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Sanjib Kumar Datta[1]
and
Lakshmi Biswas[2]

GENERALIZED RELATIVE ORDER (α, β) AND GENERALIZED RELATIVE TYPE (α, β) BASED SOME GROWTH PROPERTIES OF MEROMORPHIC FUNCTION WITH RESPECT TO AN ENTIRE FUNCTION

Abstract: In this paper our main aim is to introduce some idea about generalized relative order (α, β) and generalized relative type (α, β) of a meromorphic function with respect to an entire function where α and β are continuous non negative function in extended complex plane (∞, $+\infty$). Here we also discuss about some growth properties relating to the composition of entire and meromorphic functions on the basis of generalized relative order (α, β) and generalized relative type (α, β) as compared to the growth of their corresponding left and right factors.

Keywords and Phrases: Meromorphic Function, Analytic Function, Growth, Generalized Relative Nevanlinna Order (α, β), Generalized Relative Nevanlinna Type (α, β).

Mathematical Subject Classification (2010) No.: 30D30, 30D35.

1. Introduction, Definitions and Notations

In this paper, we assume that readers are familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions in the complex plane. Throughout this paper, by a meromorphic function f(z), we mean a meromorphic function in the complex plane. We use $T_f(r)$ and $M_f(r)$ to denote the characteristic function of a meromorphic function and the maximum modulus of an entire function. In the following, we will recall some notations about meromorphic functions and entire functions.

Let *f* be a meromorphic function defined in the open complex plane $\mathbb{C}$. For $a \in \mathbb{C} \cup \{\infty\}$ let $n_f(t, a)$ $(\overline{n}_f(t, a))$ the number of a-points (distinct a-points) of f in $|z| \leq t$, where an ∞-point is a pole of *f*. Also

$$N_f(r, a) = \int_0^r \frac{n_f(t, a) - n_f(0, a)}{t}\, dt + n_f(0, a) \log r$$

and

$$\overline{N}_f(r, a) = \int_0^r \frac{\overline{n}_f(t, a; f) - \overline{n}_f(0, a)}{t}\, dt + \overline{n}_f(0, a) \log r.$$

The function $N_f(r, a)$ $(\overline{N}_f(r, a))$ are called the counting function of a-points (distinct a-points) of *f*. In many occasions $N_f(r, \infty)$ and $\overline{N}_f(r, \infty)$ are denoted by $N_f(r)$ and $\overline{N}_f(r)$ respectively.

The function $m_f(r)$, which is called the proximity function of f is defined by

$$m_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(re^{i\theta}) \right| d\theta,$$

where

$$\log+ x = \log x, \text{ if } x \geq 1$$
$$= 0, \text{ if } 0 \leq x < 1.$$

For $a \in \mathbb{C}$ we denote by $m(r, \frac{1}{f-a})$ the function $m_f(r, a)$ and we mean by $m_f(r, \infty)$ the function $m_f(r)$.

The function $T_f(r) = m_f(r) + N_f(r)$ is called the Nevanlinna's characteristic function of f.

If f is entire, the function $T_f(r) = m_f(r)$ is called the Nevanlinna's characteristic function of f.

Moreover, if f is non constant entire then $T_f(r)$ is also strictly increasing and continuous function of r. Therefore its inverse $T_f^{-1} : (T_f(0), \infty) \to (0, \infty)$ exists and is such that $\lim_{s\to\infty} T_f^{-1}(s) = \infty$.

Now let L be a class of continuous non negative on $(-\infty, +\infty)$ function α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \to +\infty$ as $x \to +\infty$. For any $\alpha \in L$, we say that $\alpha \in L_1$, if $\alpha(cx) = (1 + o(1)\alpha)(x)$ as $x_0 \leq x \to +\infty$ for each $c \in (0, +\infty)$ and $\alpha \in L_2$, if $\alpha(\exp(cx)) = (1 + o(1))\,\alpha(\exp(x))$ as $x_0 \leq x \to +\infty$ for each $c \in (0, +\infty)$. Clearly $L_2 \subset L_1$.

Considering the above, Sheremeta introduced the concept of generalized order (α, β) of an entire function. For details about generalized order (α, β) one may see [6] . During the past decades, several authors made close investigations on the propertics of entire functions related to generalized order (α, β) in some different directions. For the purpose of further applications, in this paper we write the definition of the generalized order (α, β) of entire and meromorphic function in the following way:

Definition 1.1: (*Generalized order* (α, β) *and generalized lower order* (α, β)).*Let* $\alpha \in L_2$ *and* $\beta \in L_1$. *The Generalized order* (α, β) *and generalized lower order* (α, β) *of a meromorphic function f denoted by* $\rho_{(\alpha,\beta)}[f]$ *and* $\lambda_{(\alpha,\beta)}[f]$ *respectively are defined as*:

$$\begin{matrix}\rho_{(\alpha,\beta)}[f] \\ \lambda_{(\alpha,\beta)}[f]\end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\alpha(\exp(T_f(r)))}{\beta(r)}.$$

If f is an entire function, then

$$\begin{matrix}\rho_{(\alpha,\beta)}[f] \\ \lambda_{(\alpha,\beta)}[f]\end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\alpha(\exp(M_f(r)))}{\beta(r)}.$$

Using the inequality $T_f(r) \leq \log M_f(r) \leq 3T_f(2r)$, *for an entire function f, one may easily verify that*

$$\begin{matrix}\rho_{(\alpha,\beta)}[f] \\ \lambda_{(\alpha,\beta)}[f]\end{matrix} = \lim_{r\to\infty}^{\sup}_{\inf} \frac{\alpha(M_f(r))}{\beta(r)} = \lim_{r\to\infty}^{\sup}_{\inf} \frac{\alpha(\exp(T_f(r)))}{\beta(r)}.$$

The function f is said to be of regular generalized (α, β) *growth when generalized order* (α, β) *and generalized lower order* (α, β) *of f are the same. Functions which are not of regular generalized* (α, β) *growth are said to be of irregular generalized* (α, β) *growth.*

Definition 1.2: (*Generalized type* (α, β) *and generalized lower type* (α, β)).

Let $\alpha \in L_2$ *and* $\beta \in L_1$. *The generalized type* (α, β) *and generalized lower type* (α, β) *of a meromorphic function f having finite positive generalized order* (α, β) $(0 < \rho_{(\alpha,\beta)}[f] < \infty)$, *denoted by* $\sigma_{(\alpha,\beta)}[f]$ *and* $\bar{\sigma}_{(\alpha,\beta)}[f]$ *respectively are defined as:*

$$\begin{matrix}\sigma_{(\alpha,\beta)}[f] \\ \bar{\sigma}_{(\alpha,\beta)}[f]\end{matrix} = \lim_{r\to\infty}^{\sup}_{\inf} \frac{\exp(\alpha(\exp(T_f(r))))}{(\exp \beta(r))^{\rho(\alpha,\beta)[f]}}.$$

Analogously, to determine the relative growth of two meromorphic functions having same non zero finite generalized lower order (α, β) ,one can introduced the definition of generalized weak type (α, β) and generalized upper weak type (α, β) of a meromorphic function f having finite positive generalized lower order (α, β) in the following way:

Definition 1.3: (*Generalized upper weak type* (α, β) *and generalized weak type* (α, β)).

Let $\alpha \in L_2$ *and* $\beta \in L_1$. *The generalized upper weak type* (α, β) *and generalized weak type* (α, β) *of a meromorphic function f having finite positive*

generalized lower order $(\alpha, \beta)\,(0 < \lambda_{(\alpha,\beta)}[f] < \infty)$, *denoted by* $\tau_{(\alpha,\beta)}[f]$ *and* $\overline{\tau}_{(\alpha,\beta)}[f]$ *respectively are defined as*:

$$\begin{matrix}\tau_{(\alpha,\beta)}[f] \\ \overline{\tau}_{(\alpha,\beta)}[f]\end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\exp(\alpha(\exp(T_f(r))))}{(\exp \beta(r))^{\lambda_{(\alpha,\beta)}[f]}}.$$

It is obvious that $0 \le \overline{\tau}_{(\alpha,\beta)}[f] \le \tau_{(\alpha,\beta)}[f] \le \infty$.

Mainly the growth investigation of entire and meromorphic functions has usually been done through their maximum moduli or Nevanlinna's characteristic function in comparison with those of exponential function. But if one is paying attention to evaluate the growth rates of any entire and meromorphic function w.r.t.a new entire function, the notions of relative growth indicators will come. Now in order to make some progress in the study of relative order, one may introduce the definitions of generalized relative order (α, β) and generalized relative lower order (α, β) of a meromorphic function w.r.t. another entire function in the following way:

Definition 1.4: (*Generalized relative order* (α, β) *and generalized relative lower order* (α, β)).

Let $\alpha, \beta \in L_1$. *The Generalized relative order* (α, β) *and generalized relative lower order* (α, β) *of a meromorphic function* f *with respect to an entire function* g *denoted by* $\rho_{(\alpha,\beta)}[f]_g$ *and* $\lambda_{(\alpha,\beta)}[f]_g$ *respectively are defined as*:

$$\begin{matrix}\rho_{(\alpha,\beta)}[f]_g \\ \lambda_{(\alpha,\beta)}[f]_g\end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\alpha(T_g^{-1}(T_f(r)))}{\beta(r)}.$$

The previous definitions are easily generated as particular cases, e.g. if $g = z$, *Definition* 1.4 *reduces to Definition* 1.1. *If* $\alpha(r) = \beta(r) = \log r$, *then we get the definition of relative order of meromorphic function* f *with respect to an entire function* g *introduced by Lahiri et al. and if* $g = \exp z$

and $\alpha(r) = \beta(r) = \log r$ *then* $\rho_{(\alpha,\beta)}[f]_g = \rho(f)$. *Also if* $\alpha(r) = \log^{[p]} r$, $\beta(r) = \log^{[q]} r$ *and* $g = z$, *then Definition* 1.4 *becomes the classical one given in.*

Further if generalized relative order (α, β) *and generalized relative lower order* (α, β) *of a meromorphic function* f *with respect to an entire function* g *are the same, then* f *is called a function of regular generalized relative* (α, β) *growth w.r.t.* g. *Otherwise,* f *is called a irregular generalized relative* (α, β) *growth w.r.t.* g.

Now in order to refine the above growth scale, one may introduce the definitions of other growth indicators, such as generalized relative type (α, β) *and generalized relative lower type* (α, β) *of a meromorphic function* f *with respect to an entire function* g *which are as follows*:

Definition 1.5: (*Generalized relative type* (α, β) *and generalized relative lower type* (α, β)). *Let* $\alpha, \beta \in L_1$. *The Generalized relative type* (α, β) *denoted by* $\sigma_{(\alpha,\beta)}[f]_g$ *and generalized relative lower type* (α, β) *denoted by* $\bar{\sigma}_{(\alpha,\beta)}[f]_g$ *of a meromorphic function* f *with respect to an entire function* g *having non-zero finite generalized relative order* (α, β) *are defined as*:

$$\begin{matrix}\sigma_{(\alpha,\beta)}[f]_g \\ \bar{\sigma}_{(\alpha,\beta)}[f]_g\end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp \beta(r))^{\rho_{(\alpha,\beta)}[f]_g}}.$$

Analogously, to determine the relative growth of a meromorphic function f *having same non zero finite generalized relative lower order* (α, β) *with respect to an entire function* g, *one can introduce generalized relative upper weak type* (α, β) *denoted by* $\tau_{(\alpha,\beta)}[f]_g$ *and generalized relative weak type* (α, β) *denoted by* $\bar{\tau}_{(\alpha,\beta)}[f]_g$ *of* f *with respect to* g *of finite positive generalized relative lower order* (α, β) *in the following way*:

Definition 1.6: (*Generalized relative upper weak type* (α, β) *and generalized relative weak type* (α, β)).

Let $\alpha, \beta \in L_1$. *The Generalized relative upper weak type* (α, β) *and generalized relative weak type* (α, β) *of a meromorphic function f with respect to an entire function g having non-zero finite generalized relative lower order* (α, β) *denoted by* $\tau_{(\alpha,\beta)}[f]_g$ *and* $\overline{\tau}_{(\alpha,\beta)}[f]_g$ *respectively are defined as*:

$$\frac{\tau_{(\alpha,\beta)}[f]_g}{\overline{\tau}_{(\alpha,\beta)}[f]_g} = \lim_{r\to\infty}^{\sup\ \inf} \frac{\exp(\alpha(T_g^{-1}(T_f(r))))}{(\exp\ \beta(r))^{\lambda(\alpha,\beta)[f]_g}}.$$

However the main aim of this paper is to investigate some growth properties of entire and meromorphic functions using generalized relative order (α, β) and generalized relative type (α, β) of a meromorphic function with respect to an entire function which improve and extend some earlier result (*see, e.g.,*). Throughout this paper we assume that $\alpha, \beta \in L_1$, $\gamma \in L_2$ and all the growth indicators are non zero finite.

2. Main Results

In this section we preseent the main results of the paper.

Theorem 2.1: *Let f be a meromorphic function and g,h and k be non-constant entire functions such that* $0 < \lambda_{(\alpha,\beta)}[f(h)]_g \leq \rho_{(\alpha,\beta)}[f(h)]_g < \infty$ *and* $0 < \lambda_{(\alpha,\beta)}[f]_k \leq \rho_{(\alpha,\beta)}[f]_k < \infty$. *Then*

$$\frac{\lambda_{(\alpha,\beta)}[f(h)]_g}{\rho_{(\alpha,\beta)}[f]_k} \leq \lim_{r\to\infty} \inf \frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\alpha(T_k^{-1}(T_f(r)))} \leq \min\left\{\frac{\lambda_{(\alpha,\beta)}[f(h)]_g}{\lambda_{(\alpha,\beta)}[f]_k}, \frac{\rho_{(\alpha,\beta)}[f(h)]_g}{\rho_{(\alpha,\beta)}[f]_k}\right\}$$

$$\leq \max\left\{\frac{\lambda_{(\alpha,\beta)}[f(h)]_g}{\lambda_{(\alpha,\beta)}[f]_k}, \frac{\rho_{(\alpha,\beta)}[f(h)]_g}{\rho_{(\alpha,\beta)}[f]_k}\right\} \leq \lim_{r\to\infty} \sup \frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\alpha(T_k^{-1}(T_f(r)))} \leq \frac{\rho_{(\alpha,\beta)}[f(h)]_g}{\lambda_{(\alpha,\beta)}[f]_k}.$$

Proof: *From the definitions of* $\lambda_{(\alpha,\beta)}[f(h)]_g$, $\rho_{(\alpha,\beta)}[f(h)]_g$, $\lambda_{(\alpha,\beta)}[f]_k$, $\rho_{(\alpha,\beta)}[f]_k$ *and for ar bitrary positive* ϵ *and for all sufficiently large values of* r *we have*

$$\alpha(T_g^{-1}(T_{f(h)}(r))) \geq (\lambda_{(\alpha,\beta)}[f(h)]_g - \epsilon)\beta(r), \tag{1}$$

$$\alpha(T_g^{-1}(T_{f(h)}(r))) \leq (\rho_{(\alpha,\beta)}[f(h)]_g + \epsilon)\beta(r), \tag{2}$$

$$\alpha(T_k^{-1}(T_f(r))) \geq (\lambda_{(\alpha,\beta)}[f]_k - \epsilon)\beta(r), \tag{3}$$

and

$$\alpha(T_k^{-1}(T_f(r))) \leq (\rho_{(\alpha,\beta)}[f]_k + \epsilon)\beta(r). \tag{4}$$

Again for a sequence of values of r *tending to infinity,*

$$\alpha(T_g^{-1}(T_{f(h)}(r))) \leq (\lambda_{(\alpha,\beta)}[f(h)]_g + \epsilon)\beta(r), \tag{5}$$

$$\alpha(T_g^{-1}(T_{f(h)}(r))) \geq (\rho_{(\alpha,\beta)}[f(h)]_g - \epsilon)\beta(r), \tag{6}$$

$$\alpha(T_k^{-1}(T_f(r))) \leq (\lambda_{(\alpha,\beta)}[f]_k + \epsilon)\beta(r), \tag{7}$$

and

$$\alpha(T_k^{-1}(T_f(r))) \geq (\rho_{(\alpha,\beta)}[f]_k - \epsilon)\beta(r). \tag{8}$$

Now from equation (1) *and* (4) *it follows for all sufficiently large values of* r *that*

$$\frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\alpha(T_k^{-1}(T_f(r)))} \geq \frac{\lambda_{(\alpha,\beta)}[f(h)]_g - \epsilon}{\rho_{(\alpha,\beta)}[f]_k + \epsilon}.$$

As $\epsilon(>0)$ *is arbitrary, we obtain that*

$$\liminf_{r\to\infty} \frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\alpha(T_k^{-1}(T_f(r)))} \geq \frac{\lambda_{(\alpha,\beta)}[f(h)]_g}{\rho_{(\alpha,\beta)}[f]_k}, \tag{9}$$

which is the first part of the theorem.

Combining equation (5) *and* (3)*, we have for a sequence of values of* r *tending to infinity that*

$$\frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\alpha(T_k^{-1}(T_f(r)))} \leq \frac{\lambda_{(\alpha,\beta)}[f(h)]_g + \epsilon}{\lambda_{(\alpha,\beta)}[f]_k - \epsilon}.$$

As $\epsilon(>0)$ *is arbitrary, we obtain that*

$$\liminf_{r\to\infty} \frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\alpha(T_k^{-1}(T_f(r)))} \leq \frac{\lambda_{(\alpha,\beta)}[f(h)]_g}{\lambda_{(\alpha,\beta)}[f]_k}, \tag{10}$$

Again from equation (1) *and* (7)*, we have for a sequence of values of* r *tending to infinity that*

$$\frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\alpha(T_k^{-1}(T_f(r)))} \geq \frac{\lambda_{(\alpha,\beta)}[f(h)]_g - \epsilon}{\lambda_{(\alpha,\beta)}[f]_k + \epsilon}.$$

As $\epsilon(>0)$ *is arbitrary, we obtain that*

$$\limsup_{r\to\infty} \frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\alpha(T_k^{-1}(T_f(r)))} \geq \frac{\lambda_{(\alpha,\beta)}[f(h)]_g}{\lambda_{(\alpha,\beta)}[f]_k}, \tag{11}$$

Now from equation (3) *and* (2)*, it follows for all sufficiently large values of* r *that*

$$\frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\alpha(T_k^{-1}(T_f(r)))} \leq \frac{\rho_{(\alpha,\beta)}[f(h)]_g + \epsilon}{\lambda_{(\alpha,\beta)}[f]_k - \epsilon}.$$

As $\epsilon(>0)$ *is arbitrary, we obtain that*

$$\limsup_{r\to\infty}\frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\alpha(T_k^{-1}(T_f(r)))}\leq\frac{\rho_{(\alpha,\beta)}[f(h)]_g}{\lambda_{(\alpha,\beta)}[f]_k}, \tag{12}$$

which is the last part of the theorem.

Again from equation (2) *and* (8), *we have for a sequence of values of* r *tending to infinity that*

$$\frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\alpha(T_k^{-1}(T_f(r)))}\leq\frac{\rho_{(\alpha,\beta)}[f(h)]_g+\epsilon}{\rho_{(\alpha,\beta)}[f]_k-\epsilon}.$$

As $\epsilon(>0)$ *is arbitrary, we obtain that*

$$\liminf_{r\to\infty}\frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\alpha(T_k^{-1}(T_f(r)))}\leq\frac{\rho_{(\alpha,\beta)}[f(h)]_g}{\rho_{(\alpha,\beta)}[f]_k}, \tag{13}$$

Combining equation (4) *and* (6), *we have for a sequence of values of* r *tending to infinity that*

$$\frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\alpha(T_k^{-1}(T_f(r)))}\geq\frac{\rho_{(\alpha,\beta)}[f(h)]_g-\epsilon}{\rho_{(\alpha,\beta)}[f]_k+\epsilon}.$$

As $\epsilon(>0)$ *is arbitrary, we obtain that*

$$\limsup_{r\to\infty}\frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\alpha(T_k^{-1}(T_f(r)))}\geq\frac{\rho_{(\alpha,\beta)}[f(h)]_g}{\rho_{(\alpha,\beta)}[f]_k}, \tag{14}$$

So, the second part of the theorem follows from equation (10) *and* (13), *the third part is trivial and fourth part follows from* (11) *and* (14).

Thus, the theorem follows from (9), (10), (11), (12), (13) *and* (14).

Remark 2.1: *If we take* "$0 < \lambda_{(\alpha,\beta)}[h]_k \leq \rho_{(\alpha,\beta)}[h]_k < \infty$" *instead of* "$0 < \lambda_{(\alpha,\beta)}[f]_k \leq \rho_{(\alpha,\beta)}[f]_k < \infty$" *and other conditions remain same, the conclusion of Theorem* (2.1) *remains true with* "$\lambda_{(\alpha,\beta)}[f]_k$", "$\rho_{(\alpha,\beta)}[f]_k$" *and* "$\alpha(T_k^{-1}(T_f(r)))$" *replaced by* "$\lambda_{(\alpha,\beta)}[h]_k$","$\rho_{(\alpha,\beta)}[h]_k$" *and* "$\alpha(T_k^{-1}(T_h(r)))$" *respectively in the denominator.*

Theorem 2.2: *Let f be a meromorphic function and g,h and k be non-constant entire functions such that $0 < \lambda_{(\alpha,\beta)}[f]_k \leq \rho_{(\alpha,\beta)}[f]_k < \infty$ and $\lambda_{(\alpha,\beta)}[f(h)]_g = \infty$. Then*

$$\lim_{r\to\infty} \frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\alpha(T_k^{-1}(T_f(r)))} = \infty.$$

Proof: *If possible, let the conclusion of the theorem does not hold. Then we can find a constant $\delta > 0$ such that for a sequence of values of r tending to infinity*

$$\alpha(T_g^{-1}(T_{f(h)}(r))) \leq \delta.\, \alpha(T_k^{-1}(T_f(r))). \tag{15}$$

Again from the definition of $\rho_{(\alpha,\beta)}[f]_k$, it follows that for all sufficiently large values of r

$$\alpha(T_k^{-1}(T_f(r))) \leq (\rho_{(\alpha,\beta)}[f]_k + \epsilon)\beta(r). \tag{16}$$

From (15) *and* (16)*, for a sequence of values of r tending to infty,*

$$\alpha(T_g^{-1}(T_{f(h)}(r))) \leq \delta\, (\rho_{(\alpha,\beta)}[f]_k + \epsilon)\beta(r).$$

$$i.e., \frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\beta(r)} \leq \delta\, (\rho_{(\alpha,\beta)}[f]_k + \epsilon)$$

$$i.e., \liminf_{r\to\infty} \frac{\alpha(T_g^{-1}(T_{f(h)}(r)))}{\beta(r)} \leq \lambda_{(\alpha,\beta)}[f(h)]_k < \infty.$$

This is a contradiction. Hence, the theorem holds.

Remark 2.2: *If we take* "$0 < \lambda_{(\alpha,\beta)}[h]_k \leq \rho_{(\alpha,\beta)}[h]_k < \infty$" *instead of* "$0 < \lambda_{(\alpha,\beta)}[f]_k \leq \rho_{(\alpha,\beta)}[f]_k < \infty$" *and other conditions remain same, the conclusion of Theorem* (2.2) *remains true with* "$\alpha(T_k^{-1}(T_f(r)))$" *replaced by* "$\alpha(T_k^{-1}(T_h(r)))$" *in the denominator.*

Remark 2.3: *Theorem* (2.2) *and Remark* (2.2) *are also valid with "limit superior" instead of "limit" if* "$\lambda_{(\alpha,\beta)}[f(h)]_g = \infty$" *is replaced by* "$\rho_{(\alpha,\beta)}[f(h)]_g = \infty$" *and the other conditions remain the same.*

Theorem 2.3: *Let f be a meromorphic function and g,h and k be non-constant entire functions such that* $0 < \bar{\sigma}_{(\alpha,\beta)}[f(h)]_g \leq \sigma_{(\alpha,\beta)}[f(h)]_g < \infty$ *and* $0 < \bar{\sigma}_{(\alpha,\beta)}[f]_k \leq \sigma_{(\alpha,\beta)}[f]_k < \infty$. *Then*

$$\frac{\bar{\sigma}_{(\alpha,\beta)}[f(h)]_g}{\sigma_{(\alpha,\beta)}[f]_k} \leq \liminf_{r\to\infty} \frac{\exp(\alpha(T_g^{-1}(T_{f(h)}(r))))}{\exp(\alpha(T_k^{-1}(T_f(r))))} \leq \min\left\{\frac{\bar{\sigma}_{(\alpha,\beta)}[f(h)]_g}{\bar{\sigma}_{(\alpha,\beta)}[f]_k}, \frac{\sigma_{(\alpha,\beta)}[f(h)]_g}{\sigma_{(\alpha,\beta)}[f]_k}\right\}$$

$$\leq \max\left\{\frac{\bar{\sigma}_{(\alpha,\beta)}[f(h)]_g}{\bar{\sigma}_{(\alpha,\beta)}[f]_k}, \frac{\sigma_{(\alpha,\beta)}[f(h)]_g}{\sigma_{(\alpha,\beta)}[f]_k}\right\} \leq \limsup_{r\to\infty} \frac{\exp(\alpha(T_g^{-1}(T_{f(h)}(r))))}{\exp(\alpha(T_k^{-1}(T_f(r))))} \leq \frac{\sigma_{(\alpha,\beta)}[f(h)]_g}{\bar{\sigma}_{(\alpha,\beta)}[f]_k}.$$

Proof: *From the definitions of* $\sigma_{(\alpha,\beta)}[f(h)]_g$, $\bar{\sigma}_{(\alpha,\beta)}[f(h)]_g$, $\sigma_{(\alpha,\beta)}[f]_k$, $\bar{\sigma}_{(\alpha,\beta)}[f]_k$ *and for arbitrary positive* ϵ *and for all sufficiently large values of* r *we have*

$$\exp(\alpha(T_g^{-1}(T_{f(h)}(r)))) \geq (\bar{\sigma}_{(\alpha,\beta)}[f(h)]_g - \epsilon)(\exp\beta(r))^{\rho(\alpha,\beta)[f(h)]_g}, \quad (17)$$

$$\exp(\alpha(T_k^{-1}(T_f(r)))) \leq (\sigma_{(\alpha,\beta)}[f]_k + \epsilon)(\exp\beta(r))^{\rho(\alpha,\beta)[f]_k}, \quad (18)$$

$$\exp(\alpha(T_k^{-1}(T_f(r)))) \geq (\overline{\sigma}_{(\alpha,\beta)}[f]_k - \epsilon)(\exp \beta(r))^{\rho(\alpha,\beta)[f]_k}, \quad (19)$$

and

$$\exp(\alpha(T_g^{-1}(T_{f(h)}(r)))) \leq (\sigma_{(\alpha,\beta)}[f(h)]_g + \epsilon(\exp \beta(r))^{\rho(\alpha,\beta)[f(h)]_g}. \quad (20)$$

Again for a sequence of values of r tending to infinity,

$$\exp(\alpha(T_g^{-1}(T_{f(h)}(r)))) \leq (\overline{\sigma}_{(\alpha,\beta)}[f(h)]_g + \epsilon)(\exp \beta(r))^{\rho(\alpha,\beta)[f(h)]_g}, \quad (21)$$

$$\exp(\alpha(T_k^{-1}(T_f(r)))) \geq (\sigma_{(\alpha,\beta)}[f]_k - \epsilon)(\exp \beta(r))^{\rho(\alpha,\beta)[f]_k}, \quad (22)$$

$$\exp(\alpha(T_k^{-1}(T_f(r)))) \leq (\overline{\sigma}_{(\alpha,\beta)}[f]_k + \epsilon)(\exp \beta(r))^{\rho(\alpha,\beta)[f]_k}, \quad (23)$$

and

$$\exp(\alpha(T_g^{-1}(T_{f(h)}(r)))) \geq (\sigma_{(\alpha,\beta)}[f(h)]_g - \epsilon(\exp \beta(r))^{\rho(\alpha,\beta)[f(h)]_g}. \quad (24)$$

Now from equation (17) *and* (18) *and the condition* $\rho_{(\alpha,\beta)}[f(h)]_g = \rho_{(\alpha,\beta)}[f]_k$ *it follows for all sufficiently large values of r that*

$$\frac{\exp(\alpha(T_g^{-1}(T_{f(h)}(r))))}{\exp(\alpha(T_k^{-1}(T_f(r))))} \geq \frac{\overline{\sigma}_{(\alpha,\beta)}[f(h)]_g - \epsilon}{\sigma_{(\alpha,\beta)}[f]_k + \epsilon}.$$

As $\epsilon(> 0)$ *is arbitrary, we obtain that*

$$\liminf_{r\to\infty} \frac{\exp(\alpha(T_g^{-1}(T_{f(h)}(r))))}{\exp(\alpha(T_k^{-1}(T_f(r))))} \geq \frac{\overline{\sigma}_{(\alpha,\beta)}[f(h)]_g}{\sigma_{(\alpha,\beta)}[f]_k}, \quad (25)$$

which is the first part of the theorem.

Combining equation (21) *and* (19) *and the condition* $\rho_{(\alpha,\beta)}[f(h)]_g = \rho_{(\alpha,\beta)}[f]_k$, *we have for a sequence of values of r tending to infinity that*

$$\frac{\exp(\alpha(T_g^{-1}(T_{f(h)}(r))))}{\exp(\alpha(T_k^{-1}(T_f(r))))} \leq \frac{\overline{\sigma}_{(\alpha,\beta)}[f(h)]_g + \epsilon}{\overline{\sigma}_{(\alpha,\beta)}[f]_k - \epsilon}.$$

As $\epsilon(>0)$ *is arbitrary, we obtain that*

$$\liminf_{r\to\infty} \frac{\exp(\alpha(T_g^{-1}(T_{f(h)}(r))))}{\exp(\alpha(T_k^{-1}(T_f(r))))} \leq \frac{\overline{\sigma}_{(\alpha,\beta)}[f(h)]_g}{\overline{\sigma}_{(\alpha,\beta)}[f]_k}, \tag{26}$$

Again from equation (17) *and* (23) *and the condition* $\rho_{(\alpha,\beta)}[f(h)]_g = \rho_{(\alpha,\beta)}[f]_k$, *we have for a sequence of values of* r *tending to infinity that*

$$\frac{\exp(\alpha(T_g^{-1}(T_{f(h)}(r))))}{\exp(\alpha(T_k^{-1}(T_f(r))))} \geq \frac{\overline{\sigma}_{(\alpha,\beta)}[f(h)]_g - \epsilon}{\overline{\sigma}_{(\alpha,\beta)}[f]_k + \epsilon}.$$

As $\epsilon(>0)$ *is arbitrary, we obtain that*

$$\limsup_{r\to\infty} \frac{\exp(\alpha(T_g^{-1}(T_{f(h)}(r))))}{\exp(\alpha(T_k^{-1}(T_f(r))))} \geq \frac{\overline{\sigma}_{(\alpha,\beta)}[f(h)]_g}{\overline{\sigma}_{(\alpha,\beta)}[f]_k}, \tag{27}$$

Now from equation (19) *and* (20) *and the condition* $\rho_{(\alpha,\beta)}[f(h)]_g = \rho_{(\alpha,\beta)}[f]_k$, *it follows for all sufficiently large values of* r *that*

$$\frac{\exp(\alpha(T_g^{-1}(T_{f(h)}(r))))}{\exp(\alpha(T_k^{-1}(T_f(r))))} \leq \frac{\sigma_{(\alpha,\beta)}[f(h)]_g + \epsilon}{\overline{\sigma}_{(\alpha,\beta)}[f]_k - \epsilon}.$$

As $s(>0)$*is arbitrary, we obtain that*

$$\limsup_{r\to\infty} \frac{\exp(\alpha(T_g^{-1}(T_{f(h)}(r))))}{\exp(\alpha(T_k^{-1}(T_f(r))))} \leq \frac{\sigma_{(\alpha,\beta)}[f(h)]_g}{\sigma_{(\alpha,\beta)}[f]_k}, \tag{28}$$

which is the last part of the theorem.

Again from equation (20) *and* (22) *and the condition* $\rho_{(\alpha,\beta)}[f(h)]_g = \rho_{(\alpha,\beta)}[f]_k$, *we have for a sequence of values of* r *tending to infinity that*

$$\frac{\exp(\alpha(T_g^{-1}(T_{f(h)}(r))))}{\exp(\alpha(T_k^{-1}(T_f(r))))} \leq \frac{\sigma_{(\alpha,\beta)}[f(h)]_g + \epsilon}{\sigma_{(\alpha,\beta)}[f]_k - \epsilon}.$$

As $\epsilon(> 0)$ *is arbitrary, we obtain that*

$$\liminf_{r\to\infty} \frac{\exp(\alpha(T_g^{-1}(T_{f(h)}(r))))}{\exp(\alpha(T_k^{-1}(T_f(r))))} \leq \frac{\sigma_{(\alpha,\beta)}[f(h)]_g}{\sigma_{(\alpha,\beta)}[f]_k}, \tag{29}$$

Combining equation (18) *and* (24) *and the condition* $\rho_{(\alpha,\beta)}[f(h)]_g = \rho_{(\alpha,\beta)}[f]_k$, *we have for a sequence of values of* r *tending to infinity that*

$$\frac{\exp(\alpha(T_g^{-1}(T_{f(h)}(r))))}{\exp(\alpha(T_k^{-1}(T_f(r))))} \geq \frac{\sigma_{(\alpha,\beta)}[f(h)]_g - \epsilon}{\overline{\sigma}_{(\alpha,\beta)}[f]_k + \epsilon}.$$

As $\epsilon(> 0)$ *is arbitrary, we obtain that*

$$\limsup_{r\to\infty} \frac{\exp(\alpha(T_g^{-1}(T_{f(h)}(r))))}{\exp(\alpha(T_k^{-1}(T_f(r))))}) \geq \frac{\sigma_{(\alpha,\beta)}[f(h)]_g}{\sigma_{(\alpha,\beta)}[f]_k}, \tag{30}$$

Thus, the theorem follows from (25), (26), (27), (28), (29) *and* (30).

Remark 2.4: *If we take* "$0 < \overline{\sigma}_{(\alpha,\beta)}[h]_k \leq \sigma_{(\alpha,\beta)}[h]_k < \infty$" *and* $\rho_{(\alpha,\beta)}[f(h)]_g = \rho_{(\alpha,\beta)}[h]_k$, *instead of* "$0 < \overline{\sigma}_{(\alpha,\beta)}[f]_k \leq \sigma_{(\alpha,\beta)}[f]_k < \infty$" *and* $\rho_{(\alpha,\beta)}[f(h)]_g = \rho_{(\alpha,\beta)}[f]_k$ *other conditions remain same, the conclusion of Theorem* (2.3) *remains true with* "$\sigma_{(\alpha,\beta)}[f]_k$", "$\overline{\sigma}_{(\alpha,\beta)}[f]_k$"

and "$exp(\alpha(T_k^{-1}(T_f(r))))$" *replaced by* "$\sigma_{(\alpha,\beta)}[f]_k$", "$\overline{\sigma}_{(\alpha,\beta)}[h]_k$" "$\exp(\alpha(T_k^{-1}(T_h(r))))$" *respectively in the denominator.*

Remark 2.5: *If we take* "$0 < \overline{\tau}_{(\alpha,\beta)}[f]_k \leq \tau_{(\alpha,\beta)}[f]_k < \infty$" *and* $\rho_{(\alpha,\beta)}[f(h)]_g = \lambda_{(\alpha,\beta)}[h]_k$, *instead of* "$0 < \overline{\sigma}_{(\alpha,\beta)}[f]_k \leq \sigma_{(\alpha,\beta)}[f]_k < \infty$" *and* $\rho_{(\alpha,\beta)}[f(h)]_g = \rho_{(\alpha,\beta)}[f]_k$ *other conditions remain same, the conclusion of Theorem* (2.3) *remains true with* "$\sigma_{(\alpha,\beta)}[f]_k$", "$\overline{\sigma}_{(\alpha,\beta)}[f]_k$" *and* "$\exp(\alpha(T_k^{-1}(T_f(r))))$" *replaced by* "$\tau_{(\alpha,\beta)}[f]_k$", "$\overline{\tau}_{(\alpha,\beta)}[h]_k$" "$\exp(\alpha(T_k^{-1}(T_h(r))))$" *respectively in the denominator.*

Remark 2.6: *If we take* "$0 < \overline{\tau}_{(\alpha,\beta)}[h]_k \leq \tau_{(\alpha,\beta)}[h]_k < \infty$" *and* $\rho_{(\alpha,\beta)}[f(h)]_g = \lambda_{(\alpha,\beta)}[h]_k$, *instead of* "$0 < \overline{\sigma}_{(\alpha,\beta)}[f]_k \leq \sigma_{(\alpha,\beta)}[f]_k < \infty$" *and* $\rho_{(\alpha,\beta)}[f(h)]_g = \rho_{(\alpha,\beta)}[f]_k$ *other conditions remain same, the conclusion of Theorem* (2.3) *remains true with* "$\sigma_{(\alpha,\beta)}[f]_k$", "$\overline{\sigma}_{(\alpha,\beta)}[f]_k$" *and* "$\exp(\alpha(T_k^{-1}(T_f(r))))$" *replaced by* "$\tau_{(\alpha,\beta)}[h]_k$", "$\overline{\tau}_{(\alpha,\beta)}[h]_k$" "$\exp(\alpha(T_k^{-1}(T_h(r))))$" *respectively in the denominator.*

Now in the line of Theorem (2.3), one can easily prove the following theorem using the notion of Generalized relative upper weak type (α, β) and generalized relative weak type (α, β) and therefore the proof is omitted.

Theorem 2.4: *Let f be a meromorphic function and g, h and k be non-constant entire functions such that* $0 < \overline{\tau}_{(\alpha,\beta)}[f(h)]_g \leq \tau_{(\alpha,\beta)}[f(h)]_g < \infty$ *and* $0 < \overline{\tau}_{(\alpha,\beta)}[f]_k \leq \tau_{(\alpha,\beta)}[f]_k < \infty$ *and* $\lambda_{(\alpha,\beta)}[f(h)]_g = \lambda_{(\alpha,\beta)}[f]_k$. *Then*

$$\frac{\overline{\tau}_{(\alpha,\beta)}[f(h)]_g}{\tau_{(\alpha,\beta)}[f]_k} \leq \liminf_{r\to\infty} \frac{\exp(\alpha(T_g^{-1}(T_{f(h)}(r))))}{\exp(\alpha(T_k^{-1}(T_f(r))))} \leq \min\left\{\frac{\overline{\tau}_{(\alpha,\beta)}[f(h)]_g}{\overline{\tau}_{(\alpha,\beta)}[f]_k}, \frac{\tau_{(\alpha,\beta)}[f(h)]_g}{\tau_{(\alpha,\beta)}[f]_k}\right\}$$

$$\leq \max\left\{\frac{\overline{\tau}_{(\alpha,\beta)}[f(h)]_g}{\overline{\tau}_{(\alpha,\beta)}[f]_k}, \frac{\tau_{(\alpha,\beta)}[f(h)]_g}{\tau_{(\alpha,\beta)}[f]_k}\right\} \leq \limsup_{r\to\infty} \frac{\exp(\alpha(T_g^{-1}(T_{f(h)}(r)))}{\exp(\alpha(T_k^{-1}(T_f(r)))} \leq \frac{\tau_{(\alpha,\beta)}[f(h)]_g}{\overline{\tau}_{(\alpha,\beta)}[f]_k}.$$

Remark 2.7: *If we take* "$0 < \overline{\tau}_{(\alpha,\beta)}[h]_k \leq \tau_{(\alpha,\beta)}[h]_k < \infty$" *and* $\lambda_{(\alpha,\beta)}[f(h)]_g = \lambda_{(\alpha,\beta)}[h]_k$, *instead of* "$0 < \overline{\tau}_{(\alpha,\beta)}[f]_k \leq \tau_{(\alpha,\beta)}[f]_k < \infty$" *and* $\lambda_{(\alpha,\beta)}[f(h)]_g = \lambda_{(\alpha,\beta)}[f]_k$ *and other conditions remain same, the conclusion of Theorem* (2.4) *remains true with* "$\tau_{(\alpha,\beta)}[f]_k$", "$\overline{\tau}_{(\alpha,\beta)}[f]_k$" *and* "$\exp(\alpha(T_k^{-1}(T_f(r))))$" *replaced by* "$\tau_{(\alpha,\beta)}[f]_k$", "$\tau_{(\alpha,\beta)}[h]_k$" "$\exp(\alpha(T_k^{-1}(T_h(r))))$" *respectively* in the denominator.

Remark 2.8: *If we take* "$0 < \overline{\sigma}_{(\alpha,\beta)}[f]_k \leq \sigma_{(\alpha,\beta)}[f]_k < \infty$" *and* $\lambda_{(\alpha,\beta)}[f(h)]_g = \rho_{(\alpha,\beta)}[f]_k$, *instead of* "$0 < \overline{\tau}_{(\alpha,\beta)}[f]_k \leq \tau_{(\alpha,\beta)}[f]_k < \infty$" *and* $\lambda_{(\alpha,\beta)}[f(h)]_g = \lambda_{(\alpha,\beta)}[f]_k$ *and other conditions remain same, the conclusion of Theorem* (2.4) *remains true with* "$\tau_{(\alpha,\beta)}[f]_k$", "$\overline{\tau}_{(\alpha,\beta)}[f]_k$" *replaced by* "$\sigma_{(\alpha,\beta)}[f]_k$", "$\overline{\sigma}_{(\alpha,\beta)}[h]_k$" *respectively in the denominator.*

Remark 2.9: *If we take* "$0 < \overline{\sigma}_{(\alpha,\beta)}[f]_k \leq \sigma_{(\alpha,\beta)}[f]_k < \infty$" *and* $\lambda_{(\alpha,\beta)}[f(h)]_g = \lambda_{(\alpha,\beta)}[h]_k$, *instead of* "$0 < \overline{\tau}_{(\alpha,\beta)}[f]_k \leq \tau_{(\alpha,\beta)}[f]_k < \infty$" *and* $\lambda_{(\alpha,\beta)}[f(h)]_g = \lambda_{(\alpha,\beta)}[f]_k$ *and other conditions remain same, the conclusion of Theorem* (2.4) *remains true with* "$\tau_{(\alpha,\beta)}[f]_k$", "$\overline{\tau}_{(\alpha,\beta)}[f]_k$" *and* "$\exp(\alpha(T_k^{-1}(T_f(r))))$" *replaced by* "$\sigma_{(\alpha,\beta)}[h]_k$", "$\overline{\sigma}_{(\alpha,\beta)}[h]_k$" "$\exp(\alpha(T_k^{-1}(T_h(r))))$" *respectively in the denominator.*

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1. Department of Mathematics,
University of Kalyani P.O.: Kalyani,
Dist: Nadia, Pin: 741235, West Bengal, India
E-mail: sanjibdatta05@gmail.com

2. Kalinarayanpur Adarsha Vidyalaya
P.O.: Kalinarayanpur,
Dist.: Nadia, Pin: 741254, West Bengal, India
E-mail: kutkijit@gmail.com

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