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CLASSIFYING NUMBERS USING DIVISOR FUNCTION TO STUDY HIGHLY COMPOSITE NUMBERS AND TWIN PRIMES CONJECTURE

Abstract: In this note we classify the positive integers by using divisor function into different equivalence classes. We connect highly composite numbers with this definition. We consider pairs of consecutive odd numbers $(2n+1, 2n+3)$ with $\tau(2n+1) = \tau(2n+3)$ where τ is a divisor function in each equivalence class and call the pair as "twin odd numbers with equal number of divisors" (twin odds with ends). Using these equivalence classes we generalize the twin primes conjecture and study some special cases. We also provide computational data using Python to obtain some interesting results.

Keywords: Divisor Function, Highly Composite Numbers and Twin Primes Conjecture.

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1. Introduction

We consider the number theoretic divisor function $\tau(n)$ as follows:

Definition 1: *We define*

$$\tau(n) = \sum_{d|n} 1, d > 0$$

From the definition, $\tau(n)$ denotes the number of positive divisors of n. We consider the set of natural numbers $\mathbb{N}$ and define an equivalence relation on $\mathbb{N}$ as follows:

Two natural numbers $m \sim n$ are equivalent if and only if m and n have the same number of positive divisors. Therefore, for $m, n \in \mathbb{N}$

$$m \sim n \Leftrightarrow \tau(m) = \tau(n).$$

Clearly $\sim$ is an equivalence relation. Hence, we have got a disjoint union of equivalence classes of $\mathbb{N}$ and we denote them by $D_1, D_2, \ldots D_j \ldots$ where D_j contains the numbers with j positive divisors. Unless it is mentioned the divisors are meant to be positive divisors.

Clearly, D_1 contains only one element 1. D_2 denotes the set of prime numbers and D_3 contains squares of prime numbers.

Theorem 2: *D_j is infinite for $j > 1$.*

Proof: Since the number of primes are infinite D_2 is infinite. To show D_j is infinite we observe that for every prime p we consider $1, p, p^2, \ldots, p^{j-1}$ to conclude D_j is an infinite set. ■

In this note we study the following:

1. We relate highly composite numbers with this definition.
2. We consider subsets TO_j of D_j which consists of consecutive odd numbers and analyze them.
3. We generalize the twin primes conjecture on this basis.

Interested reader may refer to [1] to see some special cases of consecutive integers.

2. Highly Composite Numbers

In 1915, Srinivasa Ramanujan published a paper in Proceedings of London Mathematical Society with title "Highly Composite number". See [2].

A number N is highly composite if $M < N$ implies $\tau(M) < \tau(N)$. We define it another way with the following notation:

Definition 3: *A number $N \in D_n$ is said to be highly composite if every number $< N$ lies in D_k for some $k < n$.*

Example 4: *The first few highly composite numbers are* $1, 2, 4, 6, 12, 24, \ldots$ *with corresponding number of divisors* $1, 2, 3, 4, 6, 8, \ldots$.

Theorem 5: *There are infinitely many highly composite numbers.*

Proof: The proof follows by a simple reasoning. Suppose there is a highly composite number $N \in D_n$ with n divisors. Then we choose $N + K \in D_{n+k}$ which has $n + k$ divisors for some $k = 1, 2, \ldots$. Therefore, the next highly composite number will be the least number with $n + k$ divisors. This process can not be terminated. Hence, there are infinitely many composite numbers. ■

The idea of the proof can be well understood from the sequence of highly composite numbers given in the example above. Once 6 is chosen with 4 divisors the next number $6 + 6 = 12$ has $4 + 2 = 6$ divisors. So there is no highly composite number with 5 divisors !

Suppose we consider the sequence (l_j) where l_j is the least number from D_j. Clearly, highly composite numbers form a sub sequence of l_j.

Remark 6: *We can define the highly composite numbers is a maximal strictly monotonic increasing subsequence of l_j.*

Here the word maximal means the strictly monotonic sequence does not contain in any other monotonic increasing sequence. In the above table the second column denotes the first number of D_j and the third column contains the sequence of

highly composite numbers ! It is clear from the table that there are no highly composite numbers for, 250 with number of divisors 5, 13, 14, 15, 17, 19.

Table of equivalence classes containing equal number of divisors

	First number	Highly Composite Numbers	Divisors
D1	1	1	1
D2	2	2	2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127,131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241
D3	4	4	4, 9, 25, 49, 121, 169
D4	6	6	6, 8, 10, 14, 15, 21, 22, 26, 27, 33, 34, 35, 38, 39, 46, 51, 55, 57, 58, 62, 65, 69, 74, 77, 82, 85, 86, 87, 91, 93, 94, 95,106, 111, 115, 118, 119, 122, 123, 125, 129, 133, 134, 141, 142, 143, 145, 146, 155, 158, 159, 161, 166, 177, 178, 183, 185,187, 194, 201, 202, 203, 205, 206, 209, 213, 214, 215, 217, 218, 219, 221, 226, 235, 237, 247, 249
D5	16		16, 81
D6	12	12	12, 18, 20, 28, 32, 44, 45, 50, 52, 63, 68, 75, 76, 92, 98, 99, 116, 117, 124, 147, 148, 153, 164, 171, 172, 175, 188, 207,212, 236, 242, 243, 244, 245
D7	64		64
D8	24	24	24, 30, 40, 42, 54, 56, 66, 70, 78, 88, 102, 104, 105, 110, 114, 128, 130, 135, 136, 138, 152, 154, 165, 170, 174, 182,184, 186, 189, 190, 195, 222, 230, 231, 232, 238, 246, 248, 250
D9	36	36	36, 100, 196, 225
D10	48	48	48, 80, 112, 162, 176, 208
D11			No numbers found.

	First number	Highly Composite Numbers	Divisors
D12	60	60	60, 72, 84, 90, 96, 108, 126, 132, 140, 150, 156, 160, 198, 200, 204, 220, 224, 228, 234
D13			No numbers found.
D14	192		192
D15	144		144
D16	120	120	120, 168, 210, 216
D17			No numbers found
D18	180	180	180
D19			No numbers found
D20	240	240	240

Generated using Python

3. Generalization of Twin Primes

We consider special pairs of odd numbers which are existing in a very natural way and generalize the set of twin primes. We feel that these pairs of odd numbers are interesting and needs attention for further research in this area. Interestingly, twin primes conjecture turns out to be a special class in this classification.

Definition 7: *We define a pair of two consecutive odd numbers* $(2n+1, 2n+3)$ *is* ***twin odd numbers with ends*** *if both the numbers have equal number of positive divisors.*

In an obvious way all twin primes belong to this class of numbers with two divisors.

Let us denote the set of pairs of odd numbers belonging to the equivalence class D_j by TO_j. Therefore, TO_j is a subset of D_j which contains pairs of consecutive odd numbers contained in D_j.

Conjecture 8: (*Twin primes conjecture*) TO_2 *is an infinite set.*

See [3].

Theorem 9: $TO_3 = \varnothing$.

Proof: It is easy to see that for any such pair $(2n + 1, 2n + 3)$, the divisors of $2n + 1$ are $1, p, p^2$ and the divisors of $2n + 3$ are $1, q, q^2$ for some odd primes p and q. Even for any twin primes (p, q) it follows that $q^2 - p^2 > 2$.

Hence, there are no twin odd numbers with 3 positive divisors. ■

Theorem 10: $TO_5 = \varnothing$.

Proof: It follows from the observation that the prime factorization of such numbers can not contain more than two distinct primes. Following in the same argument as above we can conclude the result. ■

Problem 11: $TO_k = \varnothing$ *if* k *is odd*

Remark 12: *Above theorems show that* TO_k *is not always non-empty.*

Now we study the structure of the set TO_4. That is, twin odd numbers with 4 divisors. First we give an example to show that this set is non-empty.

For example, $(33, 35)$, $(55, 57)\ldots$ which are pairs of consecutive odd numbers with four divisors.

Theorem 13: *Any pair of twin odd numbers in* TO_4 *will have the divisor* $(positive)$ *sets*:

$$X = \{1, p_1, q_1, p_1q_1\} \text{ and } Y = \{1, p_2, q_2, p_2q_2\}$$

where p_1, q_1, p_2, q_2 *are distinct primes or* $q_1 = p_1^2$. *In the later case only one of the divisor set is of the form* $\{1, p_1, p_1^2, p_1^3\}$

Proof: Let a, b constitute the pair of twin end odds and $\tau(a) = 4 = \tau(b)$. Let $\{1, p_1, p_2, a\}$ be the set of divisors of a such that

$$1 < p_1 < p_2 < a$$

with p_1 is the least prime divisor. Then if p_2 is a prime then a is equal to $p_1 \cdot p_2$ and if p_2 is composite then $p_2 = p_1^2$ which leads to $a = p_1^3$.

Both the divisor sets can not be of the form $\{1, p_1, p_1^2, p_1^3\}$ for a prime p in which case $|a - b| > 2$ which contradicts the hypothesis $|a - b| = 2$.

This completes the proof. ■

The following tables will illustrate the possible divisor sets for $(a, b) \in TO_4$.

Table 1: Divisor sets of Type 1

1	p_1	q_1	p_1q_1	$(,)$	p_2q_2	p_2	q_2
1	3	11	33	(33, 35)	35	5	7
1	5	11	55	(55, 57)	57	3	19

Table 2: Divisor sets of Type 2

1	p_1	p_1^2	p_1^3	$(,)$	p_2q_2	p_2	q_2
1	5	5^2	125	(123, 125)	123	3	41
1	11	11^2	1331	(1331, 1333)	1333	3	41
1	19	361	6859	(6859, 6861)	6861	3	2287

Conjecture 14: *Is TO_4 an infinite set ?*

Problem 15: *Is TO_{2k} is non-empty for every $k \in \mathbb{N}$?*

With our new notation, generalized twin primes conjecture will be stated as follows:

Conjecture 16: (*Generalized twin primes conjecture*) *Is* TO_{2k} *an infinite set for every* $k \in \mathbb{N}$ *?*

The following questions will follow naturally.

Problem 17: *Find* k *for which* TO_k *is an empty set.*

Problem 18: *Find* k *for which* TO_k *is an infinite set.*

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