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# A FAMILY OF GIBONACCI SUMS: ALTERNATE GENERALIZATIONS

**Abstract:** We explore an alternate generalization of a sum involving a family of gibonacci polynomial squares and its consequences.

**Keywords:** Extended Gibonacci Polynomials, Fibonacci Polynomial, Lucas Polynomial, Binet-like Formulas.

**Mathematical Subject Classification (MSC2020) No.:** 11B37, 11B39, 11C08.

## 1. Introduction

*Extended gibonacci polynomials* $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where $x$ is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the $n$th *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the $n$th *Lucas polynomial*.

They can also be defined by the *Binet-like* formulas. Clearly, $f_n(1) = F_n$, the $n$th Fibonacci number; and $l_n(1) = L_n$, the $n$th Lucas number [1, 2].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is $no$ ambiguity; so $z_n$ will mean $z_n(x)$. In addition, we let $g_n = f_n$ or $l_n$, $\Delta = \sqrt{x^2 + 4}$ and $2\alpha = x + \Delta$ .

It follows by the Binet-like formulas that $\lim_{m\to\infty} \frac{1}{g_{m+r}} = 0$ and $\lim_{m\to\infty} \frac{g_{m+r}}{g_m} = \alpha^r$.

**1.1 Fundamental Gibonacci Identities:** Gibonacci polynomials satisfy the following properties [2, 3, 4, 5, 6, 7]:

$$g_{n+k}g_{n-k} - g_n^2 = \begin{cases} (-1)^{n+k+1} f_k^2, & \text{if } g_n = f_n \\ (-1)^{n+k} \Delta^2 f_k^2, & \text{otherwise;} \end{cases} \tag{1}$$

$$g_{n+k+r}g_{n-k} - g_{n+k}g_{n-k+r} = \begin{cases} (-1)^{n+k+1} f_r f_{2k}, & \text{if } g_n = f_n \\ (-1)^{n+k} \Delta^2 f_r f_{2k}, & \text{otherwise;} \end{cases} \tag{2}$$

where $k$ and $r$ are positive integers. These properties can be confirmed using the Binet-like formulas.

It follows from these two identities that

$$g_{(2pn+t)k+r}g_{(2pn+t-2p)k+r} - g^2_{(2pn+t-p)k+r} = \begin{cases} (-1)^{tk+r+1} f_{pk}^2, & \text{if } g_n = f_n \\ (-1)^{tk+r} \Delta^2 f_{pk}^2, & \text{otherwise;} \end{cases} \tag{3}$$

$$g_{(2pn+t)k}g_{2pn+t-2p)k+r} - g_{(2pn+t)k+r}g_{(2pn+t-2p)k} = \begin{cases} (-1)^{tk} f_r f_{2pk}, & \text{if } g_n = f_n \\ (-1)^{tk+1} \Delta^2 f_r f_{2pk}, & \text{otherwise;} \end{cases} \tag{4}$$

where $k$, $p$, $r$, and $t$ are positive integers and $t \le 2p$ [7].

## 2. A Telescoping Gibonacci Sum

Using recursion, we will now investigate a telescoping gibonacci sum.

**Lemma 1:** *Let $k$, $p$, $r$, $t$, and $\lambda$ be positive integers, where $t \le 2p$. Then*

$$\sum_{n=1}^{\infty}\left[\frac{g_{(2pn+t-2p)k}^{\lambda}}{g_{(2pn+t-2p)k+r}^{\lambda}}-\frac{g_{(2pn+t)k}^{\lambda}}{g_{(2pn+t)k+r}^{\lambda}}\right]=\frac{g_{tk}^{\lambda}}{g_{tk+r}^{\lambda}}-(-\beta)^{\lambda r}. \tag{5}$$

**Proof:** With recursion [2, 3], we will first confirm that

$$\sum_{n=1}^{m}\left[\frac{g_{(2pn+t-2p)k}^{\lambda}}{g_{(2pn+t-2p)k+r}^{\lambda}}-\frac{g_{(2pn+t)k}^{\lambda}}{g_{(2pn+t)k+r}^{\lambda}}\right]=\frac{g_{tk}^{\lambda}}{g_{tk+r}^{\lambda}}-\frac{g_{(2pm+t)k}^{\lambda}}{g_{(2pm+t)k+r}^{\lambda}}. \tag{6}$$

To realize this goal, in the interest of brevity, we let $A_m$ denote the left-hand side of this equation and $B_m$ its right-hand side. Then

$$B_m - B_{m-1} = A_m - A_{m-1}.$$

By recursion, this yields

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_1 - B_1 \\ &= 0, \end{aligned}$$

establishing the validity of equation (6).

Since, $\lim_{m\to\infty}\frac{g_m}{g_{m+r}}=\frac{1}{\alpha^r}=(-\beta)^r$, the given result now follows, as desired. □

## 3. Gibonacci Sums

Coupled with identities (3) and (4), Lemma 1 with $\lambda = 1$ plays a significant role in our explorations.

To this end, in the interest of brevity, we now let

$$\mu=\begin{cases}1, & \text{if } g_n = f_n\\ \Delta^2, & \text{otherwise;}\end{cases} \quad \text{and} \quad \nu=\begin{cases}-1, & \text{if } g_n = f_n\\ 1, & \text{otherwise.}\end{cases}$$

With these tools at our fingertips, we now embark on our discourse with the following result.

**Theorem 1:** *Let $k$, $p$, $r$, and $t$ be positive integers, where $t \le 2p$. Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk} \mu\nu f_r f_{2pk}}{g^2_{(2pn+t-p)k+r} + (-1)^{tk+r} \mu\nu f^2_{pk}} = \frac{g_{tk}}{g_{tk+r}} - (-\beta)^r. \tag{7}$$

**Proof:** Suppose $g_n = f_n$. Coupled with identities (3) and (4), Lemma 1 then yields

$$\frac{(-1)^{tk+1} f_r f_{2pk}}{f^2_{(2pn+t-p)k+r} - (-1)^{tk+r} f^2_{pk}} = \frac{f_{(2pn+t)k+r} f_{(2pn+t-2p)k} - f_{(2pn+t)k} f_{(2pn+t-2p)k+r}}{f_{(2pn+t)k+r} f_{(2pn+t-2p)k+r}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk+1} f_r f_{2pk}}{f^2_{(2pn+t-p)k+r} - (-1)^{tk+r} f^2_{pk}} = \sum_{n=1}^{\infty} \left[ \frac{f_{(2pn+t-2p)k}}{f_{(2pn+t-2p)k+r}} - \frac{f_{(2pn+t)k}}{f_{(2pn+t)k+r}} \right]$$

$$= \frac{f_{tk}}{f_{tk+r}} - (-\beta)^r.$$

On the flip side, let $g_n = l_n$. With the same two identities and Lemma 1, we then get

$$\frac{(-1)^{tk} \Delta^2 f_r f_{2pk}}{l^2_{(2pn+t-p)k+r} + (-1)^{tk+r} \Delta^2 f^2_{pk}} = \frac{l_{(2pn+t)k+r} l_{(2pn+t-2p)k} - l_{(2pn+t)k} l_{(2pn+t-2p)k+r}}{l_{(2pn+t)k+r} l_{(2pn+t-2p)k+r}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{tk} \Delta^2 f_r f_{2pk}}{l^2_{(2pn+t-p)k+r} + (-1)^{tk+r} \Delta^2 f^2_{pk}} = \sum_{n=1}^{\infty} \left[ \frac{l_{(2pn+t-2p)k}}{l_{(2pn+t-2p)k+r}} - \frac{l_{(2pn+t)k}}{l_{(2pn+t)k+r}} \right]$$

$$= \frac{l_{tk}}{l_{tk+r}} - (-\beta)^r.$$

The given result now follows by combining the two cases, as desired. □

In particular, with $r = 1$ and $k \le 3$, the theorem yields [3, 4, 7]:

**Case 1:** Let $p = 1$. Then $t \le 2$. With $t = 1$, we get

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n+1}^2 - 1} = \frac{3}{2} - \frac{\sqrt{5}}{2}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2n+1}^2 + 5} = -\frac{1}{6} + \frac{\sqrt{5}}{10};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n+1}^2 + 1} = -\frac{1}{3} + \frac{\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{4n+1}^2 - 5} = \frac{1}{12} - \frac{\sqrt{5}}{30};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{6n+1}^2 - 4} = \frac{7}{48} - \frac{\sqrt{5}}{16}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{6n+1}^2 + 20} = -\frac{3}{112} + \frac{\sqrt{5}}{80};$$

with $t = 2$, the formula yields

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n+2}^2 + 1} = -1 + \frac{\sqrt{5}}{2}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{2n+2}^2 - 5} = \frac{1}{4} - \frac{\sqrt{5}}{10};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n+3}^2 + 1} = -\frac{11}{30} + \frac{\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{4n+3}^2 - 5} = \frac{5}{66} - \frac{\sqrt{5}}{30};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{6n+4}^2 + 4} = -\frac{29}{208} + \frac{\sqrt{5}}{16}; \quad \sum_{n=1}^{\infty} \frac{1}{L_{6n+4}^2 - 20} = \frac{13}{464} - \frac{\sqrt{5}}{80}.$$

**Case 2:** With $p = 2$, we have $t \le 4$. When $t = 1$, the theorem implies

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n}^2 - 1} = \frac{1}{2} - \frac{\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{4n}^2 + 5} = -\frac{1}{18} + \frac{\sqrt{5}}{30};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{8n-1}^2 + 9} = -\frac{1}{21} + \frac{\sqrt{5}}{42}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{8n-1}^2 - 45} = \frac{1}{84} - \frac{\sqrt{5}}{210};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{12n-2}^2 - 64} = \frac{7}{864} - \frac{\sqrt{5}}{228}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{12n-2}^2 + 320} = -\frac{1}{10,080} + \frac{\sqrt{5}}{1,440};$$

with $t = 2$, we get

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n+1}^2 + 1} = -\frac{1}{3} + \frac{\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{4n+1}^2 - 5} = \frac{1}{12} - \frac{\sqrt{5}}{30};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{8n+1}^2 + 9} = -\frac{11}{210} + \frac{\sqrt{5}}{42}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{8n+1}^2 - 45} = \frac{5}{462} - \frac{\sqrt{5}}{210};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{12n+1}^2 + 64} = -\frac{29}{3,744} + \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{12n+1}^2 - 320} = \frac{13}{8,352} - \frac{\sqrt{5}}{1,440};$$

using $t = 3$, the theorem yields

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n+2}^2 - 1} = \frac{7}{18} - \frac{\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{4n+2}^2 + 5} = -\frac{1}{14} + \frac{\sqrt{5}}{30};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{8n+3}^2 + 9} = -\frac{29}{546} + \frac{\sqrt{5}}{42}; \qquad \sum_{n=1}^{\infty} \frac{1}{F_{8n+3}^2 - 45} = \frac{13}{1,218} - \frac{\sqrt{5}}{210};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{12n+4}^2 - 64} = \frac{41}{5,280} - \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{12n+4}^2 + 320} = -\frac{55}{35,424} + \frac{\sqrt{5}}{1,440};$$

and finally, letting $t = 4$, we get

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n+3}^2 + 1} = -\frac{11}{30} + \frac{\sqrt{5}}{6}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{4n+3}^2 - 5} = \frac{5}{66} - \frac{\sqrt{5}}{30};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{8n+5}^2 + 9} = -\frac{19}{357} + \frac{\sqrt{5}}{42}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{8n+5}^2 - 45} = \frac{17}{1,596} - \frac{\sqrt{5}}{210};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{12n+7}^2 + 64} = -\frac{521}{67,104} + \frac{\sqrt{5}}{288}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{12n+7}^2 - 320} = \frac{233}{150,048} - \frac{\sqrt{5}}{1,440}.$$

**3.1 Gibonacci Delights:** Using the above results, we can extract delightful dividends [3, 4].

$$\sum_{n=2}^{\infty} \frac{1}{F_{2n+1}^2 + 1} = \sum_{n=1}^{\infty} \frac{1}{F_{4n+1}^2 + 1} + \sum_{n=1}^{\infty} \frac{1}{F_{4n+3}^2 + 1} = -\frac{41}{30} + \frac{\sqrt{5}}{3};$$

$$\sum_{n=2}^{\infty} \frac{1}{F_{2n}^2 - 1} = \sum_{n=1}^{\infty} \frac{1}{F_{4n}^2 - 1} + \sum_{n=1}^{\infty} \frac{1}{F_{4n+2}^2 - 1} = \frac{8}{9} - \frac{\sqrt{5}}{3};$$

$$\sum_{n=2}^{\infty} \frac{1}{L_{2n}^2 + 5} = \sum_{n=1}^{\infty} \frac{1}{L_{4n}^2 + 5} + \sum_{n=1}^{\infty} \frac{1}{L_{4n+2}^2 + 5} = -\frac{8}{63} + \frac{\sqrt{5}}{15};$$

$$\sum_{n=2}^{\infty} \frac{1}{F_{2n+1}^2 - 1} = \sum_{n=1}^{\infty} \frac{1}{F_{4n+1}^2 - 1} + \sum_{n=1}^{\infty} \frac{1}{F_{4n+3}^2 - 1} = -\frac{7}{10} + \frac{\sqrt{5}}{3};$$

$$\sum_{n=2}^{\infty} \frac{1}{L_{2n+1}^2 - 5} = \sum_{n=1}^{\infty} \frac{1}{L_{4n+1}^2 - 5} + \sum_{n=1}^{\infty} \frac{1}{L_{4n+3}^2 - 5} = \frac{7}{44} - \frac{\sqrt{5}}{15};$$

$$\sum_{n=3}^{\infty} \frac{1}{F_{2n+1}^2 + 9} = \sum_{n=1}^{2} \left( \sum_{i=-1}^{\infty} \frac{1}{F_{8n+2i+1}^2 + 9} \right) = -\frac{228}{1,105} + \frac{2\sqrt{5}}{21};$$

$$\sum_{n=2}^{\infty} \frac{1}{L_{2n+1}^2 - 45} = \sum_{n=1}^{\infty} \left( \sum_{i=-1}^{2} \frac{1}{F_{8n+2i+1}^2 - 45} \right) = \frac{267}{6,061} - \frac{2\sqrt{5}}{105};$$

$$\sum_{n=2}^{\infty} \frac{1}{F_{6n+1}^2 + 64} = \sum_{n=1}^{\infty} \frac{1}{F_{12n+1}^2 + 64} + \sum_{n=1}^{\infty} \frac{1}{F_{12n+7}^2 + 64} = -\frac{2,255}{145,392} + \frac{\sqrt{5}}{144};$$

$$\sum_{n=2}^{\infty} \frac{1}{L_{6n+1}^2 - 320} = \sum_{n=1}^{\infty} \frac{1}{L_{12n+1}^2 - 320} + \sum_{n=1}^{\infty} \frac{1}{L_{12n+7}^2 - 320} = \frac{2,255}{725,232} - \frac{\sqrt{5}}{720};$$

$$\sum_{n=2}^{\infty} \frac{1}{F_{6n+4}^2 - 64} = \sum_{n=1}^{\infty} \frac{1}{F_{12n-2}^2 - 64} + \sum_{n=1}^{\infty} \frac{1}{F_{12n+4}^2 - 64} = \frac{377}{23,760} - \frac{\sqrt{5}}{144};$$

$$\sum_{n=2}^{\infty} \frac{1}{L_{6n+4}^2 + 320} = \sum_{n=1}^{\infty} \frac{1}{L_{12n-2}^2 + 320} + \sum_{n=1}^{\infty} \frac{1}{L_{12n+4}^2 + 320} = -\frac{377}{123,984} + \frac{\sqrt{5}}{720}.$$

We now conclude our exploration with showcasing the counterparts of equations (8)-(11) in [7]. They follow from the Theorem by letting $t = p; t = p = k;$ $t = 2p$; and $t = 2p = 2k$, respectively.

$$\sum_{n=1}^{\infty} \frac{(-1)^{pk} \mu\nu f_r f_{2pk}}{g_{2pnk+r}^2 + (-1)^{pk+r} \mu\nu f_{pk}^2} = \frac{g_{pk}}{g_{pk+r}} - (-\beta)^r. \tag{8}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{p^2} \mu\nu f_r f_{2p^2}}{g_{2p^2n+r}^2 + (-1)^{p^2+r} \mu\nu f_{p^2}^2} = \frac{g_{p^2}}{g_{p^2+r}} - (-\beta)^r. \tag{9}$$

$$\sum_{n=1}^{\infty} \frac{\mu\nu f_r f_{2pk}}{g_{pk(2n+1)+r}^2 + (-1)^r \mu\nu f_{pk}^2} = \frac{g_{2pk}}{g_{2pk+r}} - (-\beta)^r. \tag{10}$$

$$\sum_{n=1}^{\infty} \frac{\mu\nu f_r f_{2p^2}}{g_{p^2(2n+1)+r}^2 + (-1)^r \mu\nu f_{p^2}^2} = \frac{g_{2p^2}}{g_{2p^2+r}} - (-\beta)^r. \tag{11}$$

With the labels

$$A = 21,890; \qquad C = 284,240; \qquad E = 21,607,408;$$

$$B = 25,840; \qquad D = 635,664; \qquad F = 48,315,632,$$

they yield

$$\sum_{n=1}^{\infty} \frac{1}{F_{10n+1}^2 - 25} = \frac{9}{440} - \frac{\sqrt{5}}{110}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{10n+1}^2 + 125} = -\frac{2}{495} + \frac{\sqrt{5}}{550};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{18n+1}^2 - 1,156} = \frac{123}{C} - \frac{\sqrt{5}}{5,168}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{18n+1}^2 + 5,780} = -\frac{55}{D} + \frac{\sqrt{5}}{B};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{10n+6}^2 - 25} = \frac{199}{9,790} + \frac{\sqrt{5}}{110}; \qquad \sum_{n=1}^{\infty} \frac{1}{L_{10n+6}^2 + 125} = \frac{89}{A} - \frac{\sqrt{5}}{550};$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{18n+10}^2 - 1,156} = -\frac{9,349}{E} + \frac{\sqrt{5}}{5,168}; \ \sum_{n=1}^{\infty} \frac{1}{L_{18n+10}^2 - 5,780} = \frac{4,181}{F} - \frac{\sqrt{5}}{B}.$$

respectively.

It then follows that

$$\sum_{n=2}^{\infty} \frac{1}{F_{5n+1}^2 - 25} = \sum_{n=1}^{\infty} \frac{1}{F_{10n+1}^2 - 25} + \sum_{n=1}^{\infty} \frac{1}{F_{10n+6}^2 - 25} = -\frac{1,597}{39,160} + \frac{\sqrt{5}}{55};$$

$$\sum_{n=2}^{\infty} \frac{1}{F_{9n+1}^2 - 1,156} = \sum_{n=1}^{\infty} \frac{1}{F_{18n+1}^2 - 1,156} + \sum_{n=1}^{\infty} \frac{1}{F_{18n+10}^2 - 1,156} = \frac{1}{17,476,580}.$$

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