	0	0
12	0	0	1	1	0	0	0	1	1	0	0	0	0
13	0	0	0	1	1	0	1	1	0	0	0	0	0

Now delete the rows and columns of S, that corresponds to vertex of set D.

The Obtained adjacency matrix of $L(K_2 \times P_5) - D$ contains 2 block matrices, it indicates the splitting of graph in 2 components.

	2	3	4	5	7	8	9	10
2	0	1	0	0	0	0	0	0
3	1	0	1	0	0	0	0	0
4	0	1	0	1	0	0	0	0
5	0	0	1	0	0	0	0	0
7	0	0	0	0	0	1	0	0
8	0	0	0	0	1	0	1	0
9	0	0	0	0	0	1	0	1
10	0	0	0	0	0	0	1	0

As per defined partition this set D is set with minimum number of vertices that dominates all other vertices of $L(K_2 \times P_5)$, and on removal of D the $L(K_2 \times P_5)$ get splits into two components.

Thus, $\gamma_s(L(K_2 \times P_5)) = |D| = 5$.

3.2 To find the split domination number of line graph of $T(4, 4)$: $L(T(4, 4))$

As discussed in the previous example, we get following

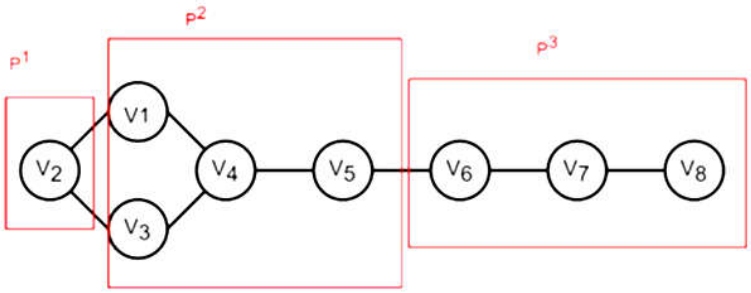


Figure 3: Line Graph of T(4,4)

$A = \{P^1, P^2, P^3\}$

Partition Matrix P:

$$\begin{array}{c} \\ P^1 \\ P^2 \\ P^3 \end{array} \begin{array}{c} \begin{array}{cccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \end{pmatrix} \end{array}$$

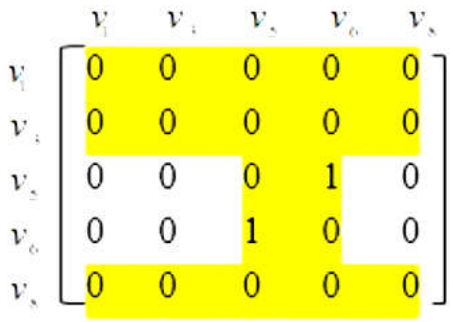
Using maximum degree in each row of P:

Dominating set $D = \{v_2, v_4, v_7\}$

The Split adjacency matrix is S:

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{array} \begin{array}{c} \begin{array}{cccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Now delete the rows and columns of S, that corresponds to vertex of set D, we get Adjacency matrix of set of vertices of $L(T(4, 4) - D$ as



In above reduced matrix three zero rows, represents 3 isolated vertex v_1, v_3 and v_8. One block of matrix represents connected component of vertex v_5 and v_6. Thus, after removing the set D the graph $L(T(4,4))$ splits into 4 components that contains 3 isolated vertices.

As per defined partition this obtained set D is minimum dominating that split's the Graph $L(T(4,4))$ in 4 components.

Thus, $\gamma_s(L(T(4,4)) = |D| = 3$

Obtained results for $\gamma_s(L(T(m,n)))$, corresponding to various values of m and n are listed below in Table 2, $m \geq 3$.

Table 2: Split domination number of Line graph of Tadpole graph

m/n	1	2	3	4	5	6	7	8	9	10	11	12	13
3	2	2	2	2	3	3	3	4	4	4	5	5	5
4	2	2	3	3	4	4	4	5	5	5	6	6	6
5	2	2	3	3	3	4	4	4	5	5	5	6	6
6	2	3	3	3	4	4	4	5	5	5	6	6	6
7	3	3	4	4	4	5	5	5	6	6	6	7	7
8	3	3	4	4	4	5	5	5	6	6	6	7	7

That matches the exact values as per our generalized formulae derived in [6] and mentioned bellow in section 4.

4. Exact values

Theorem 1: *Let* $T(m,n)$ *is Tadpole graph* $m \geq 3$ *then* $\gamma_s(L(T(m,n)))$ *is given by*

(a) $\gamma_s(L(T(3,1))) = 2$

(b) $\gamma_s(L(T(3,n))) = 1 + \left\lceil \frac{n-1}{3} \right\rceil, n > 1$

(c) $\gamma_s(L(T(m,1))) = 2\left\lceil \frac{m-1}{3} \right\rceil, m > 3$

(d) $\gamma_s(L(T(3,1))) = \begin{cases} 1 + \left\lceil \frac{m-2}{3} \right\rceil + \left\lceil \frac{n-2}{3} \right\rceil, m \neq 3k \\ 1 + \left\lceil \frac{m-3}{3} \right\rceil + \left\lceil \frac{n-1}{3} \right\rceil, m = 3k \end{cases}$ for $m \geq 4$ and $k \geq 2$

Theorem 2: $\gamma_s(L(K_2 \times P_n)) = n$

The proof of above theorems is mentioned in our previous work [6].

5. Conclusion

Finding the split domination number of any graph is a NP hard problem, there is no specific algorithm that can be used to find the split domination number of any graph. The proposed partition-based algorithmic approach is applicable not only to the specified graph but also to any graph where such defined partitions exist. Secondly, in many programmable languages, the matrix used for algorithm implementation can be declared as sparse, which uses less memory and is easier to construct with a lower time complexity. One additional feature of the suggested technique is that it provides the number of components in which the provided graph splits after deleting the split dominating set.

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