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A STUDY ON SOME ALGEBRAIC PROPERTIES OF HYBRID NUMBER

Abstract: In this paper, we define a new notion of sets, termed as symmetric hybrid number and skew symmetric hybrid number and study some of their algebraic properties. The key result of this paper is to establish that the set of non-lightlike hybrid number forms a non-abelian group under multiplication and also to find a normal subgroup of it. Moreover, the existence of rings and also their ideals is the prime concern under some additional conditions.

Keywords and Phrases: Hybrid Number, Character, Type, Symmetric Hybrid Number.

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1. Introduction, Definitions and Notations

In the last century, a lot of researchers work with some two-dimensional systems like complex, dual and hyperbolic number systems which have the most significant roles in algebraic, geometric, physics, engineering, etc. The geometry of the Euclidean plane, the Minkowski plane and the Gallian plane can be described with the help of complex numbers

$$\mathbb{C} = \{a + \mathbf{i}b : a, b \in \mathbb{R}, \mathbf{i}^2 = -1\},$$

dual numbers

$$\mathbb{D} = \{a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0\},$$

and hyperbolic numbers

$$\mathbb{H} = \{a + \mathbf{h}b : a, b \in \mathbb{R}, \mathbf{h}^2 = 1\}.$$

We know that the complex numbers, dual numbers and hyperbolic numbers can be described as the quotient of the polynomial ring $\mathbb{R}[x]$ by the ideal generated by the polynomials $x^2 + 1$, x^2 and $x^2 - 1$ respectively. i.e.,

$$\mathbb{C} = \mathbb{R}[x]/ < x^2 + 1 >,$$

$$\mathbb{D} = \mathbb{R}[x]/ < x^2 > \text{ and } \mathbb{H} = \mathbb{R}[x]/ < x^2 - 1 >.$$

In [1], [2], [3] S. Olariu defined a different generalization of n-dimensional complex numbers terming them twocomplex numbers, threecomplex numbers, etc. Actually, Olariu used the name twocomplex numbers instead of hyperbolic numbers. In these series of papers the geometrical and the algebraic properties of these numbers are thoroughly studied. The set of threecomplex numbers is defined as

$$\mathbb{C}_3 = \{z = a + \mathbf{h}b + \mathbf{k}c : a, b, c \in \mathbb{R} \text{ and } \mathbf{h}^2 = \mathbf{k}, \mathbf{k}^2 = \mathbf{h}, \mathbf{hk} = 1\}.$$

Anthony Harkin and Joseph Harkin [4] generalized the two dimensional complex numbers as

$$\mathbb{C}p = \{z = a + \mathbf{i}b : a, b \in \mathbb{R} \text{ and } \mathbf{i}^2 = p\}.$$

Here they gave some trigonometric relations for this generalization. In [5] Catoni *et al*. defined two dimensional hypercomplex numbers as

$$\mathbb{C}_{\alpha,\beta} = \{z = a + \mathbf{i}b : \mathbf{i}^2 = \alpha + \mathbf{i}\beta, a, b, \alpha, \beta \in \mathbb{R}, \mathbf{i} \notin \mathbb{R}\}$$

and extended the relationship amongst these numbers and Euclidean and semi-Euclidean geometry. This generalization is also expressible as a quotient ring $\mathbb{R}[x]/ < x^2 - \beta x - \alpha >$.

A theory of commutative two dimensional conformal hyperbolic numbers as a generalization of the theory of hyperbolic numbers is presented by Zaripov [6].

Mustafa *Ö*zdemir [7] defined a new generalization of set containing complex, hyperbolic and dual numbers as different from above generalizations. This new number system appears to be four-dimensional and it can be viewed as a two

dimensional set of numbers, since it can be represented in a generalized two dimensional plane, called hybridian plane in $\mathbb{R}^4$.

The following definitions are due to Özdemir [7].

Definition 1.1 [7]: *The set of hybrid numbers, denoted by* $\mathbb{K}$, *is defined as any combination of complex, dual and hyperbolic numbers with the relation* $\mathbf{ih} = -\mathbf{hi} = \mathbf{i} + \varepsilon$ *i.e.,*

$$\mathbb{K} = \{Z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h} : a, b, c, d \in \mathbb{R}, \mathbf{i}^2 = -1, \varepsilon^2 = 0, \mathbf{h}^2 = 1, \mathbf{ih} = -\mathbf{hi} = \mathbf{i} + \varepsilon\}.$$

The geometry corresponding to the hybrid numbers is called **Hybridian plane geometry** and it is a two-dimensional subspace of $\mathbb{R}^4$. The real part $'a'$ of the hybrid number $Z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$ is called **scalar part** and is denoted by $S(Z)$ whereas the remaining part $b\mathbf{i} + c\varepsilon + d\mathbf{h}$ is called **vector part** and is denoted by $\mathbf{V}(\boldsymbol{Z})$.

For any two hybrid numbers $Z_1 = a_0 + a_1\mathbf{i} + a_2\varepsilon + a_3\mathbf{h}$, and $Z_2 = b_0 + b_1\mathbf{i} + b_2\varepsilon + b_3\mathbf{h}$, a_i, $b_i \in \mathbb{R}$, $i = 0, 1, 2, 3$ the equality, addition, subtraction, multiplication by a scalar $s \in \mathbb{R}$ and multiplication of two hybrid numbers are defined as follows

$$Z_1 = Z_2 \Leftrightarrow a_0 = b_0,\ a_1 = b_1,\ a_2 = b_2,\ a_3 = b_3,$$

$$Z_1 \pm Z_2 = (a_0 \pm b_0) + (a_1 \pm b_1)\mathbf{i} + (a_2 \pm b_2)\ \varepsilon + (a_3 \pm b_3)\mathbf{h},$$

$$sZ_1 = sa_0 + sa_1\mathbf{i} + sa_2\varepsilon + sa_3\mathbf{h}$$

and

$$Z_1.Z_2 = (a_0 + a_1\mathbf{i} + a_2\varepsilon + a_3\mathbf{h}).(b_0 + b_1\mathbf{i} + b_2\varepsilon + b_3\mathbf{h})$$

$$= (a_0b_0 - a_1b_1 + a_2b_1 - a_1b_2 + a_3b_3) + (a_0b_2 + a_1b_0 + a_1b_3$$

$$- a_3b_1)\mathbf{i} + (a_0b_2 + a_2b_0 - a_2b_3 + a_3b_2 + a_1b_3 - a_3b_1)\varepsilon$$

$$+ (a_0b_3 + a_3b_0 + a_1b_2 + a_2b_1)\mathbf{h}.$$

The $'+'$ operation is both commutative and associative. The null element is **0** and the inverse element of Z is $-Z$. As a consequence of these properties, $(\mathbb{K}, +)$ forms an abelian group.

For the above multiplication of hybrid numbers the following table of hybrid units can be used.

.	1	**i**	ε	**h**
1	1	i	ε	**h**
i	i	1	1 **h**	ε + **i**
ε	ε	1 + **h**	0	ε
h	**h**	ε **i**	ε	1

Multiplication table of Hybrid units

From the above table it is clear that the multiplication operation in the hybrid number system is not commutative. But the multiplication is associative.

Definition 1.2 [7]: *The* ***conjugate*** *of a hybrid number* $Z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$ *is defined by*

$$\bar{Z} = S(Z) - V(Z) = a - b\mathbf{i} - c\varepsilon - d\mathbf{h}$$

This conjugation of a hybrid number is additive, involutive and multiplicative operation on $\mathbb{K}$. i.e., for any two hybrid numbers Z_1 and Z_2

a. $\overline{Z_1 + Z_2} = \overline{Z_1} + \overline{Z_2}$

b. $\overline{(\overline{Z_1})} = Z_1$

c. $\overline{Z_1.Z_2} = \overline{Z_1}.\overline{Z_2}$

Definition 1.3 [7]: *The real number* $\mathcal{C}(Z) = Z\bar{Z} = a^2 + (b-c)^2 - c^2 - d^2$ *is called the* ***character*** *of the hybrid number* Z *.*

Since, $\mathcal{C}(Z) \in \mathbb{R}$, so depending on the value of $\mathcal{C}(Z)$ a hybrid number can be categorized into three parts, **spacelike**, **timelike**, or **lightlike** according as the character is negative, positive or zero.

Also, the real number, $\sqrt{|\mathcal{C}(Z)|}$ is called the **norm** of the hybrid number Z and is denoted by $\|Z\|$.

Definition 1.4: [7] *The inverse of the hybrid number* $Z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$,

$$Z^{-1} = \frac{\bar{Z}}{\mathcal{C}(Z)}.$$

Therefore, we can conclude that a lightlike hybrid number never possesses an inverse.

Definition 1.5 [7]: *For the hybrid number* $Z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$, *the vector* $\varepsilon_Z = ((b-c), c, d)$ *is called the* ***hybrid vector*** *of* Z *.*

Definition 1.6 [7]: *The real number* $\mathcal{C}_\varepsilon(Z) = -(b-c)^2 + c^2 + d^2$ *is called the* ***type*** *of the hybrid number* Z *.*

Depending on the real value of the type of a hybrid nunber, it is classified as **complike** (**elliptic**), **hyperlike** (**hyperbolic**) or **duallike** (**parabolic**) for $\mathcal{C}_\varepsilon(Z) < 0$, $\mathcal{C}_\varepsilon(Z) > 0$ and $\mathcal{C}_\varepsilon(Z) = 0$ respectively. Also, the real number $\sqrt{|\mathcal{C}_\varepsilon(Z)|}$ is called the norm of the hybrid vector of Z and is denoted by $\mathcal{N}(Z)$.

Continuing the above discussion we may define following sets.

Definition 1.7: *A hybrid number is called symmetric if it's imaginary part and dual part both are same.*

i.e. the hybrid number, $Z_S = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$ *is called symmetric hybrid number if* $b = c$.

The following example is a symmetric hybrid number.

Example 1.1: *The hybrid number* $Z_S = 2 + 3\mathbf{i} + 3\varepsilon + \mathbf{h}$ *is a symmetric.*

Definition 1.8: *The set of symmetric hybrid number with free hyperbolic unit is called null-hyperbolic and is denoted by* $Z_{S\bar{H}}$.

Example 1.2: $-1 + i + \varepsilon$ *is an example of null-hyperbolic symmetric hybrid number.*

Definition 1.9: *A hybrid number is called skew-symmetric hybrid number if it's scalar, dual and hyperbolic coefficients are vanish.*

Example 1.3: $Z = 2\mathbf{i}$ *is an example of a skew-symmetric hybrid number.*

2. Lemmas

In this section we present the following lemma which will be needed in the sequel.

Lemma 2.1: *For any two hybrid numbers* Z_1, Z_2 *the following equality holds*

$$\mathcal{C}(Z_1.Z_2) = \mathcal{C}(Z_1).\mathcal{C}(Z_2)$$

3. Theorems

In this section we present the main results of the paper.

Theorem 3.1: *Symmetric hybrid numbers never be elliptic.*

Proof: The type of any symmetric hybrid number $Z_S = a + c\mathbf{i} + c\varepsilon + d\mathbf{h}$ is

$$\mathcal{C}_\varepsilon(Z_S) = -(c - c)^2 + c^2 + d^2 = c^2 + d^2 \geq 0.$$

So, a symmetric hybrid number is either hyperbolic or parabolic, but it never be elliptic. □

Remark 3.1.1: *The character for any symmetric hybrid number* $Z_S = a + c\mathbf{i} + c\varepsilon + d\mathbf{h}$ *is* $\mathcal{C}(Z_S) = a^2 - c^2 - d^2$. *Thus,* Z_S *may be spacelike, timelike or lightlike for* $a^2 < c^2 + d^2$, $a^2 > c^2 + d^2$ *or* $a^2 = c^2 + d^2$ *respectively.*

Theorem 3.2: *A skew-symmetric hybrid number never be spacelike as well as hyperbolic.*

Proof: Character of a skew-symmetric hybrid number, $Z = b\mathbf{i}$ is $\mathcal{C}(Z) = b^2 \geq 0$ and therefore it is either timelike or lightlike, it never be spacelike.

Now, the type of a skew-symmetric hybrid number $Z = b\mathbf{i}$ is $\mathcal{C}_\varepsilon(z) = -b^2 \leq 0$ and hence it is either elliptic or parabolic but never be hyperbolic. □

Theorem 3.3: *The set* Z_S *is a subgroup of* $\mathbb{K}$ *under* $'+'$.

Proof: The proof of the theorem is trivial. □

Remark 3.3.1: *The group* $(Z_S, +)$ *is abelian as hybrid addition is commutative.*

Theorem 3.4: *The set of null-hyperbolic symmetric hybrid numbers is a normal subgroup of symmetric hybrid numbers.*

Proof: Clearly, $Z_{S\bar{H}}$ is a non empty subset of Z_S as $\mathbf{0} \in Z_{S\bar{H}}$.

Let,

$$Z_1 = a_1 + c_1\mathbf{i} + c_1\varepsilon,\ Z_2 = a_2 + c_2\mathbf{i} + c_2\varepsilon \in Z_{S\bar{H}}.$$

Now, $Z_1 + Z_2 = (a_1 + a_2) + (c_1 + c_2)\mathbf{i} + (c_1 + c_2)\varepsilon \in Z_{S\bar{H}}$ and $Z_1^{-1} = -a_1 - c_1\mathbf{i} - c_1\varepsilon \in Z_{S\bar{H}}$.

Therefore, $(Z_{S\bar{H}}, +)$ is a subgroup of $(Z_S, +)$ and as $(Z_S, +)$ is abelian implies $(Z_{S\bar{H}}, +)$ is normal in $(Z_S, +)$. □

Theorem 3.4 leads us to the following remarks.

Remark 3.4.1: *The set of null-hyperbolic symmetric hybrid numbers is also a normal subgroup of hybrid numbers.*

Remark 3.4.2: *Let* $G = (\mathbb{K}, +)$ *and* $H = (Z_{S\bar{H}}, +)$. *Then* G/H *forms the quotient group.*

Remark 3.4.3: *Let,* $L = (Z_S, +)$ *and* $H = (Z_{S\bar{H}}, +)$. *Then* L/H *forms the quotient group.*

Remark 3.4.4: *Since,* $H = (Z_{S\bar{H}}, +)$ *is a normal subgroup of* $G = (\mathbb{K}, +)$ *and* $L = (Z_S, +)$ *is a subgroup* $G = (\mathbb{K}, +)$ *with* $H \subset L \subset G$ *then* L/H *is a subgroup of* G/H.

Theorem 3.5: *The set of all symmetric hybrid numbers* Z_S *does not form a group under the operation 'hybrid Multiplication'.*

Proof: Let, $Z_1 = a_1 + c_1\mathbf{i} + c_1\varepsilon + d_1\mathbf{h}$ and $Z_2 = a_2 + c_2\mathbf{i} + c_2\varepsilon + d_2\mathbf{h} \in Z_s$. Then

$$Z_1.Z_2 = a_1a_2 + c_1a_2\mathbf{i} + c_1a_2\varepsilon + d_1a_2\mathbf{h} + a_1c_2\mathbf{i} - c_1c_2 + c_1c_2(\mathbf{h} + 1)$$

$$+ d_1c_2(-\mathbf{i} - \varepsilon) + a_1c_2\varepsilon + c_1c_2(1 - \mathbf{h}) + d_1c_2\varepsilon + a_1d_2\mathbf{h}$$

$$+ c_1d_2(\mathbf{i} + \varepsilon) - c_1d_2\varepsilon + d_1d_2$$

$$= (a_1a_2 + c_1c_2 + d_1d_2) + (c_1a_2 + a_1c_2 - d_1c_2 + c_1d_2)\mathbf{i}$$
$$+ (c_1a_2 + a_1c_2)\varepsilon + (d_1a_2 + a_1d_2)\mathbf{h}.$$

Clearly, $Z_1.Z_2 \notin Z_S$.

Hence, $(Z_S, .)$ is not a group. □

The following example ensures the above fact.

Example 3.1: *Let,* $Z_1 = 1 + \mathbf{i} + \varepsilon + \mathbf{h}$, $Z_2 = \mathbf{i} + \varepsilon + 2\mathbf{h} \in Z_S$ *Then,*

$$Z_1.Z_2 = \mathbf{i} - 1 + \mathbf{h} + 1 - \varepsilon - \mathbf{i} + \varepsilon + 1 - \mathbf{h} + \varepsilon + 2\mathbf{h}$$

$$+ 2(\mathbf{i} + \varepsilon) - 2\varepsilon + 2$$

$$= 3 + 2\mathbf{i} + \varepsilon + 2\mathbf{h},$$

not a symmetric hybrid number.

The following remark is immediate of the above.

Remark 3.5.1: *The set of non-lightlike symmetric hybrid number whose dual and hyperbolic coefficients are in a constant ratio forms an abelian group under hybrid multiplication.*

Theorem 3.6: *The set of all non-lightlike hybrid number forms a group under the hybrid multiplication.*

Proof: Let $\mathbb{K}_{NL} = \{Z \in \mathbb{K} : \mathcal{C}(Z) \neq 0\}$ and $Z_1, Z_2 \in \mathbb{K}_{NL}$. Then, $\mathcal{C}(Z_1) \neq 0$, $\mathcal{C}(Z_2) \neq 0$.

Using Lemma 2.1, $\mathcal{C}(Z_1.Z_2) \neq 0$ and hence, $Z_1.Z_2 \in \mathbb{K}_{NL}$.

Here $Z_e = 1 + 0.\mathbf{i} + 0.\varepsilon + 0.\mathbf{h} \in \mathbb{K}_{NL}$ acts as the identity in $\mathbb{K}_{NL}$. Let Z^{-1} be the inverse of $Z \in \mathbb{K}_{NL}$.

Therefore, $\mathcal{C}(Z.Z^{-1}) = \mathcal{C}(Z_e) = 1 \Rightarrow \mathcal{C}(Z).\mathcal{C}(Z^{-1}) = 1$ and $\mathcal{C}(Z^{-1}) = \dfrac{1}{\mathcal{C}(Z)} \neq 0$ [since $\mathcal{C}(Z) \neq 0$].

Thus, $Z^{-1} \in \mathbb{K}_{NL}$.

As the associative property is the hereditary property implies the set of non-lightlike hybrid numbers $\mathbb{K}_{NL}$ forms a group under multiplication. □

The following corollary is immediate of the above.

Corollary 3.6.1: *The group $\mathbb{K}_{NL}$ is non-commutative.*

Proof: We know that $(\mathbb{K}_{NL}, .)$ is a group.

Now, let $Z_1 = a_1 + b_1\mathbf{i} + c_1\varepsilon + d_1\mathbf{h}$, $Z_2 = a + b\mathbf{i} + c\varepsilon + d\mathbf{h} \in \mathbb{K}_{NL}$.

Then

$$Z_1.Z_2 = (a_1 + b_1\mathbf{i} + c_1\varepsilon + d_1\mathbf{h}).(a_2 + b_2\mathbf{i} + c_2\varepsilon + d_2\mathbf{h})$$

$$= (a_2a_1 - b_2b_1 + c_2b_1 + b_2c_1 + d_2d_1) + (b_2a_1 + a_2b_1 + d_2b_1 - b_2d_1)\mathbf{i}$$

$$+ (c_2a_1 + d_2b_1 + a_2c_1 - d_2c_1 - b_2d_1 + c_2d_1)\varepsilon$$

$$+ (d_2a_1 - c_2b_1 + b_2c_1 + a_2d_1)\mathbf{h}$$

whereas

$$Z_2.Z_1 = (a_2a_1 - b_2b_1 + c_2b_1 + b_2c_1 + d_2d_1)$$

$$+ (b_2a_1 + a_2b_1 - d_2b_1 + b_2d_1)\mathbf{i}$$

$$+ (c_2a_1 - d_2b_1 + a_2c_1 + d_2c_1 + b_2d_1 - c_2d_1)\varepsilon$$

$$+ (d_2a_1 + c_2b_1 - b_2c_1 + a_2d_1)\mathbf{h}$$

Thus, $Z_1.Z_2 \neq Z_2.Z_1$ and hence, $(\mathbb{K}_{NL}, .)$ is a non-abelian group. □

Theorem 3.7: *The set of all hybrid numbers of unit character is normal in* $\mathbb{K}_{NL}$.

Proof: Let $\mathbb{K}_{\mathcal{C}_1} = \{Z \in \mathbb{K} : \mathcal{C}(Z) = 1\}$,

which is non-empty as $\mathbf{1} = 1 + 0.\mathbf{i} + 0.\varepsilon + 0.\mathbf{h} \in \mathbb{K}_{\mathcal{C}_1}$. Let $Z_1, Z_2 \in \mathbb{K}_{\mathcal{C}_1}$, then $\mathcal{C}(Z_1.Z_2) = 1$.

Thus, $Z_1.Z_2 \in \mathbb{K}_{\mathcal{C}_1}$.

Let Z_1^{-1} be the inverse of $Z_1 \in \mathbb{K}_{\mathcal{C}_1}$.

Now, $\mathcal{C}(Z_1.Z_1^{-1}) = \mathcal{C}(Z_e) = 1$

i.e., $\mathcal{C}(Z_1^{-1}) = \dfrac{1}{\mathcal{C}(Z_1)} = 1$ [as $\mathcal{C}(Z_1) = 1$]

Thus, $Z_1^{-1} \in \mathbb{K}_{\mathcal{C}_1}$.

Hence, $K_{\mathcal{C}_1}$ is a subgroup of $\mathbb{K}_{NL}$. Let $W \in \mathbb{K}_{NL}$ and $Z \in \mathbb{K}_{\mathcal{C}_1}$

Now, $\mathcal{C}(WZW^{-1}) = \mathcal{C}(W)\mathcal{C}(Z)\mathcal{C}(W^{-1}) = \mathcal{C}(W)\mathcal{C}(W^{-1}) = \mathcal{C}(\mathbf{1}) = 1$, non zero.

Therefore, $WZW^{-1} \in \mathbb{K}_{\mathcal{C}_1}$ and hence $\mathbb{K}_{\mathcal{C}_1}$ is a normal subgroup of $\mathbb{K}_{NL}$

This proves the theorem. □

Remark 3.7.1: *The set of lightlike hybrid numbers does not form a group under hybrid multiplication as the inverse of any element does not exist.*

Theorem 3.8: *The set* $\mathbb{K}_{\mathbb{Q}} = \{Z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h} : a, b, c, d \in \mathbb{Q}\}$ *forms a subring of* $\mathbb{K}$.

Proof: Clearly $\mathbf{0} \in \mathbb{K}_{\mathbb{Q}}$ and let $Z_1 = a_1 + b_1\mathbf{i} + c_1\varepsilon + d_1\mathbf{h}$ and $Z_2 = a_2 + b_2\mathbf{i} + c_2\varepsilon + d_2\mathbf{h} \in \mathbb{K}_{\mathbb{Q}}$

Now, $Z_1 - Z_2 = (a_1 - a_2) + (b_1 - b_2)\mathbf{i} + (c_1 - c_2)\varepsilon + (d_1 - d_2)\mathbf{h} \in \mathbb{K}_{\mathbb{Q}}$ and also

$$Z_1.Z_2 = (a_1a_2 - b_1b_2 + c_1b_2 + b_1c_2 + d_1d_2) + (b_1a_2 + a_1b_2 - d_1b_2 + b_1d_2)\mathbf{i}$$

$$+ (c_1a_2 - d_1b_2 + a_1c_2 + d_1c_2 + b_1d_2 - c_1d_2)\varepsilon$$

$$+ (d_1a_2 + c_1b_2 - b_1c_2 + a_1d_2)\mathbf{h} \in \mathbb{K}_{\mathbb{Q}}$$

Therefore, $\mathbb{K}_{\mathbb{Q}}$ is a subring of the ring of $\mathbb{K}$.

This completes the proof. □

Remark 3.8.1: *The set* $\mathbb{K}_{\mathbb{Z}} = \{z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h} : a, b, c, d \in \mathbb{Z}\}$ *is also a subring of* $\mathbb{K}$.

Proof: The proof is omitted as it is similar to previous. □

Theorem 3.9: *The set* $\mathbb{K}_{RI} = \{Z = \frac{a}{2} + \frac{a+b}{2}\mathbf{i} + \frac{a}{2}\varepsilon + \frac{b}{2}\mathbf{h} : a, b \in \mathbb{Z}\}$ *is a right ideal of* $\mathbb{K}$.

Proof: Clearly $\mathbb{K}_{RI}$ is a subring of $\mathbb{K}$.

Let $Z = \frac{a}{2} + \frac{a+b}{2}\mathbf{i} + \frac{a}{2}\varepsilon + \frac{b}{2}\mathbf{h} \in \mathbb{K}_{RI}$ and $W = p + q\mathbf{i} + r\varepsilon + s\mathbf{h} \in K$.

Then, $Z.W = \frac{ap}{2} + \frac{ap+bp}{2}\mathbf{i} + \frac{ap}{2}\varepsilon + \frac{bp}{2}\mathbf{h} + \frac{aq}{2}\mathbf{i} - \frac{aq+bq}{2}$

$$+ \frac{aq}{2}(\mathbf{h}+1) + \frac{bq}{2}(-\varepsilon - \mathbf{i}) + \frac{ar}{2}\varepsilon + \frac{ar+br}{2}(1-\mathbf{h})$$

$$+ \frac{br}{2}\varepsilon + \frac{as}{2}\mathbf{h} + \frac{as+bs}{2}(\varepsilon + \mathbf{i}) + \frac{as}{2}(-\varepsilon) + \frac{bs}{2}$$

$$= (\frac{ap - bq + ar + bs + br}{2}) + (\frac{ap + aq + as + bs + bp - bq}{2})\mathbf{i}$$

$$= (\frac{ap - bq + ar + br + bs}{2})\varepsilon + (\frac{bp + aq - ar - br + as}{2})\mathbf{h}$$

Therefore, $Z.W \in \mathbb{K}_{RI}$ and hence, $\mathbb{K}_{RI}$ is a right ideal of $\mathbb{K}$. □

Corollary 3.9.1: *The subring $\mathbb{K}_{RI}$ is not an left ideal of $\mathbb{K}$.*

Proof: Let $Z = \frac{a}{2} + \frac{a+b}{2}\mathbf{i} + \frac{a}{2}\varepsilon + \frac{b}{2}\mathbf{h} \in \mathbb{K}_{RI}$ and $W = p + q\mathbf{i} + r\varepsilon + s\mathbf{h} \in \mathbb{K}$
Then

$$W.Z = \frac{pa}{2} + \frac{aq}{2}\mathbf{i} + \frac{ar}{2}\varepsilon + \frac{as}{2}\mathbf{h} + \frac{ap+bq}{2}\mathbf{i} - \frac{aq+bq}{2}$$

$$+ \frac{ar+br}{2}(\mathbf{h}+1) + \frac{as+bs}{2}(-\varepsilon - \mathbf{i}) + \frac{ap}{2}\varepsilon + \frac{aq}{2}(1-\mathbf{h})$$

$$+ \frac{as}{2}\varepsilon + \frac{pb}{2}\mathbf{h} + \frac{qb}{2}(\varepsilon + \mathbf{i}) + \frac{br}{2}(-\varepsilon) + \frac{bs}{2}$$

$$= (\frac{ap - bq + ar + br + bs}{2}) + (\frac{aq + ap + bp - as - bs + bq}{2})\mathbf{i}$$

$$= (\frac{ar - bs + ap + bq - br}{2})\varepsilon + (\frac{as + ar + br - aq + bp}{2})\mathbf{h}$$

$$= \frac{u}{2} + \frac{v}{2}\,\mathbf{i} + \frac{x}{2}\,\varepsilon + \frac{y}{2}\,\mathbf{h} \qquad (say)$$

where $u = ap - bq + ar + br + bs; v = aq + ap + bp - as - bs + bq$;
$x = ar - bs + ap + bq - br$; and $y = as + ar + br - aq + bp$

Clearly, $u \neq x$ and so $W.Z \notin \mathbb{K}_{RI}$.

Therefore, $\mathbb{K}_{RI}$ is not left ideal. □

The following example validate the above result.

Example 3.2: *Consider* $Z = \mathbf{i} + \mathbf{h} \in \mathbb{K}$, $W = (-\frac{1}{2})\mathbf{i} + \frac{1}{2}\,\mathbf{h} \in \mathbb{K}_{RI}$.

But, $W.Z = 1 - \mathbf{i} - \varepsilon \notin \mathbb{K}_{RI}$.

Theorem 3.10: *The set* $\mathbb{K}_{LI} = \{Z = \frac{a}{2} + \frac{a-b}{2}\,\mathbf{i} + \frac{a}{2}\,\varepsilon + \frac{b}{2}\,\mathbf{h} : a, b \in \mathbb{K}\}$ *is a left ideal of* $\mathbb{K}$.

Proof: Let $Z = \frac{a}{2} + \frac{a-b}{2}\,\mathbf{i} + \frac{a}{2}\,\varepsilon + \frac{b}{2}\,\mathbf{h} \in \mathbb{K}_{LI}$ and $W = p + q\mathbf{i} + r\varepsilon + s\mathbf{h} \in \mathbb{K}$. Then

$$W.Z = \frac{ap + bq + ar - br + aq + bs}{2} + \frac{aq + ap - bp - as + bs + bq}{2}\,\mathbf{i}$$

$$+ \frac{ar - as + bs + ap + as + bq - br}{2}\,\varepsilon$$

$$+ \frac{as + ar - br - aq + bp}{2}\,\mathbf{h}$$

Therefore, $W.Z \in \mathbb{K}_{LI}$ and hence, $\mathbb{K}_{LI}$ is a left ideal of $\mathbb{K}$.

This proves the theorem. □

Corollary 3.10.1: *The subring* $\mathbb{K}_{LI}$ *is not a right ideal of* $\mathbb{K}$.

Proof:

$$W.Z = \frac{ap + bq + ar - br + bs}{2} + \frac{ap - bp + aq - bq + as - bs}{2}\mathbf{i}$$

$$+ \frac{ap - bq + ar + br - bs}{2}\varepsilon + \frac{bp + aq - ar + br - as}{2}\mathbf{h}$$

$$= \frac{u_1}{2} + \frac{v_1}{2}\mathbf{i} + \frac{x_1}{2}\varepsilon + \frac{y_1}{2}\mathbf{h}$$

Clearly, $u_1 \neq x_1$ and so $ZW \notin \mathbb{K}_{LI}$.

So, $\mathbb{K}_{LI}$ is not right ideal.

Example 3.3: *Consider* $Z = \frac{1}{2} - \mathbf{i} + \frac{1}{2}\varepsilon + \frac{3}{2}\mathbf{h}$; $W = \mathbf{i} + \mathbf{h}$, *then* $ZW = 3 - 2\mathbf{i} - 3\varepsilon + \mathbf{h} \notin \mathbb{K}_{LI}$.

Clearly, $u1 = x1$ and so $ZW \in / \mathrm{K}LI$.

Conclusion

In this work we define a pair of sets called symmetric & skew-symmetric hybrid number and we show that the set of null-hyperbolic hybrid number is a normal subgroup of the group of symmetric hybrid numbers under the hybrid addition whereas the symmetric hybrid number does not form a subgroup under hybrid multiplication. After that, we prove that the set of all non-lightlike hybrid numbers forms an abelian group under hybrid multiplication, not only that the set of hybrid numbers of unit character is normal in the set of all hybrid number with non-zero character. Then, we find some subgroups of the set of hybrid numbers. In addition, we concentrated on finding right ideal, left ideal of this system.

4. Future Prospects

In the line of work as carried out in the paper one may think of to explore some properties by taking into account some different kinds of ideals and also may try to investigate the results in higher dimensional system.

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Conflict of Interest

All the authors of this paper declare that they have no conflict of interest.

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