Journal of Indian Acad. Math. ISSN: 0970-5120
Vol. 46, No. 2 (2024) pp. 55-65

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A FAMILY OF JACOBSTHAL SUMS: ALTERNATE GENERALIZATIONS

Abstract: We explore an alternate generalization of a sum involving Jacobsthal polynomial squares and its consequences.

Keywords: Jacobsthal Polynomial, Jacobsthal-Lucas Polynomial, Extended Gibonacci Polynomials, Fibonacci Polynomial, Lucas Polynomial.

Mathematical Subject Classification No.: MSC2020, 11B37, 11B39, 11C08.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 3].

On the other hand, let $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial*. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$; and $j_n(1) = L_n$ [2, 3].

Gibonacci and Jacobsthal polynomials are linked by the relationships $J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$ and $j_n(x) = x^{n/2} l_n(1/\sqrt{x})$ [3, 4].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $c_n = J_n$ or j_n, $\Delta = \sqrt{x^2 + 4}$, $2\beta = x - \Delta$, $D = \sqrt{4x + 1}$, and $2\omega = 1 - D$. Then $\beta(1/\sqrt{x}) = \dfrac{1 - D}{2\sqrt{x}} = \dfrac{\omega}{\sqrt{x}}$.

2. A Gibonacci Sum: An Alternate Generalization

Before presenting an interesting gibonacci sum, again in the interest of brevity and expediency, we now let [5, 6]

$$\mu = \begin{cases} 1, & if\ g_n = f_n \\ \Delta^2, & otherwise; \end{cases} \qquad \nu = \begin{cases} -1, & if\ g_n = f_n\ or\ c_n = J_n \\ 1, & if\ g_n = l_n\ or\ c_n = j_n; \end{cases} \quad \text{and}$$

$$D^* = \begin{cases} 1, & if\ c_n = J_n \\ D^2, & otherwise. \end{cases}$$

With these tools as building blocks, we established the following gibonacci sum in [6], the cornerstone of our discourse.

Theorem 1: *Let k, p, r, and t be positive integers, where $t \le 2p$. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk} \mu\nu f_r f_{2pk}}{g^2_{(2pn+t-p)k+r} + (-1)^{tk+r} \mu\nu f^2_{pk}} = \frac{g_{tk}}{g_{tk+r}} - (-\beta)^r. \tag{1}$$

The objective of our discourse is to explore the Jacobsthal counterpart of this delightful sum.

3. A Jacobsthal Polynomial Sum

To achieve our goal, we will employ the gibonacci-Jacobsthal relationships in Section 1. Again, in the interest of conciseness and clarity, we let A denote the fractional expression on the left side of the given gibonacci equation and B that on its right side, and LHS and RHS the left-hand side and right-hand side of the corresponding Jacobsthal equation, as in [4, 5].

With this brief background, we now begin our endeavor.

Proof: Case 1: Suppose $g_n = f_n$. We have $A = \dfrac{(-1)^{tk+1} f_r f_{2pk}}{f^2_{(2pn+t-p)k+r} - (-1)^{tk+r} f^2_{pk}}$.

Replacing x with $1/\sqrt{x}$ and then multiplying the numerator and denominator with $x^{(2pn+t)k+r-1}$. We then get

$$A = \frac{(-1)^{tk+1} x^{(2pn+t-p)k+r/2}[x^{(r-1)/2} f_r][x^{(2pk-1)/2} f_{2pk}]}{\{x^{[(2pn+t-p)k+r-1]/2} f_{(2pn+t-p)k+r}\}^2 - (-1)^{tk+r} x^{(2pn+t-2p)k+r}[x^{(pk-1)/2} f_{pk}]^2};$$

$$= \frac{(-1)^{tk+1} x^{(2pn+t-2p)k+r/2} J_r J_{2pk}}{J^2_{(2pn+t-p)k+r} - (-1)^{tk+r} x^{(2pn+t-2p)k+r} J^2_{pk}};$$

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{(-1)^{tk+1} x^{(2pn+t-2p)k+r/2} J_r J_{2pk}}{J^2_{(2pn+t-p)k+r} - (-1)^{tk+r} x^{(2pn+t-2p)k+r} J^2_{pk}}.$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Turning to the right side, we have $B = \dfrac{f_{tk}}{f_{tk+r}} - (-\beta)^r$.

Now, replace x with $1/\sqrt{x}$, and then multiply the numerator and denominator with $x^{(tk+r-1)/2}$. This yields

$$B = \frac{x^{r/2}[x^{(tk-1)/2} f_{tk}]}{[x^{(tk+r-1)/2} f_{tk+r}]} - \frac{(-\omega)^r}{x^{r/2}};$$

$$\text{RHS} = \frac{x^{r/2} J_{tk}}{J_{tk+r}} - \frac{(-\omega)r}{x^{r/2}},$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Combining the two sides, we get the Jacobsthal version of equation (1):

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk+1} x^{(2pn+t-2p)k} J_r J_{2pk}}{J^2_{(2pn+t-p)k+r} - (-1)^{tk+r} x^{(2pn+t-2p)k+r} J^2_{pk}} = \frac{J_{tk}}{J_{tk+r}} - \frac{(-\omega)^r}{x^r}, \tag{2}$$

where $c_n = c_n(x)$.

Next, we explore the Jacobsthal-Lucas version of theorem 1.

Case 2: With $g_n = l_n$, we have $A = \dfrac{(-1)^{tk} \Delta^2 f_r f_{2pk}}{l^2_{(2pn+t-p)k+r} + (-1)^{tk+r} \Delta^2 f^2_{pk}}$. As above, x with $1/\sqrt{x}$ but then multiplying the numerator and denominator with $x^{(2pn+t-p)k+r}$, we then have

$$A = \frac{(-1)^{tk} D^2 \cdot x^{(2pn+t-2p)k+r/2}[x^{(r-1)/2} f_r][x^{(2pk-1)/2} f_{2pk}]}{\{x^{[(2pn+t-p)k+r]/2} l_{(2pn+t-p)k+r}\}^2 + (-1)^{tk+r} D^2 x^{(2pn+t-2p)k+r}[x^{(pk-1)/2} f_{pk}]^2};$$

$$= \frac{(-1)^{tk} D^2 x^{(2pn+t-2p)k+r/2} J_r J_{2pk}}{j^2_{(2pn+t-p)k+r} + (-1)^{tk+r} D^2 x^{(2pn+t-2p)k+r} J^2_{pk}};$$

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{(-1)^{tk} D^2 x^{(2pn+t-2p)k+r/2} J_r J_{2pk}}{j^2_{(2pn+t-p)k+r} + (-1)^{tk+r} D^2 x^{(2pn+t-2p)k+r} J^2_{pk}}.$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Correspondingly, we have $B = \dfrac{l_{tk}}{l_{tk+r}} - (-\beta)^r$. Replacing x with $1/\sqrt{x}$ and then multiplying the numerator and denominator with $x^{(tk+r)/2}$, we get

$$B = \frac{x^{r/2}[x^{tk/2} l_{tk}]}{[x^{(tk+r)/2} l_{tk+r}]} - \frac{(-\omega)^r}{x^{r/2}};$$

$$\text{RHS} = \frac{x^{r/2} j_{tk}}{j_{tk+r}} - \frac{(-\omega)r}{x^{r/2}},$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Equating the two sides yields the corresponding Jacobsthal-Lucas version of equation (1):

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk} D^2 x^{(2pn+t-2p)k} J_r J_{2pk}}{j^2_{(2pn+t-p)k+r} + (-1)^{tk+r} D^2 x^{(2pn+t-2p)k+r} J^2_{pk}} = \frac{j_{tk}}{j_{tk+r}} - \frac{(-\omega)^r}{x^r}, \qquad (3)$$

where $c_n = c_n(x)$.

This equation, coupled with formula (2), yields the desired Jacobsthal counterpart of Theorem 1, as the following theorem showcases.

Theorem 2: *Let k, p, r, and t be positive integers, where $t \le 2p$. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk} D^* \nu x^{(2pn+t-2p)k} J_r J_{2pk}}{c^2_{(2pn+t-p)k+r} + (-1)^{tk+r} D^* \nu x^{(2pn+t-2p)k+r} J^2_{pk}} = \frac{c_{tk}}{c_{tk+r}} - \frac{(-\omega)^r}{x^r}. \qquad (4)\ \square$$

Employing the gibonacci-Jacobsthal relationships in a compact way, we now present an alternate proof of this theorem.

3.1 A Delightful Alternate Method: To begin, first, we let

$$d^* = \frac{1-\nu}{4} = \begin{cases} 1/2, & \text{if } g_n = f_n \\ 0, & \text{otherwise.} \end{cases}$$

It follows from the gibonacci-Jacobsthal links in Section 1 that

$$f_n(1/\sqrt{x}) = \frac{J_n(x)}{x^{(n-1)/2}}; \quad l_n(1/\sqrt{x}) = \frac{j_n(x)}{x^{n/2}}; \quad g_n(1/\sqrt{x}) = \frac{c_n(x)}{x^{n/2-d^*}}.$$

With these tools at our fingertips, we are now ready for a sophisticated proof of Theorem 2.

Proof: Replacing x with $1/\sqrt{x}$ in the rational expression on the left side of equation (1) and using the above substitutions, we get

$$A = \frac{(-1)^{tk}\mu\nu[J_r / x^{(r-1)/2}][J_{2pk} / x^{(2pk-1)/2}]}{\left[c_{(2pn+t-p)k+r} / x^{\frac{(2pn+t-p)k+r}{2}-d^*}\right]^2 + (-1)^{tk+r}\mu\nu\left[\frac{J_{pk}}{x^{(pk-1)/2}}\right]^2};$$

$$= \frac{(-1)^{tk}\mu\nu x^{(2pn+t-p)k+r-2d^*-\frac{2pk+r-2}{2}} J_r J_{2pk}}{c^2_{(2pn+t-p)k+r} + (-1)^{tk+r}\mu\nu x^{[(2pn+t-p)k-2d^*-(pk-1)+r]} J^2_{pk}};$$

$$= \frac{(-1)^{tk} D^*\nu x^{(2pn+t-2p)k+\frac{r}{2}} J_r J_{2pk}}{c^2_{(2pn+t-p)k+r} + (-1)^{tk+r} D^*\nu x^{(2pn+t-2p)k+r} J^2_{pk}};$$

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{(-1)^{tk} D^*\nu x^{(2pn+t-2p)k+\frac{r}{2}} J_r J_{2pk}}{c^2_{(2pn+t-p)k} - (-1)^{tk} D^*\nu x^{(2pn+t-2p)k+r} J^2_{pk}}.$$

where $c_n = c_n(x)$.

The right side of equation (1) yields

$$B = \frac{g_{tk}}{g_{tk+r}} - (-\beta)^r;$$

$$= \frac{c_{tk} / x^{tk/2-d^*}}{c_{tk+r} / x^{(tk+r)/2-d^*}} - \frac{(-\omega)^r}{x^{r/2}};$$

$$\text{RHS} = \frac{x^{r/2} c_{tk}}{c_{tk+r}} - \frac{(-\omega^*)^r}{x^{r/2}},$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Combing the two sides yields the same Jacobsthal version, as expected:

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk} D^* \nu x^{(2pn+t-2p)k} J_r J_{2pk}}{c^2_{(2pn+t-p)k} - (-1)^{tk} D^* \nu x^{(2pn+t-2p)k+r} J^2_{pk}} = \frac{c_{tk}}{c_{tk+r}} - \frac{(-\omega^*)^r}{x^r},$$

where $c_n = c_n(x)$. □

Finally, we now explore a host of gibonacci and Jacobsthal implications of Theorem 2. To this end, we define the following labels:

A	=	10,080;	F	=	25,840;	K	=	117,390;	Q	=	1,040,130;
B	=	13,650;	G	=	29,241;	L	=	149,872;	R	=	1,392,300;
C	=	15,504;	H	=	30,030;	M	=	253,890;	S	=	7,514,766;
D	=	15,810;	I	=	57,330;	N	=	263,169;	T	=	8,912,862;
E	=	15,840;	J	=	67,184;	P	=	873,810;	V	=	66,584,322.

3.2 Gibonacci and Jacobsthal Consequences: With $J_n(1) = F_n$, $j_n(1) = L_n$, $J_n(2) = J_n$, $j_n(2) = j_n$, Theorem 2 yields

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk+1} F_r F_{2pk}}{F^2_{(2pn+t-p)k+r} - (-1)^{tk+r} F^2_{pk}} = \frac{F_{tk}}{F_{tk+r}} - (-\beta)^r; \tag{5}$$

$$\sum_{n=1}^{\infty} L \frac{(-1)^{tk} 5F_r F_{2pk}}{F^2_{(2pn+t-p)k+r} + (-1)^{tk+r} 5F^2_{pk}} = \frac{L_{tk}}{L_{tk+r}} - (-\beta)^r; \tag{6}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk+1} 2^{(2pn+t-2p)k} J_r J_{2pk}}{J^2_{(2pn+t-p)k+r} - (-1)^{tk+r} 2^{(2pn+t-2p)k+r} J^2_{pk}} = \frac{J_{tk}}{J_{tk+r}} - \frac{1}{2^r}; \tag{7}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk} 9 \cdot 2^{(2pn+t-2p)k} J_r J_{2pk}}{j^2_{(2pn+t-p)k+r} + (-1)^{tk+r} 9 \cdot 2^{(2pn+t-2p)k+r} J^2_{pk}} = \frac{j_{tk}}{j_{tk+r}} - \frac{1}{2^r}. \tag{8}$$

In particular, with $p \in \{2, 3\}, k \le 3, r = 1$, and $t \le 2$ equations (5) and (6) yield

$$\sum_{n=1}^{\infty} \frac{1}{F^2_{4n} - 1} = \frac{1}{2} - \frac{\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{4n} + 5} = -\frac{1}{18} + \frac{\sqrt{5}}{30};$$

$$\sum_{n=1}^{\infty} \frac{1}{F^2_{8n-1} + 9} = -\frac{1}{21} + \frac{\sqrt{5}}{42}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{8n-1} - 45} = \frac{1}{84} - \frac{\sqrt{5}}{210};$$

$$\sum_{n=1}^{\infty} \frac{1}{F^2_{12n-2} - 64} = \frac{7}{864} - \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{12n-2} + 320} = -\frac{1}{672} + \frac{\sqrt{5}}{1,440};$$

$$\sum_{n=1}^{\infty} \frac{1}{F^2_{4n+1} + 1} = -\frac{1}{3} + \frac{\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{4n+1} - 5} = -\frac{1}{12} - \frac{\sqrt{5}}{30};$$

$$\sum_{n=1}^{\infty} \frac{1}{F^2_{8n+1} + 9} = -\frac{11}{210} + \frac{\sqrt{5}}{42}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{8n+1} - 45} = \frac{5}{462} - \frac{\sqrt{5}}{210};$$

$$\sum_{n=1}^{\infty} \frac{1}{F^2_{12n+1} + 64} = -\frac{29}{3,744} + \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{12n+1} - 320} = \frac{13}{8,352} - \frac{\sqrt{5}}{1,440};$$

$$\sum_{n=1}^{\infty} \frac{1}{F^2_{6n-1} - 4} = \frac{3}{16} - \frac{\sqrt{5}}{16}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{6n-1} + 20} = -\frac{1}{48} + \frac{\sqrt{5}}{80};$$

$$\sum_{n=1}^{\infty} \frac{1}{F^2_{12n-3} + 64} = -\frac{1}{144} + \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L^2_{12n-3} - 320} = \frac{1}{576} - \frac{\sqrt{5}}{1,440};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{18n-5}^2 - 1,156} = \frac{7}{C} - \frac{\sqrt{5}}{5,168}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{18n-5}^2 + 5,780} = -\frac{3}{3,556} + \frac{\sqrt{5}}{F};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{6n}^2 + 4} = -\frac{1}{8} + \frac{\sqrt{5}}{16}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{6n}^2 - 20} = \frac{1}{32} - \frac{\sqrt{5}}{80};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{12n-1}^2 + 64} = -\frac{1}{1,440} + \frac{\sqrt{5}}{1,440}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{12n-1}^2 - 320} = \frac{5}{E} - \frac{\sqrt{5}}{1,440};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{18n-2}^2 + 1,156} = -\frac{1}{J} + \frac{\sqrt{5}}{5,168}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{18n-2}^2 - 5,780} = \frac{13}{L} - \frac{\sqrt{5}}{F}.$$

Using equations (7) and (8), we now present their Jacobsthal counterparts:

$$\sum_{n=1}^{\infty} \frac{2^{4n-3}}{J_{4n}^2 - 2^{4n-2}} = \frac{1}{10}; \qquad \sum_{n=1}^{\infty} \frac{2^{4n-3}}{j_{4n}^2 + 9 \cdot 2^{4n-2}} = \frac{1}{150};$$

$$\sum_{n=1}^{\infty} \frac{2^{2(4n-3)}}{J_{8n-3}^2 + 25 \cdot 2^{8n-5}} = \frac{1}{510}; \qquad \sum_{n=1}^{\infty} \frac{2^{2(4n-3)}}{j_{8n-3}^2 - 225 \cdot 2^{8n-5}} = \frac{1}{3,570};$$

$$\sum_{n=1}^{\infty} \frac{2^{3(4n-3)}}{J_{12n-2}^2 - 441 \cdot 2^{12n-8}} = \frac{1}{B}; \qquad \sum_{n=1}^{\infty} \frac{2^{3(4n-3)}}{j_{12n-2}^2 + 3,969 \cdot 2^{12n-8}} = \frac{1}{R};$$

$$\sum_{n=1}^{\infty} \frac{2^{4n-2}}{J_{4n+1}^2 + 2^{4n-1}} = \frac{1}{30}; \qquad \sum_{n=1}^{\infty} \frac{2^{4n-2}}{j_{4n+1}^2 - 9 \cdot 2^{4n-1}} = \frac{1}{126};$$

$$\sum_{n=1}^{\infty} \frac{2^{2(4n-2)}}{J_{8n+1}^2 + 25 \cdot 2^{8n-3}} = \frac{1}{1,870}; \qquad \sum_{n=1}^{\infty} \frac{2^{2(4n-2)}}{j_{8n+1}^2 - 225 \cdot 2^{8n-3}} = \frac{1}{D};$$

$$\sum_{n=1}^{\infty} \frac{2^{3(4n-2)}}{J_{12n+1}^2 + 441 \cdot 2^{12n-5}} = \frac{1}{K}; \qquad \sum_{n=1}^{\infty} \frac{2^{3(4n-2)}}{j_{12n+1}^2 - 3,969 \cdot 2^{12n-5}} = \frac{1}{Q};$$

$$\sum_{n=1}^{\infty} \frac{2^{6n-5}}{J_{6n-1}^2 - 9 \cdot 2^{6n-4}} = \frac{1}{42}\ ; \qquad \sum_{n=1}^{\infty} \frac{2^{6n-5}}{j_{6n-1}^2 + 81 \cdot 2^{6n-4}} = \frac{1}{630};$$

$$\sum_{n=1}^{\infty} \frac{2^{2(6n-5)}}{J_{12n-9}^2 + 441 \cdot 2^{12n-9}} = \frac{1}{8,190}\ ; \quad \sum_{n=1}^{\infty} \frac{2^{2(6n-5)}}{j_{12n-9}^2 - 3,969 \cdot 2^{12n-9}} = \frac{1}{I};$$

$$\sum_{n=1}^{\infty} \frac{2^{3(6n-5)}}{J_{18n-5}^2 - G \cdot 2^{18n-14}} = \frac{1}{P}\ ; \qquad \sum_{n=1}^{\infty} \frac{2^{3(6n-5)}}{j_{18n-5}^2 + N \cdot 2^{18n-14}} = \frac{1}{T}$$

$$\sum_{n=1}^{\infty} \frac{2^{6n-4}}{J_{6n-1}^2 + 9 \cdot 2^{6n-3}} = \frac{1}{126}\ ; \qquad \sum_{n=1}^{\infty} \frac{2^{6n-4}}{j_{6n-1}^2 - 81 \cdot 2^{6n-3}} = \frac{1}{882};$$

$$\sum_{n=1}^{\infty} \frac{2^{2(6n-4)}}{J_{12n-1}^2 + 441 \cdot 2^{12n-7}} = \frac{1}{H}\ ; \qquad \sum_{n=1}^{\infty} \frac{2^{2(6n-4)}}{j_{12n-1}^2 - 3,969 \cdot 2^{12n-7}} = \frac{1}{M};$$

$$\sum_{n=1}^{\infty} \frac{2^{3(6n-4)}}{J_{18n-2}^2 + G \cdot 2^{18n-11}} = \frac{1}{S}\ ; \qquad \sum_{n=1}^{\infty} \frac{2^{3(6n-4)}}{j_{18n-2}^2 + N \cdot 2^{18n-11}} = \frac{1}{V},$$

respectively.

Finally, we encourage gibonacci enthusiasts to explore the gibonacci and Jacobsthal sums with $p = 5, k, r \in \{1, 2\}$, and $t \le 5$.

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(*Received*, April 6, 2024)