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HYPERBOLIC HYBRID VALUED PROBABILISTIC MEASURES UNDER THE FLAVOUR OF KOLMOGOROV'S AXIOMS

Abstract: The main aim of this paper is to introduce a notion of a hyperbolic hybrid valued probabilistic measurable space to generalize 'Kolmogorov's system of axioms'. The probability which we define here may take values $\mathbf{e}_+$ and $\mathbf{e}_-$ for the certain event other than **1** which is the key difference from the probability in $\mathbb{R}$, where $\mathbf{e}_+$ and $\mathbf{e}_-$ are very special kind of zero divisors in the ring of hybrid numbers. In this work we also prove the usual properties of probability theory like extended addition theorem, Boole's inequality, continuity theorem, Bonferroni's inequalities etc. by this new measure.

Keywords and Phrases: Kolmogorov's Axioms, Measurable Spaces, Hybrid Numbers, Idempotent Representation, Hyperbolic Hybrid Valued Probabilistic Measure.

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1. Introduction, Notations and Definitions

A probabilistic measurable space $(\Omega, \sum, P_{\mathbb{R}})$ is a triplet formed by a set Ω which has no structure but represents the sample space of a random experiment, a σ-field of subsets Σ of Ω and a measure $P_{\mathbb{R}}$ on the measurable space (Ω, Σ) which satisfies $P_{\mathbb{R}}(\Omega) = 1$. In 1933 Russian mathematician Andrei Nikolaevitch Kolmogorov introduced an alternative approach in formalizing the probability by

three axioms known as Kolmogorov's axioms which stated as follows:

Axiom-I: The probability of an event is always a non-negative real number.

Axiom-II: The probability of that at least one of the elementary events in the entire sample space is 1.

Axiom-III: Probability of countable union events is same as the countable sum of the probability of each events.

Kolmogorov's empowered us to study the probability theory in the context of measure theory. We can derive many important results from these axioms for an example the law of large numbers. More precisely, we can say a probability space is a measurable space with total mass equal to 1 and a random variable which is a real valued measurable function [6], [8], [10], [7]. Kolmogorov introduced the set Ω as the space of elementary events.

In the last century, a lot of researcher's works with some two-dimensional systems like complex, dual and hyperbolic number systems which have the most significant roles in algebraic, geometric, physics, engineering, etc. The geometry of the Euclidean plane, the Minkowski plane and the Gallian plane can be described with the help of complex numbers

$$\mathbb{C} = \{a + \mathbf{i}b : a, \ \ b \in \mathbb{R}, \mathbf{i}^2 = -1\},$$

dual numbers

$$\mathbb{D} = \{a + \varepsilon b : a, b \in \mathbb{R}, \ \ \varepsilon^2 = 0\}$$

and hyperbolic numbers

$$\mathbb{H} = \{a + \mathbf{h}b : a, b \in \mathbb{R}, \mathbf{h}^2 = 1\}.$$

We know that the complex numbers, dual numbers and hyperbolic numbers can be described as the quotient of the polynomial ring $\mathbb{R}[x]$ by the ideal generated by the polynomials $x^2 + 1$, x^2 and $x^2 - 1$ respectively, i.e.,

$$\mathbb{C} = \mathbb{R}[x]/ < x^2 + 1 >, \quad \mathbb{D} = \mathbb{R}[x]/ < x^2 >, \qquad \mathbb{H} = \mathbb{R}[x]/ < x^2 - 1 >$$

For a piece of detailed information about complex, dual and hyperbolic

numbers one can read some outstanding articles and books [1], [5], [11], [2], [3], [4]. Now any combination of the complex, dual and hyperbolic numbers satisfying the relation $\mathbf{ih} = -\mathbf{hi} = \mathbf{i} + \varepsilon$ forms a new non-commutative ring [9] and opens a new direction of thoughts, which we discuss in the following section.

Definition 1.1: [9] *The set of hybrid numbers, denoted by* $\mathbb{K}$, *contains complex, dual and hyperbolic numbers as well as any combination of these three types of numbers with* $\mathbf{ih} = -\mathbf{hi} = \mathbf{i} + \varepsilon$ *i.e.,*

$$\mathbb{K} = \{Z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h} : a, b, c, d \in \mathbb{R}, \mathbf{i}^2 = -1,\ \varepsilon^2 = 0, \mathbf{h}^2 = 1\ , \quad \mathbf{ih} = -\mathbf{hi} = \mathbf{i} + \varepsilon\}$$

The geometry corresponding to the hybrid numbers is called **Hybridian plane geometry**. This plane is a two-dimensional subspace of $\mathbb{R}^4$. The real part *'a'* of the hybrid number $Z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$ is called **scalar part** and is denoted by $S(Z)$ whereas the remaining part $b\mathbf{i} + c\varepsilon + d\mathbf{h}$ is called **vector part** and is denoted by $\mathbf{V(Z)}$.

Definition 1.2: [9] **(*Conjugate of a hybrid number*)** *The conjugate of a hybrid number* $Z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$ *is denoted by* $\overline{Z}$ *and is defined by*

$$\overline{Z} = S(Z) - V(Z) = a - b\mathbf{i} - c\varepsilon - d\mathbf{h}\ .$$

This conjugation of a hybrid number is additive, involutive and multiplicative operation on $\mathbb{K}$. i.e., for any two hybrid numbers Z_1 and Z_2

a. $\overline{Z_1 + Z_2} = \overline{Z_1} + \overline{Z_2}$

b. $\overline{(\overline{Z_1})} = Z_1$

c. $\overline{Z_1.Z_2} = \overline{Z_1}.\overline{Z_2}$

Definition 1.3: [9] **(*Character of a hybrid number*)** *The real number* $C(Z) = Z\overline{Z} = a^2 + (b-c)^2 - c^2 - d^2$ *is called the character of the hybrid number Z.*

Since, $C(Z) \in \mathbb{R}$, so depending on the value of $C(Z)$ a hybrid number can be categorized into three parts, spacelike, timelike, or lightlike according as the character is negative, positive or zero.

Also, the real number $\sqrt{|C(Z)|}$ is called the **norm** of the hybrid number Z and is denoted by $\| Z \|$.

Definition 1.4: [9] *The inverse of the hybrid number* $Z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$, $\| Z \| \neq 0$ *is defined as*

$$Z^{-1} = \frac{\bar{Z}}{C(Z)}.$$

Therefore, we can conclude that a lightlike hybrid number never possesses an inverse. So if we consider a set of zero divisors $\mathfrak{Z}$ of the ring of hybrid numbers then it contains all non-zero lightlike hybrid numbers.

Definition 1.5: [9] **(*Hybrid vector*)** *For the hybrid number* $Z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$, *the vector* $\varepsilon_Z = ((b-c),(c,d)$ *is called the hybrid vector of* Z.

Definition 1.6: [9] **(*Type of a hybrid number*)** *The real number* $\mathcal{T}_\varepsilon(Z) = -(b-c)^2 + c^2 + d^2$ *is called the type of the hybrid number Z.*

Depending on the real value of the type of a hybrid number, it is classified as complike (elliptic), hiperlike (hyperbolic) or duallike (parabolic) for $\mathcal{T}_\varepsilon(Z) < 0$, $\mathcal{T}_\varepsilon(Z) > 0$ or $\mathcal{T}_\varepsilon(Z) = 0$ respectively. Also, the real number, $\sqrt{|\mathcal{T}_\varepsilon(Z)|}$ is called the norm of the hybrid vector of Z and is denoted by $\mathcal{N}(Z)$.

1.1 Idempotent Representation of hyperbolic Hybrid Numbers: Let us denote the set of hyperbolic hybrid numbers as $\mathcal{P}$. Every hyperbolic hybrid number $Z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$ can be written [9] as

$$Z = s + \mathbf{H}v$$

where

$$\mathbf{H} = \frac{b\mathbf{i} + c\varepsilon + d\mathbf{h}}{\mathcal{N}(Z)} \quad \text{and} \quad \mathbf{H}^2 = 1.$$

Now, let us consider two numbers $\mathbf{e}_+ = \frac{1+\mathbf{H}}{2}$ and $\mathbf{e}_- = \frac{1-\mathbf{H}}{2}$. These two numbers are satisfying the equalities, $\| \mathbf{e}_+ \| = \| \mathbf{e}_- \| = \mathbf{e}_+\mathbf{e}_- = 0$, $\mathbf{e}_+^2 = \mathbf{e}_+$ and $\mathbf{e}_-^2 = \mathbf{e}_-$ and is called idempotent elements. Since, $\| \mathbf{e}_+ \| = \| \mathbf{e}_- \| = 0$, so $\mathbf{e}_+, \mathbf{e}_- \in \mathfrak{Z}$ and both are mutually complementary idempotent elements. Thus, the two principal ideals $\mathcal{P}_{\mathbf{e}+} := \mathbf{e}_+.\mathcal{P}$ and $\mathcal{P}_{\mathbf{e}-} := \mathbf{e}_-.\mathcal{P}$ in the ring $\mathcal{P}$ have the following properties

$$\mathcal{P}_{\mathbf{e}+} \cap \mathcal{P}_{\mathbf{e}-} = \{0\} \quad \text{and} \quad \mathcal{P} = \mathcal{P}_{\mathbf{e}+} \cup \mathcal{P}_{\mathbf{e}-} \tag{1.1}$$

and property 1.1 is known as the idempotent decomposition of $\mathcal{P}$. With the help of these idempotent elements $\mathbf{e}_+$ and $\mathbf{e}_-$, every hyperbolic hybrid number can be uniquely expressed as their linear combination

$$Z = (s+v)\mathbf{e}_+ + (s-v)\mathbf{e}_- = z_+\mathbf{e}_+ + z_-\mathbf{e}_- \tag{1.2}$$

where $z_+ = s + v \in \mathbb{R}$ and $z_- = s - v \in \mathbb{R}$ and this representation is called the idempotent representation of a hyperbolic hybrid number.

Lemma 1.1: The *sets* $\mathcal{P}_{\mathbf{e}+} = \{r_1\mathbf{e}_+ : r_1 \in \mathbb{R}\} = \mathbb{R}\mathbf{e}_+$ *and* $\mathcal{P}_{\mathbf{e}-} = \{r_2\mathbf{e}_- : r_2 \in \mathbb{R}\} = \mathbb{R}\mathbf{e}_-$ *satisfy the following properties*

a. $\mu \in \mathcal{P}\mathbf{e}_+ \Longleftrightarrow \mu\mathbf{e}_+ = \mu$;

b. $\nu \in \mathcal{P}_{\mathbf{e}-} \Longleftrightarrow \nu\mathbf{e}_- = \nu$.

Lemma 1.2: *Let us consider a set* $\mathcal{P}^+ = \{z_+\mathbf{e}_+ + z_-\mathbf{e}_- : z_+, z_- \geq 0\}$ *and a relation* $\succeq$ *on* $\mathcal{P}$ *in such a way that for any two hyperbolic hybrid numbers* ζ_1 *and* ζ_2, $\zeta_1 \succeq \zeta_2$ *if and only if* $\zeta_2 - \zeta_1 \in \mathcal{P}^+$.

Clearly this relation is a reflexive, antisymmetric and transitive relation and hence it is a poset (partial order relation) on $\mathcal{P}$.

The poset have the following properties:

For $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \mathcal{P}$ and $\xi \in \mathcal{P}^+$,

a. If $\zeta_1 \preceq \zeta_2$ and $\xi\zeta_1 \preceq \xi\zeta_2$.

b. If $\zeta_1 \preceq \zeta_2$ and $\zeta_3 \preceq \zeta_4$ then $\zeta_1 + \zeta_3 \preceq \zeta_2 + \zeta_4$.

c. If $\zeta_1 \preceq \zeta_2$ then $-\zeta_2 \preceq -\zeta_1$.

Before proving our main results, we have to go through following definition.

Definition 1.7: *Consider the set* $\mathcal{P} \cup \{0, 1\}$ *as* $\mathfrak{H}$ *and let* (Ω, Σ) *be a measurable space, a function* $P_{\mathfrak{H}} : \Sigma \mapsto \mathfrak{H}$ *with the properties*:

1. *for any event* $A \in \Sigma$, $P_{\mathfrak{H}}(A) \succeq 0$,

2. *for the certain event* Ω, $P_{\mathfrak{H}}(\Omega) = \mathfrak{p}$, *where* $\mathfrak{p}$ *is either* 1 *or* $\mathbf{e}_+$ *or* $\mathbf{e}_-$, *and*

3. *for a given sequence* $\{A_n\} \subset \Sigma$ *of pairwise disjoint events,*

$$P_{\mathfrak{H}}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P_{\mathfrak{H}}(A_n)$$

is called a $\mathfrak{H}$*-valued probabilistic measure, or a* $\mathfrak{H}$*-valued probability, on the* σ*-algebra of events* Σ *and the triplet* $(\Omega, \Sigma, P_{\mathfrak{H}})$ *is called a* $\mathfrak{H}$*-probabilistic space.*

Since, $\mathcal{P}_{\mathfrak{H}}(A) \in \mathfrak{H}$. So, it can be expressed as

$$\mathcal{P}_{\mathfrak{H}}(A) = \mathcal{P}_1(A)\mathbf{e}_+ + \mathcal{P}_2(A)\mathbf{e}_-.$$

Now property 1. of $\mathfrak{H}$-valued probabilistic measure implies that

$$P_1(A) \geq 0 \quad \text{and} \quad P_2(A) \geq 0\,,\ \forall A \in \Sigma\,.$$

From property 2, we get that

$$P_{\mathfrak{H}}(\Omega) = \mathfrak{p} = P_1(\Omega)\mathbf{e}_+ + P_2(\Omega)\mathbf{e}_-,$$

i.e.,

(I) If $\mathfrak{p} = 1$ then $P_1(\Omega) = 1, P_2(\Omega) = 1\,.$

(II) If $\mathfrak{p} = \mathbf{e}_+$ then $P_1(\Omega) = 1,\ P_2(\Omega) = 0\,.$

(III) If $\mathfrak{p} = \mathbf{e}_-$ then $P_1(\Omega) = 0,\ P_2(\Omega) = 1\,.$

The property 3. of $P_{\mathfrak{H}}$ implies

$$P_i\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P_i(A_n) \quad \text{for } i = 1 \text{ and } 2.$$

Therefore one can say that, in general, the $\mathfrak{H}$-valued probabilis- tic measure is equivalent if we consider a pair of unrelated usual $\mathbb{R}$-valued measures on the same measurable space.

Here we can observe that P_1 is a probabilistic measure in the cases (I) and (II); P_2 is a probabilistic measure in the cases (I) and (III) whereas P_2 and P_1 are trivial measures for the cases (II) and (III) respectively. The cases (II) and (III) can be seen as two different embedding of the $\mathbb{R}$-valued probabilistic measures into our new concept of $\mathfrak{H}$-valued probabilistic measures: we identify such real-valued measures with $\mathfrak{H}$-probabilistic measures which takes as its values only zero divisors.

Now we are in a position to prove our main results.

2. Main Results

In this section, we have discussed the usual properties of $\mathfrak{H}$-valued

Probabilistic Measures.

Theorem 2.1: *Let* $(\Omega, \Sigma, P_{\mathfrak{H}})$ *be a* $\mathfrak{H}$*-probabilistic space. Then*

(i) *the* $\mathfrak{H}$*-valued probability of the null event and a complementary event of* $A \in \Sigma$ *are respectively,* 0 *and* $\mathfrak{p} - P_{\mathfrak{H}}(A)$.

(ii) *if* $A_1, A_2, ..., A_n \in \Sigma$ *are n events, then*

$$P_{\mathfrak{H}}\left(\bigcup_{i=1}^{n} A_n\right) = \sum_{i=1}^{n} P_{\mathfrak{H}}(A_i) - \sum_{1\le i<j\le n} P_{\mathfrak{H}}(A_i \cap A_j)$$

$$+ \sum_{1\le i<j<k\le n} P_{\mathfrak{H}}(A_i \cap A_j \cap A_k) \tag{2.1}$$

$$- \ldots + (-1)^{n-1} P_{\mathfrak{H}}(A_1 \cap A_2 \cap .... \cap A_n).$$

(iii) *if* $A,\ B \in \Sigma$ *with* $A \subset B$ *then*

$$P_{\mathfrak{H}}(A) \preceq P_{\mathfrak{H}}(B).$$

i.e., $P_{\mathfrak{H}}(A)$ *and* $P_{\mathfrak{H}}(B)$ *are comparable with respect to the partial order* $\preceq$.

(iv) *given n events* $A_1, A_2, \ldots, A_n$ *there follows*:

$$P_{\mathfrak{H}}\left(\bigcup_{i=1}^{n} A_i\right) \preceq \sum_{i=1}^{n} P_{\mathfrak{H}}(A_i). \tag{2.2}$$

(v) *for any n events* $A_1, A_2, \ldots, A_n$ *the following relation holds*

$$P_{\mathfrak{H}}\left(\bigcup_{i=1}^{n} A_i\right) \succeq \sum_{i=1}^{n} P_{\mathfrak{H}}(A_i) - \sum_{1\le i<j\le n} P_{\mathfrak{H}}(A_i \cap A_j) \tag{2.3}$$

(vi) ***Continuity theorem of the*** $\mathfrak{H}$***-probability***:

if $A_1 \subset A_2 \subset \ldots \subset A_n \subset \ldots$ *and* $A := A_1 \cup A_2 \cup \ldots \cup A_n \cup \ldots$ *then*

$$\lim_{n\to\infty} P_{\mathfrak{H}}(A_n) = P_{\mathfrak{H}}(A) = P_{\mathfrak{H}}\left(\bigcup_{n=1}^{\infty} A_n\right).$$

(vii) *for any two events* A *and* B,

$$P_{\mathfrak{H}}(A^C \cap B) = P(B) - P(A \cap B)$$

Proof: (i) We know that $A \cup A^C = \Omega$, $A \cap A^C = \varnothing$ and so that

$$P_{\mathfrak{H}}(A) + P_{\mathfrak{H}}(A^C) = P_{\mathfrak{H}}(\Omega) = \mathfrak{p}$$

$$P_{\mathfrak{H}}(A^C) = \mathfrak{p} - P_{\mathfrak{H}}(A) .$$

Now, $P_{\mathfrak{H}}(\varnothing) = P_{\mathfrak{H}}(\Omega^C) = \mathfrak{p} - P(\Omega) = 0$.

Hence, it follows the results.

(ii) By the help of mathematical induction we can prove the theorem. Since for $n = 2$, $A_1 \cup A_2 = A_1 \cup ({A_1}^C \cap A_2)$ and also $A_2 = (A_1 \cap A_2) \cup ({A_1}^C \cap A_2)$, then

$$\begin{aligned} P_{\mathfrak{H}}(A_1 \cup A_2) &= P_{\mathfrak{H}}(A_1) + P_{\mathfrak{H}}({A_1}^C \cap A_2) \\ &= P_{\mathfrak{H}}(A_1) + P_{\mathfrak{H}}(\mathrm{A}_2) - P_{\mathfrak{H}}(A_1 \cap A_2). \end{aligned}$$

Therefore, the equation (2.1) holds for $n = 2$.

Now, suppose that the equation (2.1) is true for $n = r$.

i.e,.

$$P_{\mathfrak{H}}\left(\bigcup_{i=1}^{r} A_i\right) = \sum_{i=1}^{r} P_{\mathfrak{H}}(A_i) - \sum_{1\le i<j\le r} P_{\mathfrak{H}}(A_i \cap A_j)$$

$$+ \sum_{1\le i<j<k\le r} P_{\mathfrak{H}}(A_i \cap A_j \cap A_k) - \ldots + (-1)^{r-1} P_{\mathfrak{H}}(A_1 \cap A_2 \cap \ldots. \cap A_r).$$

Then, for $n = r + 1$

$$P_{\mathfrak{H}}(A_1 \cup \ldots \cup A_r \cup A_{r+1}) = P_{\mathfrak{H}}\left(\left[\bigcup_{i=1}^{r} A_i\right] \cup A_{r+1}\right)$$

$$= P_{\mathfrak{H}}\left(\bigcup_{i=1}^{r} A_i\right) + P_{\mathfrak{H}}(A_{r+1}) - P_{\mathfrak{H}}\left(\left[\bigcup_{i=1}^{r} A_i\right] \cap A_{r+1}\right)$$

$$= \sum_{i=1}^{r+1} P_{\mathfrak{H}}(A_i) - \sum_{1\le i<j\le r} P_{\mathfrak{H}}(A_i \cap A_j) + \sum_{1\le i<j<k\le r} P_{\mathfrak{H}}(A_i \cap A_j \cap A_k)$$

$$- \ldots + (-1)^{r-1} P_{\mathfrak{H}}(A_1 \cap A_2 \cap \ldots \cap A_r) - \left[\sum_{i=1}^{r} P_{\mathfrak{H}}\,(A_i A_{r+1})\right.$$

$$- \sum_{1\le i<j\le r} P_{\mathfrak{H}}(A_i A_j A_{r+1})$$

$$\left. + \ldots + (-1)^{r-1} P_{\mathfrak{H}}(A_1 \cap A_2 \cap \ldots \cap A_{r+1})\right]$$

$$= \sum_{i=1}^{r+1} P_{\mathfrak{H}}(A_i) - \sum_{1\le i<j\le r+1} P_{\mathfrak{H}}(A_i \cap A_j)$$

$$+ \sum_{1\le i<j<k\le r+1} P_{\mathfrak{H}}(A_i \cap A_j \cap A_k)$$

$$- \ldots + (-1)^{n} P_{\mathfrak{H}}(A_1 \cap A_2 \cap \ldots \cap A_{r+1})$$

Thus, the equation (2.1) is also true for $n = r + 1$ and hence this proves the theorem.

This result is the extension of 'Addition Theorem' in probability.

(iii) As $B = B \cap \Omega = B \cap (A \cup A^C) = (B \cap A) \cup (B \cap A^C) = A \cup (A^C \cap B)$. Since, $A \cap (A^C \cap B) = \varnothing$, thus, $P_{\mathfrak{H}}(B) = P_{\mathfrak{H}}(A) + P_{\mathfrak{H}}(A^C \cap B)$, and also since $P_{\mathfrak{H}}(A^C \cap B) \succeq 0$, so we get $P_{\mathfrak{H}}(A) \preceq P_{\mathfrak{H}}(B)$.

(iv) Since for any two events A_1 and A_2, we have

$$P_{\mathfrak{H}}(A_1 \cup A_2) = P_{\mathfrak{H}}(A_1) + P_{\mathfrak{H}}(A_2) - P_{\mathfrak{H}}(A_1 \cap A_2)$$

$$\preceq P_{\mathfrak{H}}(A_1) + P_{\mathfrak{H}}(A_2) \qquad [\because P_{\mathfrak{H}}(A_1 \cap A_2) \succeq 0]$$

Thus, the result (2.2) is true for $n = 2$.

Let us suppose that the result (2.2) is true for $n = m$.

Now, for $n = m + 1$,

$$P_{\mathfrak{H}}(A_1 \cup A_2 \cup \ldots \cup A_{m+1}) \preceq P_{\mathfrak{H}}(A_1 \cup A_2 \cup \ldots \cup A_m) + P_{\mathfrak{H}}(A_{m+1})$$

$$\preceq P_{\mathfrak{H}}(A_1) + P_{\mathfrak{H}}(A_2) + \ldots + P_{\mathfrak{H}}(A_{m+1})$$

Thus, the result (2.2) is true for all $n \in \mathbb{N}$ and hence the result known as 'Boole's inequality' also hold in $\mathfrak{H}$-probabilistic space.

From the addition theorem, we get

$$P_{\mathfrak{H}}(A_1 \cup A_2 \cup A_3) = P_{\mathfrak{H}}(A_1) + P_{\mathfrak{H}}(A_2) + P_{\mathfrak{H}}(A_3) - P_{\mathfrak{H}}(A_1 \cap A_2)$$

$$- P_{\mathfrak{H}}(A_2 \cap A_3) - P_{\mathfrak{H}}(A_3 \cap A_1) + P_{\mathfrak{H}}(A_1 \cap A_2 \cap A_3)$$

$$P_{\mathfrak{H}}\left(\bigcup_{i=1}^{3} A_i\right) \succeq \sum_{i=1}^{3} P_{\mathfrak{H}}(A_i) \sum_{1 \le i < j \le 3} P_{\mathfrak{H}}(A_i \cap A_j)$$

Thus, the result (2.3) is true for $n = 3$. Now, we will prove the result by the help of mathematical induction.

Let us suppose that the result (2.3) is true for $n = k$, i.e.,

$$P_{\mathfrak{H}}\left(\bigcup_{i=1}^{k} A_i\right) \succeq \sum_{i=1}^{k} P_{\mathfrak{H}}(A_i) - \sum_{1\le i<j\le k} P_{\mathfrak{H}}(A_i \cap A_j)$$

Now, for $n = k + 1$

$$P_{\mathfrak{H}}\left(\bigcup_{i=1}^{k+1} A_i\right) = P_{\mathfrak{H}}\left(\left[\bigcup_{i=1}^{k} A_i\right] \cup A_{k+1}\right)$$

$$= P_{\mathfrak{H}}\left(\bigcup_{i=1}^{k} A_i\right) + P_{\mathfrak{H}}(A_{k+1}) - P_{\mathfrak{H}}\left(\left[\bigcup_{i=1}^{k} A_i\right] \cap A_{k+1}\right)$$

$$= P_{\mathfrak{H}}\left(\bigcup_{i=1}^{k} A_i\right) + P_{\mathfrak{H}}(A_{k+1}) - P_{\mathfrak{H}}\left(\bigcup_{i=1}^{k} (A_i \cap A_{k+1})\right)$$

$$\succeq \left[\sum_{i=1}^{k} P_{\mathfrak{H}}(A_i) - \sum_{1\le i<j\le k} P_{\mathfrak{H}}(A_i \cap A_j)\right] + P_{\mathfrak{H}}(A_{k+1}) - P_{\mathfrak{H}}\left(\bigcup_{i=1}^{k} (A_i \cap A_{k+1})\right)$$

From the result-(iv), we get

$$-P_{\mathfrak{H}}\left(\bigcup_{i=1}^{k} (A_i \cap A_{k+1})\right) \succeq -\sum_{i=1}^{k} P_{\mathfrak{H}}(A_i \cap A_{k+1})$$

and therefore,

$$P_{\mathfrak{H}}\left(\bigcup_{i=1}^{k+1} A_i\right) \succeq \sum_{i=1}^{k+1} P_{\mathfrak{H}}(A_i) - \sum_{1\le i<j\le k} P_{\mathfrak{H}}(A_i \cap A_j)$$

$$- \sum_{i=1}^{k} P_{\mathfrak{H}}(A_i \cap A_{k+1})$$

$$P_{\mathfrak{H}}\left(\bigcup_{i=1}^{k+1} A_i\right) \succeq \sum_{i=1}^{k+1} P_{\mathfrak{H}}(A_i) - \sum_{1 \le i < j \le k+1} P_{\mathfrak{H}}(A_i \cap A_j)$$

Thus, the theorem is also true $n = k + 1$ and hence this proves the result.

(vi) Since the given sequence of events is expanding, so we get

$$\bigcup_{i=1}^{n} A_i = A_n \qquad \text{and} \qquad \lim_{n \to \infty} A_n = \bigcup_{i=1}^{\infty} A_n \tag{2.4}$$

Now if we take some events as

$$B_1 = A_1, B_2 = A_2 \cap A_1^C, \ldots, B_n = A_n \cap A_{n-1}^C ,$$

then $B_i \cap B_j = \phi$, for all $i \neq j$ and $i,\ j = 1, 2, \ldots, n$ and hence

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n \quad \text{and} \quad P_{\mathfrak{H}}\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} P_{\mathfrak{H}}(B_n) \tag{2.5}$$

Now, from (2.4) and (2.5), we get

$$P_{\mathfrak{H}}\left(\lim_{n \to \infty} A_n\right) = P_{\mathfrak{H}}\left(\bigcup_{n=1}^{\infty} A_n\right)$$

$$= \sum_{n=1}^{\infty} P_{\mathfrak{H}}(B_n)$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} P_{\mathfrak{H}}(B_i)$$

$$= \lim_{n\to\infty} P_{\mathfrak{H}}\left(\bigcup_{i=1}^{n} B_i\right)$$

$$= \lim_{n\to\infty} P_{\mathfrak{H}}\left(\bigcup_{i=1}^{n} A_i\right) = \lim_{n\to\infty} P_{\mathfrak{H}}(A)$$

This completes the proof.

(vii) Since, $A^C \cap B$ and $A \cap B$ are disjoint events and $(A^C \cap B) \cup (A \cap B) = B$, then

$$P_{\mathfrak{H}}(B) = P_{\mathfrak{H}}(A^C \cap B) + P_{\mathfrak{H}}(A \cap B)$$

or, $$P_{\mathfrak{H}}(A^C \cap B) = P_{\mathfrak{H}}(B) - P_{\mathfrak{H}}(A \cap B).$$

□

Corollary 2.1.1: *Since, for given* $A \in \Sigma$, $A \subseteq \Omega$, *therefore* $P_{\mathfrak{H}}(A)$ *is always comparable with* $P_{\mathfrak{H}}(\Omega)$ *and also,* $P_{\mathfrak{H}}(A) \preceq P_{\mathfrak{H}}(\Omega) = \mathfrak{p}$. *Hence, for any* $A \in \Sigma$, $0 \preceq P_{\mathfrak{H}}(A) \preceq \mathfrak{p}$.

Corollary 2.1.2: *If* $P\mathfrak{H}(\Omega) = e_+$ *then for any random event* A *there holds that* $P_{\mathfrak{H}}(A)$ *is of the form* λe_+ *with* $\lambda \in [0,1]$ *. If* $P\mathfrak{H}(\Omega) = e_-$ *then for any random events* A *there holds that* $P_{\mathfrak{H}}(A)$ *is of the form* μe_- *with* $\mu \in [0,1]$.

Corollary 2.1.3: *For any n events* $A_1, A_2, \ldots, A_n$

$$P_{\mathfrak{H}}(A_1 \cap A_2 \cap \ldots \cap A_n) \succeq \mathfrak{p} - \sum_{i=1}^{n} P_{\mathfrak{H}}(A_i^C) \tag{2.6}$$

and $$P_{\mathfrak{H}}(A_1 \cap A_2 \cap \ldots \cap A_n) \succeq \sum_{i=1}^{n} P_{\mathfrak{H}}(A_i) - (n-1)\mathfrak{p} \tag{2.7}$$

Proof: Applying the 'Boole's inequality' on the events $A_1^C, A_2^C, \ldots, A_n^C$, then we get

$$P_{\mathfrak{H}}(A_1^C \cup A_2^C \cup \ldots \cup A_n^C) \preceq \sum_{i=1}^{n} P_{\mathfrak{H}}(A_i^C)$$

or,

$$P_{\mathfrak{H}}\left[(A_1 \cap A_2 \cap \ldots \cap A_n)^C\right] \preceq \sum_{i=1}^{n} P_{\mathfrak{H}}(A_i^C)$$

or,
$$P_{\mathfrak{H}}(A_1 \cap A_2 \cap \ldots \cap A_n) \succeq \mathfrak{p} - \sum_{i=1}^{n} P_{\mathfrak{H}}(A_i^C) \tag{2.8}$$

Again,

$$\sum_{i=1}^{n} P_{\mathfrak{H}}(A_i^C) = \mathfrak{p} - P_{\mathfrak{H}}(A_1)\mathfrak{p} - P_{\mathfrak{H}}(A_2) + \ldots \mathfrak{p} - P_{\mathfrak{H}}(A_n)$$

$$= n\mathfrak{p} - \sum_{i=1}^{n} P_{\mathfrak{H}}(A_i^C) \tag{2.9}$$

Now, combining (2.8) and (2.9), we get

$$P_{\mathfrak{H}}(A_1 \cap A_2 \cap \ldots \cap A_n) \succeq \sum_{i=1}^{n} P_{\mathfrak{H}}(A_i) - (n-1)\mathfrak{p}\,.$$

□

The results (2.6) and (2.7) is the 'Bonferroni's inequalities' in $\mathfrak{H}$-valued probabilistic space.

Corollary 2.1.4: *If* $A_1 \supset A_2 \supset \ldots \supset A_n \supset \ldots$

and

$$A := A_1 \cap A_2 \cap \ldots \cap A_n \cap \ldots \cap, \textit{then}$$

$$\lim_{n\to\infty} P_{\mathfrak{H}}(An) = P_{\mathfrak{H}}(A) = P_{\mathfrak{H}}\left(\bigcap_{n=1}^{\infty} A_n\right).$$

3. Future Prospect

In the line of the works as carried out in the paper one may think of to establish the hyperbolic hybrid valued conditional probability and may look into the matter of independency of events and also may try to revisit some well known results like multiplication theorem, law of total probability and Bayes' theorem. This may be regarded as an active area of research to the future workers in this branch.

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