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SUMS INVOLVING EXTENDED GIBONACCI POLYNOMIALS

Abstract: We explore three infinite gibonacci polynomial sums, and their Pell and Jacobsthal versions.

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1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*.

Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 4,].

On the other hand, let $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial*. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$; and $j_n(1) = L_n$ [3, 4].

Gibonacci and Jacobsthal polynomials are linked by the relationships $J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$ and $j_n(x) = x^{n/2} l_n(1/\sqrt{x})$ [4].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $c_n = J_n$ or j_n, $\Delta = \sqrt{x^2 + 4}$, $E = \sqrt{x^2 + 1}$, and $D = \sqrt{4x + 1}$, where $c_n = c_n(x)$.

1.1 Fundamental Gibonacci Identities

Using Binet-like formulas, we can establish the following gibonacci identities [4]:

$$f_a l_a = f_{2a}; \tag{1}$$

$$l_{a-1} + l_{a+1} = \Delta^2 f_a; \tag{2}$$

$$l_{a+b} + (-1)^b l_{a-b} = l_a l_b; \tag{3}$$

$$l_{a+b} - (-1)^b l_{a-b} = \Delta^2 f_a f_b. \tag{4}$$

Identities (3) and (4) imply that

$$l_{2a} + 2(-1)^a = l_a^2; \tag{5}$$

$$l_{2a} - 2(-1)^a = \Delta^2 f_a^2. \tag{6}$$

respectively.

These identities play a major role in our discourse.

2. Gibonacci Sums

With these identities at our fingertips, we now begin our exploration with the following result.

Theorem 1: *Let*

$$\nu_n(x) = \begin{cases} x, & if \ \ n = 1 \\ 1, & otherwise. \end{cases}$$

Then
$$\sum_{n=1}^{\infty} \frac{\nu_n f_{2n-1}}{l_{2^n} + 1} = \frac{1}{\Delta}. \tag{7}$$

Proof: Let $m \geq 2$. Using recursion [4], we will first establish that

$$\sum_{n=1}^{m} \frac{\nu_n f_{2n-1}}{l_{2^n} + 1} = \frac{f_{2^m}}{l_{2^m} + 1}. \tag{8}$$

To this end, we let A_m denote the left side (LHS) of this equation and B its right side (RHS). With $m \geq 2$, identity (5) yields $l_{2^m} = l_{2^{m-1}}^2 - 2$. Identity (1), coupled with this result, then gives

$$\begin{aligned} B_m - B_{m-1} &= \frac{f_{2^m}}{l_{2^m} + 1} - \frac{f_{2^m-1}}{l_{2^m-1} + 1} = \frac{f_{2^m}}{l_{2^m} + 1} - \frac{(l_{2^m-1} - 1) f_{2^m-1}}{(l_{2^m-1} - 1)(l_{2^m-1} + 1)}; \\ &= \frac{f_{2^m}}{l_{2^m} + 1} - \frac{(l_{2^m-1} - 1) f_{2^m-1}}{(l_{2^m-1} - 1)(l_{2^m-1} + 1)} = \frac{f_{2^m-1}}{l_{2^m} + 1}; \\ &= A_m - A_{m-1}. \end{aligned}$$

With recursion, this yields

$$\begin{aligned} A_m - B_m &= A_{m-1} - B_{m-1} = \cdots = A_2 - B_2 \\ &= \left(\frac{x}{l_2 + 1} + \frac{x}{l_4 + 1} \right) - \frac{f_4}{l_4 + 1} \\ &= 0. \end{aligned}$$

Thus, $A_m = B_m$, as claimed.

Since, $\lim_{m\to\infty} \frac{f_m}{l_m + 1} = \frac{1}{\Delta}$, the given result now follows from equation (8), as desired. □

It follows from equation (7) that [2, 4, 6],

$$\sum_{n=1}^{\infty} \frac{F_{2^n-1}}{L_{2^n} + 1} = \frac{\sqrt{5}}{5}. \tag{9}$$

The next result invokes identities (1), (3), and (5).

Theorem 2:

$$\sum_{n=1}^{\infty} \frac{l_{2n+1}}{f_{3.2^n}} = \frac{\Delta^2}{l_3}. \tag{10}$$

Proof: With $a = 3{\cdot}2^{m-1}$ and $b = 2^{m-1}$, it follows from identity (3) that

$$l_{2^{m+1}} + l_{2^m} = l_{3{\cdot}2^{m-1}} l_{2^{m-1}}. \tag{11}$$

Using recursion [4] and identities (1), (3), and (11), we will now confirm that

$$\sum_{n=1}^{m} \frac{l_{2n+1}}{f_{3.2^n}} = \frac{l_4}{f_6} + \frac{l_2}{f_6} - \frac{l_{2^m}}{f_{3.2^m}}, \tag{12}$$

where $m \geq 2$, Letting $A_m = \text{LHS}$ and $B = \text{RHS}$ of this equation, we then get

$$\begin{aligned} B_m - B_{m-1} &= \frac{l_{2^{m-1}}}{f_{3.2^{m-1}}} - \frac{l_{2^m}}{f_{3.2^m}} = \frac{l_{3.2^{m-1}} l_{2^{m-1}}}{f_{3.2^m}} - \frac{l_{2^m}}{f_{3.2^m}}; \\ &= \frac{l_{2^{m+1}} + l_{2^m}}{f_{3.2^m}} - \frac{l_{2^m}}{f_{3.2^m}} = \frac{l_{2^{m+1}}}{f_{3.2^m}}; \\ &= A_m - A_{m-1}. \end{aligned}$$

Recursively, this yields

$$A_m - B_m = A_{m-1} - B_{m-1} = \cdots = A_2 - B_2$$

$$= \left(\frac{l_4}{f_6} + \frac{l_8}{f_{12}}\right) - \left(\frac{l_4}{f_6} + \frac{l_2}{f_6} - \frac{l_4}{f_{12}}\right) = \left(\frac{l_4}{f_6} + \frac{l_8}{f_{12}}\right) - \left(\frac{l_4}{f_6} + \frac{l_8}{f_{12}}\right)$$

$$= 0,$$

confirming the validity of equation (12).

Since, $\lim_{m\to\infty} \frac{l_m}{f_{3m}} = 0$, equation (12) yields the desired result. □

It then follows that

$$\sum_{n=1}^{\infty} \frac{l_{2^n+1}}{f_{3.2^n}} = \frac{l_4}{f_6} + \frac{l_2}{f_6} = \left(\frac{l_4}{f_6} + \frac{l_8}{f_{12}}\right) - \left(\frac{l_4}{f_6} + \frac{l_2}{f_6} - \frac{l_4}{f_{12}}\right) = \frac{\Delta^2}{l_3}$$

$$\sum_{n=0}^{\infty} \frac{l_{2^n+1}}{f_{3.2^n}} = \frac{l_2}{f_3} + \frac{\Delta^2}{l_3}.$$

Consequently, we have [5, 7]

$$\sum_{n=1}^{\infty} \frac{L_{2^n+1}}{F_{3.2^n}} = \frac{5}{4}; \qquad \sum_{n=0}^{\infty} \frac{L_{2^n+1}}{F_{3.2^n}} = \frac{11}{4},$$

respectively.

The following result showcases an application of identity (5).

Theorem 3:

$$\sum_{n=1}^{\infty} \frac{f_{2^n-1}^2}{l_{2^n}^2 - 1} = \frac{\Delta^2 + 1}{\Delta^2(l_2^2 - 1)}. \tag{13}$$

Proof: Using recursion [4], we will first establish that

$$\sum_{n=1}^{m} \frac{f_{2^n-1}^2}{l_{2^n}^2 - 1} = \frac{1}{l_2^2 - 1} + \frac{1}{\Delta^2}\left(\frac{1}{l_2^2 - 1} - \frac{1}{l_{2^m}^2}\right), \tag{14}$$

where $m \geq 2$.

With $m \geq 2$ and $a = 2^{m-1}$, it follows by identities (5) and (6) that $l_{2^m} - 2 = \Delta^2 f_{2^m-1}^2$ and $l_{2^m} + 2 = f_{2^m-1}^2$, respectively. Then

$$\Delta^2 f_{2^m-1}^2 (l_{2^m-1}^2 - 1) = (l_{2^m} - 2)(l_{2^m} + 1) = l_{2^m}^2 - l_{2^m} - 2 = l_{2^m}^2 - l_{2^m-1}.$$

Let $A_m = \text{LHS}$ and $B_m = \text{RHS}$ of equation (13). Then

$$B_m - B_{m-1} = \frac{1}{\Delta^2}\left(\frac{1}{l_{2^m-1}^2 - 1} - \frac{1}{l_{2^m}^2 - 1}\right) = \frac{l_{2^m}^2 - l_{2^m-1}^2}{\Delta^2 (l_{2^m-1}^2 - 1)(l_{2^m}^2 - 1)}$$

$$= \frac{\Delta^2 f_{2^m-1}^2 (l_{2^m-1}^2 - 1)}{\Delta^2 (l_{2^m-1}^2 - 1)(l_{2^m}^2 - 1)} = \frac{l_{2^m-1}^2}{l_{2^m}^2 - 1}$$

$$= A_m - A_{m-1}.$$

Recursively, this yields

$$A_m - B_m = A_{m-1} - B_{m-1} = \cdots = A_2 - B_2$$

$$= \left(\frac{f_1^2}{l_2^2 - 1} + \frac{f_2^2}{l_4^2 - 1}\right) - \left[\frac{1}{l_2^2 - 1} + \frac{1}{\Delta^2}\left(\frac{1}{l_2^2 - 1} - \frac{1}{l_4^2 - 1}\right)\right]$$

$$= \left(\frac{f_1^2}{l_2^2 - 1} + \frac{f_2^2}{l_4^2 - 1}\right) - \left[\frac{1}{l_2^2 - 1} + \frac{l_4^2 - l_2^2}{\Delta^2 (l_2^2 - 1)(l_4^2 - 1)}\right]$$

$$= \left(\frac{f_1^2}{l_2^2 - 1} + \frac{f_2^2}{l_4^2 - 1}\right) - \left[\frac{1}{l_2^2 - 1} + \frac{\Delta^2 f_2^2 (l_2^2 - 1)}{\Delta^2 (l_2^2 - 1)(l_4^2 - 1)}\right]$$

$$= \left(\frac{f_1^2}{l_2^2 - 1} + \frac{f_2^2}{l_4^2 - 1} \right) - \left(\frac{1}{l_2^2 - 1} + \frac{f_2^2}{l_4^2 - 1} \right)$$

$$= 0,$$

confirming the validity of equation (14), as desired.

Since $\lim_{m \to \infty} \frac{1}{l_{2^m}^2 - 1} = 0$, the given result now follows from it. □

This theorem implies [4, 5, 7],

$$\sum_{n=1}^{\infty} \frac{F_{2^{n-1}}^2}{L_{2^n}^2 - 1} = \frac{3}{20}.$$

We now explore the Pell versions of equations (7), (10), and (13).

3. Pell Consequences

Using the Pell-gibonacci relationship $b_n(x) = g_n(2x)$ [4], we get the desired counterparts and their numeric versions:

$$\sum_{n=1}^{\infty} \frac{\nu_n(2x) p_{2^n - 1}}{q_{2^n} + 1} = \frac{1}{2E}; \qquad \sum_{n=1}^{\infty} \frac{\nu_n(2) P_{2^n - 1}}{2Q_{2^n} + 1} = \frac{\sqrt{2}}{4};$$

$$\sum_{n=1}^{\infty} \frac{q_{2^n + 1}}{p_{3.2^n}} = \frac{4E^2}{q_3}; \qquad \sum_{n=1}^{\infty} \frac{Q_{2^n + 1}}{P_{3.2^n}} = \frac{2}{7};$$

$$\sum_{n=1}^{\infty} \frac{p_{2^n - 1}^2}{q_{2^n}^2 - 1} = \frac{4E^2 + 1}{4E^2 (q_2^2 - 1)}; \qquad \sum_{n=1}^{\infty} \frac{P_{2^n - 1}^2}{4Q_{2^n}^2 - 1} = \frac{9}{64}.$$

respectively.

Next, we explore the Jacobsthal versions of the gibonacci sums (7), (10), and (13) using the Jacobsthal-gibonacci relationships in Section 1.

4. Jacobsthal Consequences

In the interest of brevity and clarity, we let A denote the left-hand side of each equation and B its right-hand side, and LHS and RHS those of the corresponding Jacobsthal equation, respectively.

4.1 Jacobsthal Version of Equation (7): Proof: Let $A = \dfrac{\nu_n f_{2^n-1}}{l_{2^n} + 1}$. Replacing x with $1/\sqrt{x}$, and multiplying the numerator and denominator of the resulting expression with x^{2^n-1}, we get

$$A = \frac{\nu_n(1/\sqrt{x})x(x^{2^n-1-1}f_{2^n-1}}{x^{2^n-1}l_{2^n} + x^{2^n-1}};$$

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{\nu_n(1/\sqrt{x})xJ_{2^n-1}}{j_{2^n} + x^{2^n-1}},$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Now, let $B = \dfrac{1}{\Delta}$. Replacing x with $1/\sqrt{x}$, we get

$$\text{RHS} = \frac{\sqrt{x}}{D}.$$

Equating the two sides, we get the desired Jacobsthal version:

$$\sum_{n=1}^{\infty} \frac{\nu_n(1/\sqrt{x})xJ_{2^n-1}}{j_{2^n} + x^{2^n-1}} = \frac{\sqrt{x}}{D}. \tag{15}$$

□

In particular, this yields [2, 4, 6]

$$\sum_{n=1}^{\infty} \frac{F_{2^n-1}}{L_{2^n} + 1} = \frac{\sqrt{5}}{5},$$

as in equation (9).

4.2 Jacobsthal Version of Equation (10): Proof: Let $A = \dfrac{l_{2^{n+1}}}{f_{3.2^n}}$. Now, replace x with $1/\sqrt{x}$, and multiply the numerator and denominator of the resulting expression with $x^{(3.2^n-1)/2}$. Then

$$A = \frac{x^{(2^n-1)/2}[x^{2^{(n+1)/2}} l_{2^{n+1}}}{x^{(3.2^n-1)/2} f_{3.2^n}};$$

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{x^{(2^n-1)/2} j_{2^{n+1}}}{J_{3.2^n}},$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Next, we let $B = \dfrac{\Delta^2}{l_3}$. Replacing x with $1/\sqrt{x}$, and then multiplying the numerator and denominator with $\sqrt{x}$, we get

$$B = \frac{D^2}{x l_3(1/\sqrt{x})};$$

$$\text{RHS} = \frac{D^2\sqrt{x}}{j_3}.$$

Equating the two sides yields the desired Jacobsthal version:

$$\sum_{n=1}^{\infty} \frac{x^{(2^n-1)/2} j_{2^{n+1}}}{J_{3.2^n}} = \frac{D^2\sqrt{x}}{j_3}. \tag{16}$$

□

This implies [5, 7],

$$\sum_{n=1}^{\infty} \frac{L_{2^{n+1}}}{F_{3.2^n}} = \frac{5}{4},$$

as we found earlier.

Finally, we explore the Jacobsthal counterpart of equation (13).

4.3 Jacobsthal Version of Equation (13): Proof: Let $A = \dfrac{f_{2^{n+1}}^2}{l_{2^n}^2 - 1}$.

Replacing x with $1/\sqrt{x}$, and then multiplying the numerator and denominator of the resulting expression with x^{2^n}, we then get

$$A = \frac{x^{2^{n-1}+1}\left[x^{(2^{n-1})/2} f_{2^{n-1}}\right]^2}{(x^{2^n/2} l_{2^n})^2 - x^{2^n}};$$

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{x^{2^{n-1}+1} j_{2^{n-1}}^2}{J_{2^n}^2 - x^{2^n}},$$

where $g_n = g_n(1/\sqrt{x})$ and $c_n = c_n(x)$.

Next, we let $B = \dfrac{\Delta^2 + 1}{\Delta^2(l_2^2 - 1)}$. Now replace x with $1/\sqrt{x}$, and then multiplying the numerator and denominator with x^2. This gives,

$$B = \frac{x\left(\frac{D^2}{x} + 1\right)x^2}{D^2\left[\left(x^{2/2} l_2\right)^2 - x^2\right]};$$

$$\text{RHS} = \frac{(D^2 + x)x^2}{D^2(j_2^2 - x^2)}.$$

Combining the two sides yields the desired Jacobsthal version:

$$\sum_{n-1}^{\infty} \frac{x^{2n-1-1} J_{2^{n-1}}^2}{j_{2^n}^2 - x^{2^n}} = \frac{(D^2 + x)x^2}{D^2(j_2^{2^n} - x^2)}. \tag{17}$$

□

This yields [4, 5, 7]

$$\sum_{n=1}^{\infty} \frac{F_{2^n-1}^2}{L_{2^n-1}^2 - 1} = \frac{3}{20},$$

as found earlier.

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