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L. Priyanandini Devi[1]
and
M. Ranjit Singh[2]

KKM-MAPS IN NORMED SPACES

Abstract: In 1928, Sperner established a combinatorial lemma. In 1929, from this Sperner's lemma, Knaster, Kuratowski and Mazurkiewicz established the well-known KKM Theorem on the closed cover of a simplex. In this paper we extend and generalize the results of KKM-map related to normed space and a few results are derived as corollary.

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1. Introduction

In 1961 Ky Fan generalized the KKM Theorem to a subset of any topological vector space. There are many generalizations and applications of this theorem in fixed point theory, approximation theory, minimax theory and variational inequalities. We can mention authors like Prolla [9], Li [8], Carbone [1] extended and improve KKM theorem in normed linear spaces'. Before we state the theorem we recall some well known definitions and other relevant results.

Let X be a subset of a vector space E. A map $F: X \to 2^E$ is called a KKM-map if $co\{x_1,...,x_n\} \subseteq \bigcup_{i=1}^{n} F(x_i)$ for each finite subset $\{x_1,\ldots,x_n\}$ of X.

For details on KKM maps we refer Granas [2].

A function f is said to be *strongly continuous* if $x_n \to x$ weakly implies that $fx_n \to fx$ strongly. A function $g : X \to E$, where X is non-empty convex set, is said to be *almost affine* if $|| g(t) - y || \leq || g(t_1) - y || + (1 - \lambda) || g(t_2) - y ||$ for all $t_1, t_2 \in X, 0 < \lambda < 1, t = \lambda t_1 + (1-\lambda)t_2$ and $y \in E$.

Theorem 1.1[8]: Let C be a nonempty convex set in a normed vector space X and let $f : C \to X$ be a mapping such that

(i) for every $y \in C, \{x \in C : ||f(x) - y|| \geq ||f(x) - x||\}$ is a closed subset of C;

(ii) there exists a compact convex subset D of C, such that the set $\bigcap_{y \in D} \{x \in C : ||f(x) - y|| \geq ||f(x) - x||\}$ is contained in a compact subset of C.

Then there exists a point $x_0 \in C$ such that $||f(x_0) - x_0|| = d(f(x_0), C)$.

Theorem 1.2 [5]: Let Y be a convex set in a topological vector space E and X a nonempty subset of Y. For each $x \in$ X, let F(x) be a relatively closed subset of Y such that the convex hull of every finite subset $\{x_1, ..., x_n\}$ of X is contained in the corresponding union $\bigcup_{i=1}^{n} F(x_i)$. If there is a nonempty subset X_0 of X such that the intersection $\bigcap_{x \in X_0} F(x)$ is compact and X_0 is contained in a compact convex subset of Y then $\bigcap_{x \in X} F(x) \neq \varphi$.

Theorem 1.3 [9]: Let C be a nonempty compact convex subset of a normed space X and $g : C \to C$ a continuous, almost affine, onto mapping. Then, for each continuous mapping $f : C \to X$ there exists $x \in C$ satisfying $||g(x) - f(x)|| = d(f(x), C)$.

2. Main Results

In [8], Li studied some applications of KKM theorem to approximation theory and fixed point there. Here, the author used one map satisfying some conditions. The aim of this paper is to generalize some results of [8] by using one more almost affine self map. This work also extends and improve KKM theorem in normed linear space.

Theorem 2.1: Let C be a nonempty convex subset of a normed linear space X and g be almost affine self map of C onto C. Let $f : C \to X$ be a map satisfying the following conditions

(i) for every $x \in C$, $\{y \in C : ||g(y) - f(y)|| \le ||g(x) - f(y)|| \}$ is closed in C;

(ii) C has a nonempty compact convex subset C_o such that the set $\{y \in C : ||g(y) - f(y)|| \le ||g(x) - f(y)|| \text{ for all } x \in C_0\}$ is compact.

Then there will be a point $y_0 \in C$ such that $||g(y_0) - f(y_0)|| = d(f(y_0), C)$.

Proof: Let $F : C \to 2^X$ be defined by

$$F(x) = \{y \in C : ||g(y) - f(y)|| \le ||g(x) - f(y)||\}, \text{ for every } x \in C.$$

Then by condition (i), F(x) is closed in C. It is obvious that $x \in F(x)$ for all $x \in C$. Next, we have to show that $F : C \to 2^X$ is a KKM map that is, $c_0\{x_1,...., x_n\} \subset \bigcup_{i=1}^{n} F(x_i)$ for any finite subset $\{x_1,..., x_n\}$ of X.

Let $z = \sum_{i=1}^{n} \lambda_i x_i$, where $0 \le \lambda_i \le 1$, $\sum_{i=1}^{n} \lambda_i = 1$, $i = 1, 2, ..., n$.

If $z \notin \bigcup_{i=1}^{n} F(x_i)$, then $||g(z) - f(z)|| > ||g(x_i) - f(z)||\}$ for $i = 1, 2, ..., n$.

$$\Rightarrow \lambda_1||g(z) - f(z)|| + ... + \lambda_n||g(z) - f(z)|| > \lambda_1||g(x_1) - f(z)|| + ... + \lambda_n||g(x_n) - f(z)||$$

$$\Rightarrow \sum_{i=1}^{n} \lambda_i||g(z) - f(z)|| > ||g(\lambda_1 x_1 + ... + \lambda_n x_n) - f(z)|| \quad [\because \text{ g is almost affine map}]$$

$$\Rightarrow ||g(z) - f(z)|| > ||g(z) - f(z)|| \quad [\text{since} \quad z = \sum_{i=1}^{n} \lambda_i x_i, \sum_{i=1}^{n} \lambda_i = 1] \quad \text{which is meaningless.}$$

Hence, $z \in \bigcup_{i=1}^{n} F(x_i)$. Thus, $F : C \to 2^X$ is a KKM-map.

By (ii), the set

$$\bigcap_{x\in C_0} F(x) = \{y \in C : ||g(y) - f(y)|| \le ||g(x) - f(y)|| \text{ for all } x \in C_0\} \text{ is compact.}$$

Then by Theorem 1.2, $\bigcap_{x\in C} F(x) \neq \varphi$.

Therefore, there exists a point $y_0 \in C$ such that $y_0 \in \bigcap_{x\in C} F(x)$ which implies $||g(y_0) - f(y_0)|| \le ||g(x) - f(y_0)|| = \min_{x\in C} ||g(x) - f(y_0)|| = d(f(y_0), C)$ since g is onto. Hence proved.

Remark: If $g = I$, identity map then we get Theorem 1.1 of Li [8] as a corollary of this Theorem 2.1.

Corollary 2.2: Let C be a nonempty convex subset of a normed linear space X and g be continuous almost affine self map of C onto C. Let $f : C \to X$ be a continuous map satisfying the condition

(i) C has a nonempty compact convex subset C_o such that the set $\{y \in C : ||g(y) - f(y)|| \le ||g(x) - f(y)|| \text{ for all } x \in C_0\}$ is compact.

Then there will be a point $y_0 \in C$ such that $||g(y_0) - f(y_0)|| = d(f(y_0), C)$.

Proof: For every $x \in C$, let $F : C \to 2^X$ be defined by

$$F(x) = \{y \in C : ||g(y) - f(y)|| \le ||g(x) - f(y)||\}$$

Since f and g are continuous, F(x) is closed and hence (i) of above Theorem 2.1 is satisfied. Therefore, the result follows.

Remark: In case when C itself is also a compact convex set the above result reduces to Theorem1.3 (Theorem 1of Prolla [9]). In addition to this, if $g = I$, identity function condition, we have the Theorem of Ky Fan [4] (Theorem 2.1, p.74 [12])

Corollary 2.3: Let C be a nonempty convex subset of a normed linear space X and *g* be strongly continuous, almost affine self map of C onto C. Let $f : C \to X$ be a strongly continuous map satisfying the following condition

C has a nonempty weakly compact convex subset C_o such that the set $\{y \in C : ||g(y) - f(y)|| \le ||g(x) - f(y)|| \text{ for all } x \in C_0\}$ is weakly compact.

Then there will be a point $y_0 \in C$ such that $||g(y_0) - f(y_0)|| = d(f(y_0), C)$.

Proof: Suppose $F : C \to 2^X$ be defined by

$$F(x) = \{y \in C : ||g(y) - f(y)|| \le ||g(x) - f(y)||\}$$

Since f and g are strongly continuous functions, F(x) is weakly closed in C and $x \in F(x)$ for all $x \in C$. As g is almost affine map, so F is a KKM map.

Also $\bigcap_{x \in C_0} F(x) = \{y \in C : ||g(y) - f(y)|| \le ||g(x) - f(y)|| \text{ for all } x \in C_0\}$ is weakly compact. Satisfying the above corollary 2.2 for a weak topology of X, the result follows.

Remark: In case when C itself is a nonempty weakly compact set corollary 2.3 reduces to Theorem 3 of Carbone [1]. Further, when $g = I$, we have the result of Kapoor (Theorem 3.25, p.133[12]).

Corollary 2.4: Let C be a nonempty convex subset of a normed linear space X and g be almost affine self map of C onto C. Let $f : C \to C$ be a one to one map satisfying the following condition

(i) for every $x \in C$, $\{y \in C : ||g(y) - f(y)|| \le ||g(x) - f(y)|| \}$ is closed in C;

(ii) C has a nonempty compact convex subset C_0 such that the set $\{y \in C : ||g(y) - f(y)|| \le ||g(x) - f(y)|| \text{ for all } x \in C_0\}$ is compact.

Then there will be a coincidence point $y_0 \in C$ i.e., $g(y_0) = f(y_0)$ for some $y_0 \in X$.

Proof: Let us define $F : C \to 2^C$ by

$$F(x) = \{y \in C : ||g(y) - f(y)|| \le ||g(x) - f(y)|| \} \text{ for all } x \in C.$$

By condition (i), F(x) is closed in C.

Hence, by Theorem 2.1, $\|g(y_0) - f(y)\| = d(f(y_0, C)\|$

Since f is self map and one to one, for each $y_0 \in C$ implies $f(y_0) \in C$.

Hence, we get $f(y_0) = g(y_0)$, that is y_0 is a coincidence point.

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1. Dr. L.Priyanandini Devi
Research Scholar in Mathematics, Manipur University
Presently: Science Graduate Teacher (Math),
Thoubal Leitanthem Kanhai Junior High School (Govt. School) ,
Thoubal, Manipur.795138
Mob.No.8415096943
E-mail: priyadevilaishangbam@gmail.com

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2. Prof. M. Ranjit Singh
Dept.of Mathematics,
Manipur University,Canchipur-795003,Imphal
Mob.No. 9436891017
E-mail: mranjitmu@rediffmail.com