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KKM-MAPS VIA SET VALUED MAPS

Abstract: In this paper we extend some theorems relating to KKM maps in three set valued maps and a few results are derived as corollary.

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1. Introduction and Preliminaries

In some of the research works of different researchers in KKM theory we see that they proved theorems of one set valued map and two set valued map by using KKM map. In 1968, Browder [2] proved the existence of fixed point for a set valued map by using the partition of unity. We can mention authors like Brosowski [1], Granas [5], Yannelis and Prabhakar [8], Lin [6] extended and improved KKM theorem on set valued maps.

Before giving our main results we quote some related definition, theorems and corollaries as follows.

By a *set valued map F* from a set *X* to a set *Y*, denoted by $F : X \to 2^Y$, we mean a correspondence which with each $x \in X$ associates a set $F(x) \in 2^Y$ (the space of subsets of *Y*).

Let X be a subset of a vector space E. A map $\mathrm{F} : \mathrm{X} \to 2^{\mathrm{E}}$ is called a

KKM-map if $co\{x_1,...,x_n\} \subseteq \bigcup_{i=1}^{n} F(x_i)$ for each finite subset $\{x_1,\ldots,x_n\}$ of X.

For details on KKM maps we refer Granas [5].

In 1961, Ky Fan [3] gave the following theorem.

Theorem 1.1: (Fan-KKM or KKMF). Let X be an arbitrary subset of a topological vector space E. To each $x \in X$, let F(x) be a closed in E and let the convex hull of every finite subset $\{x_1,..., x_n\}$ of X is contained in the corresponding union $\bigcup_{i=1}^{n} F(x_i)$. If F(x) is compact for at least one $x \in X$ then $\bigcap_{x\in X} F(x) \neq \varphi$.

It is to be noted that the above theorem has applications in various fields including fixed point theory, variational inequality and game theory.

In 1968, Browder [2] proves the following theorem.

Theorem 1.2: Let X be a nonempty compact convex subset of Hausdorff topological vector spaces E. Let $T : X \to 2^X$ be a satisfy

(i) T(x) is convex and nonempty for each $x \in X$;

(ii) $T^{-1}(y)$ is open in X for each $y \in X$.

Then there exist an $y_0 \in X$ such that $y_0 \in T(y_0)$.

In 1984, Ky Fan [4] gave the following theorem as an extension of Theorem 1.1, where compactness is relaxed. This important result was used by several researchers in recent years.

Theorem 1.3: Let Y be a convex set in a topological vector space E and X a nonempty subset of Y. For each $x \in$ X, let F(x) be a relatively closed subset of Y such that the convex hull of every finite subset $\{x_1,..., x_n\}$ of X is contained in the corresponding union $\bigcup_{i=1}^{n} F(x_i)$. If there is a nonempty subset X_0 of X such that the intersection $\bigcap_{x\in X_0} F(x)$ is compact and X_0 is contained in a compact convex subset of Y then $\bigcap_{x\in X} F(x) \neq \varphi$.

In 1985, Yannelis and Prabhakar [8] has given the following result.

Theorem 1.4: Let X be a nonempty compact convex subset of Hausdorff topological vector spaces E. Let $T : X \to 2^X$ satisfy

(i) $x \notin co(T(x))$ for each $x \in X$;

(ii) $T^{-1}(y)$ is open in X for each $y \in X$.

Then there exist an $x_0 \in X$ such that $T(x_0) = \varphi$.

In 1986, Lin [6] gave the following theorem.

Theorem 1.5: Let X be a nonempty convex subset of Hausdorff topological vector spaces E. Let $T : X \to 2^X$ satisfy

(i) $x \notin co(T(x))$ for each $x \in X$;

(ii) $T^{-1}(y) = \{x \in X \mid y \in T(x)\}$ is open in X for each $y \in X$;

(iii) X has a nonempty compact convex subset X_0 such that $B = \{x \in X : y \notin T(x) \text{ for all } y \in X_0\}$ is compact.

Then there exist an $x_0 \in X$ such that $T(x_0) = \varphi$.

The above result remains true if the following condition is given in place of (iii).

$(iii)'$ Let X_0 be a nonempty compact convex subset of X, and K a nonempty compact set of X. For every $x \in X$ K there exists a $y \in X_0$ such that $y \in T(x)$.

Corollary 1.6 [7]: Let X be a nonempty convex subset of Hausdorff topological vector spaces E. Let $T : X \to 2^X$ satisfy

(i) $x \notin T(x)$ for each $x \in X$;

(ii) T(x) is convex or empty, for each $x \in X$;

(iii) $T^{-1}(y)$ is open in X for each $y \in X$;

(iv) X has a nonempty compact convex subset X_0 such that $B = \{x \in X: y \notin T(x) \text{ for all } y \in X_0\}$ is compact.

Then there exists an $x_0 \in X$ such that $T(x_0) = \varphi$.

The following is an extension of Theorem 1.2.

Theorem 1.7[7]: Let X be a nonempty convex subset of Hausdorff topological vector spaces E. Let $S, T : X \to 2^X$ satisfy

(i) $S(x) \subset T(x)$ for each $x \in X$;

(ii) $x \notin T(x)$ for each $x \in X$;

(iii) T(x) is convex or empty for each $x \in X$;

(iv) $S^{-1}(y) = \{x \in X \mid y \in S(x)\}$ is open in X for each $y \in X$;

(v) X has a nonempty compact convex subset X_0 such that $C = \{x \in X : y \notin S(x) \text{ for all } y \in X_0\}$ is compact.

Then there exist an $x_0 \in X$ such that $S(x_0) = \varphi$.

Theorem 1.8[7]: Let X be a nonempty convex subset of Hausdorff topological vector spaces E. Let $S, T : X \to 2^X$ satisfy the following conditions.

(i) $S(x) \subset T(x)$ for each $x \in X$;

(ii) $S(x) \neq \varphi$ for each $x \in X$,

(iii) T(x) is convex for each $x \in X$,

(iv) $S^{-1}(y)$ is open in X for each $x \in X$,

(v) X has a nonempty compact convex subset X_0 such that $C = \{x \in X : y \notin S(x) \text{ for all } y \in X_0\}$ is compact.

Then T has a fixed point in X.

2. Main Results

We develop KKM-maps using set valued maps. Further, we give theorems, corollaries and give also results on existence of fixed points under certain condition.

Theorem 2.1: Let X be nonempty convex subset of topological vector spaces E. Let $S, T, R : X \to 2^X$ be set-valued maps satisfying the following conditions:

(i) $S(x) \subset T(x) \subset R(x)$ for every $x \in X$;

(ii) For every $x \in X$, $x \in S(x)$;

(iii) For every $y \in X$, $\{x \in X \mid y \in R(x)\}$ is closed in X;

(iv) For every $x \in X$, the set $\{y \in X \mid y \notin T(x)\}$ is convex or empty;

(v) X has a nonempty compact convex subset X_0 such that $\bigcap_{y \in X_0} \{x \in X : y \in R(x)\}$ is compact.

Then there exist an $x_0 \in X$ such that $y \in R(x_0)$ for every $y \in X$.

Proof: Let $F : X \to 2^X$ be defined by $F(y) = \{x \in X : y \in R(x)\}$, for all $y \in X$. Then F(y) is closed in X by condition (iii). Also $y \in F(y)$ and $F(y) \neq \varphi$ by conditions (i) and (ii). Next we have to prove that $F : X \to 2^X$ is a KKM-map. Suppose $y_1,...,y_n \in X$ and $0 \le \lambda_i \le 1$, $i = 1,...., n$. Let $z = \sum_{i=1}^{n} \lambda_i y_i$. We need to show that $z \in \bigcup_{i=1}^{n} F(y_i)$.

On the contrary, let $z \notin \bigcup_{i=1}^{n} F(y_i)$. It implies that $z \notin F(y_i) = \{x \in X : y_i \in R(x)\}$ for each $y_i, i = 1,...,n$. Then, $y_i \notin R(z)$ i.e., $y_i \notin T(z)$ for $i = 1,..., n$ by condition (i). By condition (iv) applied to this z, the set $\{y \in X : y \notin T(z)\}$ is convex or empty for all $z \in X$. If the set $\{y \in X \mid y \notin T(z)\}$ is empty, the result follows. If not, the set $\{y \in X \mid y \notin T(z)\}$ is convex.

Then, $z = \sum_{i=1}^{n} \lambda_i y_i \in \{y \in X : y \notin T(z)\}$ which implies $z \notin T(z)$. So, $z \notin S(z)$, a contradiction to (ii). Hence, $z \in \bigcup_{i=1}^{n} F(y_i)$ i.e., F is KKM map. By condition (v), $\bigcap_{y \in X_0} F(y)$ is compact.

Hence the conditions of theorem 1.3 are satisfied for the topology of E.

Therefore, $\bigcap_{y \in X} F(y) \neq \varphi$ i.e., $\exists\ x_0 \in X$ s.t. $x_0 \in \bigcap_{y \in X} F(y)$ i.e., $x_0 \in F(y) \Rightarrow y \in R(x_0)$, $\forall\ y \in X$.

Remark: If X be a non empty compact convex subset of a topological vector spaces E, the condition (v) is redundant and accordingly may be dropped.

Corollary 2.2: Let X be non-empty convex subset of topological vector space E. Let $S, T : X \to 2^X$ be set -valued maps satisfying

(i) $S(x) \subset T(x)$ for each $x \in X$;

(ii) For each $x \in X$, $x \in S(x)$;

(iii) For each $y \in X$, the set $\{x \in X \mid y \in T(x)\}$ is closed in X;

(iv) For each $x \in X$, the set $\{y \in X \mid y \notin S(x)\}$ is convex or empty;

(v) X has a nonempty compact convex subset X_0 such that $\bigcap_{y \in X_0} \{x \in X : y \in T(x)\}$ is compact.

Then there exist an $x_0 \in X$ such that $y \in T(x_0)$ for every $y \in X$.

Proof: One can prove very easily by letting S = T and T = R in above theorem.

Corollary 2.3: Let X be non empty convex subsets of topological vector spaces E. Let $S : X \to 2^X$ be set-valued map satisfying the following conditions:

(i) For every $x \in X$, $x \in S(x)$;

(ii) For every $y \in X$, the set $\{x \in X \mid y \in S(x)\}$ is closed in X;

(iii) For every $x \in X$, the set $\{y \in X \mid y \notin S(x)\}$ is convex or empty;

(iv) X has a nonempty compact convex subset X_0 such that $\bigcap_{y \in X_0} \{x \in X : y \in S(x)\}$ is compact.

Then there exists an $x_0 \in X$ such that $y \in S(x_0)$ for every $y \in X$.

Proof: By letting S = T = R in above theorem, the proof follows.

Corollary 2.4: Let X be nonempty convex subsets of vector spaces E. Let $S, T, R : X \to 2^X$ be set valued maps satisfying the following conditions.

(i) $S(x) \subset T(x) \subset R(x)$ for each $x \in X$;

(ii) For every $x \in X$, $x \in S(x)$;

(iii) For every $y \in X$, $\{x \in X : y \in R(x)\}$ is weakly closed in X;

(iv) For each $x \in X$, the set $\{y \in X : y \notin T(x)\}$ is convex or empty;

(v) X has a nonempty compact convex subset X_0 such that $\bigcap_{y \in X_0} \{x \in X: y \in R(x)\}$ is weakly compact.

Then there exists an $x_0 \in X$ such that $y \in R(x_0)$ for every $y \in X$.

Proof: Let $F : X \to 2^X$ be defined by $F(y) = \{x \in X : y \in R(x)\}$ for all $y \in X$. Then F(y) is weakly closed in X by condition (iii). Further by condition (v), $\bigcap_{y \in X_0} F(y)$ is weakly compact.

Hence the conditions of theorem 2.1 are satisfied for the weak topology of E.

Therefore, there exist $x_0 \in X$ such that $x_0 \in \bigcap_{y \in X} F(y)$. That is $x_0 \in F(y)$ implies $y \in R(x_0)$ for every $y \in X$.

Corollary 2.5: Let X be nonempty compact convex subset of topological vector spaces E. Let $S, T, R : X \to 2^X$ be set-valued maps satisfying the following conditions.

(i) $S(x) \subset T(x) \subset R(x)$ for each $x \in X$;

(ii) For each $x \in X$, $x \in S(x)$;

(iii) For each $x \in X$, $\overline{\{y \in X : y \in R(x)\}}$ is a subset of X;

(iv) For each $y \in X$, the set $\{x \in X : y \notin T(x)\}$ is convex or empty.

Then $\bigcap_{y \in X} \overline{\{x \in X : y \in R(x)\}} \neq \varphi$.

Proof: Its proof follows from that of Theorem 2.1.

Theorem 2.6: Let X be a nonempty convex subset of a topological vector space E. Let $S, T, R : X \to 2^X$ be set valued maps satisfying the following conditions.

(i) $S(x) \subset T(x) \subset R(x)$ for each $x \in X$;

(ii) For each $y \in X$, the set $\{x \in X : y \in S(x)\}$ is open in X;

(iii) For each $x \in X$, the set $\{y \in X : y \in T(x)\}$ is convex and $\{y \in X : y \in S(x)\}$ is nonempty;

(iv) X has a nonempty compact convex subset X_0 such that the set $\{x \in X : y \notin S(x) \text{ for all } y \in X_0\}$ is compact.

Then R has a fixed point i.e., $x_0 \in R(x_0)$ for all $x_0 \in X$.

Proof: Let $G(x) = X - S(x)$, $H(x) = X - T(x)$ and $I(x) = X - R(x)$.

Then $I(x) \subset H(x) \subset G(x)$. Let $F : X \to 2^X$ be a map defined by

$F(y) = \{x \in X : y \in G(x)\} = \{x \in X : y \notin S(x)\}$ for all $y \in X$.

Then F(y) is closed in X by condition (ii). By condition (iii), the set $\{y \in X : y \notin H(x)\} = \{y \in X : y \in T(x)\}$ is convex for each $x \in X$

By condition (iv), $\bigcap_{y \in X_0} F(y) = \{x \in X : y \in G(x) \text{ for all } y \in X_0\}$

$= \{x \in X : y \notin S(x) \text{ for all } y \in X_0\}$ is compact.

If we assume that R has no fixed point i.e., $x \notin R(x)$ for all $x \in X$. Then $x \in I(x)$ for all $x \in X$. Theorem 2.1 implies that there exist an $x_0 \in X$ such that $y \in G(x_0)$ i.e., $y \notin S(x_0)$ for all $y \in X$.

Therefore, $\{y \in X : y \in S(x_0)\} = \varphi$, a contradiction to (iii).

Thus, there exists a $x_0 \in X$ such that $x_0 \in R(x_0)$. Hence, R has a fixed point.

Theorem 2.7: Let X be nonempty convex subset of topological vector spaces E. Let $S, T : X \to 2^X$ be set-valued maps satisfying the following conditions.

(i) For each $x \in X, S(x) \subset T(x)$;

(ii) $x \notin co(T(x))$ for each $x \in X$;

(iii) $S^{-1}(y) = \{x \in X : y \in S(x)\}$ is open in X for each $y \in X$;

(iv) X has a nonempty compact convex subset X_0 such that the set $\bigcap_{y\in X_0}\{x\in X: y\notin S(x)\}$ is compact.

Then there exists an $x_0\in X$ such $S(x_0)=\varphi$.

Proof: Let $F(y)=X-S^{-1}(y)=\{x\in X: y\notin S(x)\}$ for each $y\in X$. Then F(y) is closed in X for each $y\in X$ by condition (iii).

We have to show that F is KKM map. Let $z\in co\{y_1,...,y_n\}$. If $z\notin\bigcup_{i=1}^{n}F(y_i)$, then $z\notin F(y_i)$ which implies $z\in S^{-1}(y_i)\ \Rightarrow y_i\in S(z)\ \Rightarrow y_i\in T(z)$ by condition (i), for all $i=1,...,n$. $\Rightarrow\ co\{y_1,...,y_n\}\in coT(z)$ which implies $z\in coT(z)$ a contradiction to (ii)

Hence, F is KKM-map. Further, by condition (iv), the set $\bigcap_{y\in X_0}F(y)$ is compact. Therefore, by theorem 1.3, there exists $x_0\in\bigcap_{y\in X_0}F(y)$ which implies $x_0\notin S^{-1}y$ for all $y\in X$. Then, $y\notin S(x_0)$ for all $y\in X$. Hence, $S(x_0)=\varphi$.

REFERENCES

[1] B. Brosowski (1969): Fixpunktsatze in der approximations-theorie, *Mathematica* (*Cluj*)., Vol. 11, pp. 195-220.

[2] F. E. Browder (1968): The fixed point theory of multivalued mappings in topological vector spaces, *Math. Ann.*, Vol. 177, pp. 283-301.

[3] Ky Fan (1961): A generalization of Tychonoff's fixed point theorem, *Math. Ann.*, Vol. 142, pp. 305-310.

[4] Ky Fan (1984): Some properties of convex sets related to fixed point theorems *Math. Ann.*, Vol. 266, pp. 519-537.

[5] A.Granas (1982): KKM-maps and their applications to non linear problems, *The Scottish Book,* Ed. R. D Mauldin, Birkhauser, Boston, pp. 45-61.

[6] T. C. Lin (1986): Convex sets, fixed points, variational and minimax inequalities, *Bull. Austr. Math. Soc.*, Vol. 34, pp. 107-117.

[7] S. Singh, B. Watson and P. Srivastava (1997): Fixed Point Theory and Best Approximation, The KKM-map Principle, Kluwer Academic Publishers, Dordrecht.

[8] N. C. Yannelis and N. D. Prabhakar (1985): On a market equilibrium theorem with an infinite number of commodities, *J. Math .Anal. Appl.*, Vol. 108, pp. 595-599.

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