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GIBONACCI SUMS INVOLVING TRIANGULAR NUMBERS

Abstract: We explore consequences of a generalization of an infinite gibonacci sum involving triangular numbers.

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1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. They can also be defined by the *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 3].

In the interest of brevity, clarity, and convenience, we omit the argument in

the functional notation, when there is no ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $\Delta = \sqrt{x^2 + 4}$ and $2\alpha = x + \Delta$. With $2\alpha = 1 + \sqrt{5}$, have $\alpha^3 = 2 + \sqrt{5}$, $\alpha^6 = 9 + 4\sqrt{5}$, and $2\alpha^{10} = 123 + 55\sqrt{5}$.

Triangular numbers t_n are positive integers that can pictorially be represented by equilateral triangular arrays. They can be defined recursively by $t_n = t_{n-1} + n$, where $t_1 = 1$ and $n \geq 2$ [2]. The first five triangular numbers are 1, 3, 6, 10, and 15.

2. Gibonacci Sums

A generalization of a sum involving gibonacci polynomial squares is studied in [6]. To showcase it, we first let

$$\mu = \begin{cases} 1, & if\ g_n = f_n \\ \Delta^2, & otherwise; \end{cases} \quad \text{and} \quad \nu^* = \begin{cases} 1, & if\ g_n = f_n \\ -1, & othewrwise. \end{cases}$$

With these tools at our fingertips, we now present the cornerstone of our discourse [6].

Theorem 1: *Let k, p, r, and t be positive integers, where $t \leq 2p$. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk} \mu \nu^* f_r f_{2pk}}{g^2_{(2pn+t-p)k} - (-1)^{tk} \mu \nu^* f^2_{pk}} = \frac{g_{tk+r}}{g_{tk}} - \alpha^r. \tag{1}$$

The objective of our discourse is to explore consequences of this theorem when k, p, r, and t are triangular numbers ≤ 15. To this end, in the interest of brevity and expediency, we use the labels below in the explorations to follow.

A_1	=	10,336;	B_7	=	4,870,800;
A_2	=	11,552;	B_8	=	6,656,320;
A_3	=	11,556;	B_9	=	6,674,746;
A_4	=	13,500;	B_{10}	=	6,677,056;
A_5	=	13,530;	B_{11}	=	13,271,424;

A_6	=	15,125;	B_{12}	=	13,354,112;
A_7	=	18,040;	B_{13}	=	13,358,736;
A_8	=	20,736;	B_{14}	=	14,930,352;
A_9	=	20,801;	B_{15}	=	26,708,224;
A_{10}	=	25,840;	B_{16}	=	29,860,704;
A_{11}	=	27,060;	B_{17}	=	33,385,280;
A_{12}	=	30,258;	B_{18}	=	52,721,408;
A_{13}	=	35,424;	B_{19}	=	59,721,408;
A_{14}	=	60,885;	B_{20}	=	66,747,456;
A_{15}	=	67,650;	B_{21}	=	66,770,568;
B_1	=	103,361;	B_{22}	=	67,186,584;
B_2	=	104,005;	B_{23}	=	149,303,520;
B_3	=	187,575;	C_1	=	276,058,890,624;
B_4	=	248,050;	C_2	=	868,059,063;603;
B_5	=	802,780;	C_3	=	1,380,294,453,120;
B_6	=	1,845,492;	C_4	=	3,474,165,668,824.

Case 1: Suppose $p = 1$. Recalling that $t = 1$ and with $k, r \in \{1, 3, 6, 10\}$, the theorem yields [6]:

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n}^2 + 1} = -\frac{1}{2} + \frac{\sqrt{5}}{2}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2n}^2 - 5} = \frac{1}{2} - \frac{\sqrt{5}}{10};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n+2}^2 + 1} = -1 + \frac{\sqrt{5}}{2}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2n+2}^2 - 5} = \frac{1}{4} - \frac{\sqrt{5}}{10};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{6n}^2 + 4} = -\frac{1}{8} + \frac{\sqrt{5}}{16}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{6n}^2 - 20} = \frac{1}{32} - \frac{\sqrt{5}}{80};$$

$$\sum_{n=1}^{\infty}\frac{1}{F_{12n}^2-64}=\frac{1}{128}-\frac{\sqrt{5}}{288};\qquad \sum_{n=1}^{\infty}\frac{1}{L_{12n}^2+320}=-\frac{1}{648}+\frac{\sqrt{5}}{1,440};$$

$$\sum_{n=1}^{\infty}\frac{1}{F_{20n}^2-3,025}=\frac{1}{6,050}-\frac{\sqrt{5}}{A_5};\qquad \sum_{n=1}^{\infty}\frac{1}{L_{20n+1}^2+A_6}=-\frac{5}{A_{12}}+\frac{\sqrt{5}}{A_{15}}.$$

Case 2: Let $p=3$. With $k,r\in\{1,3,6,10\}$ and $t\in\{1,3,6\}$, the theorem implies [4, 5]:

$$\sum_{n=1}^{\infty}\frac{1}{F_{6n-2}^2+4}=-\frac{1}{16}+\frac{\sqrt{5}}{16};\qquad \sum_{n=1}^{\infty}\frac{1}{L_{6n-2}^2-20}=\frac{1}{16}-\frac{\sqrt{5}}{80};$$

$$\sum_{n=1}^{\infty}\frac{1}{F_{6n}^2+4}=-\frac{1}{8}+\frac{\sqrt{5}}{16};\qquad \sum_{n=1}^{\infty}\frac{1}{L_{6n}^2-20}=\frac{1}{32}-\frac{\sqrt{5}}{80};$$

$$\sum_{n=1}^{\infty}\frac{1}{F_{6n+3}^2-4}=\frac{9}{64}-\frac{\sqrt{5}}{16};\qquad \sum_{n=1}^{\infty}\frac{1}{L_{6n+3}^2+20}=-\frac{1}{36}+\frac{\sqrt{5}}{80};$$

$$\sum_{n=1}^{\infty}\frac{1}{F_{3(6n-2)}^2+3,025}=-\frac{1}{2,584}+\frac{\sqrt{5}}{5,168};\quad \sum_{n=1}^{\infty}\frac{1}{L_{3(6n-2)}^2-A_6}=\frac{1}{A_1}-\frac{\sqrt{5}}{A_{10}};$$

$$\sum_{n=1}^{\infty}\frac{1}{F_{3(6n)}^2+3,025}=-\frac{1}{2,312}+\frac{\sqrt{5}}{5,168};\qquad \sum_{n=1}^{\infty}\frac{1}{L_{3(6n)}^2-A_6}=\frac{1}{A_2}-\frac{\sqrt{5}}{A_{10}};$$

$$\sum_{n=1}^{\infty}\frac{1}{F_{3(6n+3)}^2-3,025}=\frac{2,889}{B_{10}}-\frac{\sqrt{5}}{5,168};\qquad \sum_{n=1}^{\infty}\frac{1}{L_{3(6n+3)}^2+A_6}=-\frac{1}{A_3}+\frac{\sqrt{5}}{A_{10}};$$

$$\sum_{n=1}^{\infty}\frac{1}{F_{6(6n-2)}^2-B_{10}}=\frac{1}{B_{11}}-\frac{\sqrt{5}}{B_{16}};\qquad \sum_{n=1}^{\infty}\frac{1}{L_{6(6n-2)}^2+B_{17}}=-\frac{1}{B_{22}}+\frac{\sqrt{5}}{B_{23}};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{6(6n)}^2 - B_{10}} = \frac{1}{B_{12}} - \frac{\sqrt{5}}{B_{16}}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{6(6n)}^2 + B_{17}} = -\frac{1}{B_{21}} + \frac{\sqrt{5}}{B_{23}};$$

Case 3: Let $p = 6$. With $k \in \{1, 3\}$, $r \in \{1, 3, 6\}$, and $t \in \{1, 3, 6, 10\}$, we get:

$$\sum_{n=1}^{\infty} \frac{1}{F_{12n-5}^2 + 64} = -\frac{1}{288} + \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{12n-5}^2 - 320} = \frac{1}{288} - \frac{\sqrt{5}}{1,440};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{12n-3}^2 + 64} = -\frac{1}{144} + \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{12n-3}^2 - 320} = \frac{1}{576} - \frac{\sqrt{5}}{1,440};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{12n}^2 - 64} = \frac{1}{128} - \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{12n}^2 + 320} = -\frac{1}{648} + \frac{\sqrt{5}}{1,440};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{12n+4}^2 - 64} = \frac{41}{5,280} - \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{12n+4}^2 + 320} = -\frac{55}{A_{13}} - \frac{\sqrt{5}}{1,440};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{3(12n-5)}^2 + 64} = -\frac{1}{B_{14}} + \frac{\sqrt{5}}{B_{16}}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{3(12n-5)}^2 - 320} = \frac{1}{B_{18}} - \frac{\sqrt{5}}{B_{23}};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{3(12n-3)}^2 + B_{10}} = -\frac{1}{B_{13}} + \frac{\sqrt{5}}{B_{16}}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{3(12n-3)}^2 - B_{17}} = \frac{1}{B_{21}} - \frac{\sqrt{5}}{B_{23}};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{3(12n)}^2 - B_{10}} = \frac{1}{B_{12}} - \frac{\sqrt{5}}{B_{16}}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{3(12n)}^2 + B_{17}} = -\frac{1}{B_{21}} + \frac{\sqrt{5}}{B_{23}};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{3(12n+2)}^2 - B_{10}} = \frac{B_1}{C_3} - \frac{\sqrt{5}}{B_{16}}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{3(12n+2)}^2 + B_{17}} = -\frac{1}{B_{11}} + \frac{\sqrt{5}}{B_{23}};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{3(12n-5)}^2 + B_{10}} = -\frac{1}{B_{14}} + \frac{\sqrt{5}}{B_{16}}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{3(12n-5)}^2 - B_{17}} = \frac{1}{B_{19}} - \frac{\sqrt{5}}{B_{23}};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{3(12n+4)}^2 - B_{10}} = \frac{B_1}{C_3} - \frac{\sqrt{5}}{B_{16}}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{3(12n+4)}^2 + B_{17}} = -\frac{B_2}{C_4} + \frac{\sqrt{5}}{B_{23}}.$$

Case 4: Next, we let $p = 10$. Then $t \le 20$. With $k = 1 = r$ and $t \in \{1, 3, 6, 10, 15\}$, the theorem yields:

$$\sum_{n=1}^{\infty} \frac{1}{F_{20n-9}^2 + 3,025} = -\frac{1}{A_5} + \frac{\sqrt{5}}{A_5}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{20n-9}^2 - A_6} = \frac{1}{A_5} - \frac{\sqrt{5}}{A_{15}};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{20n-7}^2 + 3,025} = -\frac{1}{6,765} + \frac{\sqrt{5}}{A_5}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{20n-7}^2 - A_6} = \frac{1}{A_{11}} - \frac{\sqrt{5}}{A_{15}};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{20n-4}^2 - 3,025} = \frac{3}{A_7} - \frac{\sqrt{5}}{A_5}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{20n-4}^2 + A_6} = -\frac{2}{A_{14}} + \frac{\sqrt{5}}{A_{15}};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{20n}^2 - 3,025} = \frac{1}{B_4} - \frac{\sqrt{5}}{A_5}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{20n}^2 + A_6} = -\frac{1}{A_{12}} + \frac{\sqrt{5}}{A_{15}};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{20n+5}^2 + 3,025} = -\frac{31}{B_3} + \frac{\sqrt{5}}{A_5}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{20n+5}^2 - A_6} = \frac{61}{B_6} - \frac{\sqrt{5}}{A_{15}}.$$

We now encourage gibonacci enthusiasts to compute the sums when $k = 3$ and $t \in \{1, 3, 6, 10, 15\}$.

Case 5: Finally, we let $p = 15$, and restrict $k = 1 = r$ and $t \in \{1, 3, 6, 10, 15\}$. With the labels

$$\begin{aligned} D1 &= 372,100; \quad E1 = 3,720,992; \\ D2 &= 744,200; \quad E2 = 3,744,180; \\ D3 &= 832,040; \quad E3 = 40,936,368, \end{aligned}$$

we then have:

$$\sum_{n=1}^{\infty}\frac{1}{F_{30n-14}^2+D_1}=-\frac{1}{2D_3}+\frac{\sqrt{5}}{2D_3};\qquad \sum_{n=1}^{\infty}\frac{1}{L_{30n-14}^2-5D_1}=\frac{1}{2D_3}-\frac{\sqrt{5}}{10D_3};$$

$$\sum_{n=1}^{\infty}\frac{1}{F_{30n-12}^2+D_1}=-\frac{1}{D_3}+\frac{\sqrt{5}}{2D_3};\qquad \sum_{n=1}^{\infty}\frac{1}{L_{30n-12}^2-5D_1}=\frac{1}{4D_3}-\frac{\sqrt{5}}{10D_3};$$

$$\sum_{n=1}^{\infty}\frac{1}{F_{30n-8}^2-D_1}=\frac{9}{8D_3}-\frac{\sqrt{5}}{2D_3};\qquad \sum_{n=1}^{\infty}\frac{1}{L_{30n-8}^2+5D_1}=-\frac{1}{E_2}+\frac{\sqrt{5}}{10D_3};$$

$$\sum_{n=1}^{\infty}\frac{1}{F_{30n-5}^2-D_1}=\frac{123}{11D_3}-\frac{\sqrt{5}}{2D_3};\qquad \sum_{n=1}^{\infty}\frac{1}{L_{30n-5}^2+5D_1}=-\frac{1}{E_3}+\frac{\sqrt{5}}{10D_3};$$

$$\sum_{n=1}^{\infty}\frac{1}{F_{30n}^2+D_1}=-\frac{1}{D_2}+\frac{\sqrt{5}}{2D_3};\qquad \sum_{n=1}^{\infty}\frac{1}{L_{30n}^2-5D_1}=-\frac{1}{E_1}+\frac{\sqrt{5}}{10D_3}.$$

Again, we encourage the gibonacci enthusiasts to compute the sums with $k\in\{3,6\}$.

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